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MODIFIED GENERALIZED $\alpha - \psi -$GERAGHTY CONTRACTION TYPE MAPS IN METRIC SPACE AND RELATED FIXED POINTS

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Abstract: In this paper, we introduce the notion of modified generalized $\alpha - \psi -$Geraghty contraction type maps in the context of metric space and establish some fixed point theorems for such maps. Our results provide the fixed point results of K. Anthony Singh [8] and Popescu [16] as direct corollaries. Some examples are also given to illustrate the validity of our results.

Keywords and Phrases: Metric space, fixed point, $\alpha -$orbital admissible mapping with respect to η, triangular $\alpha -$orbital admissible mapping with respect to η , generalized $\alpha -$Geraghty contraction type map, extended generalized $\alpha - \psi -$Geraghty contraction type map, modified generalized $\alpha - \psi -$Geraghty contraction type map.

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1. Introduction

The celebrated Banach contraction principle which forms the foundation of the metric fixed point theory, is one of the most widely used fixed point theorems in all analysis. Over the years, this result has been generalized in different directions by various authors and researchers. Among such results, the works of Geraghty [6], Amini-Harandi and Emami [2], Caballero et al. [4], Gordji et al. [7] etc. may be mentioned. Samet et al. [18] introduced the concepts of $\alpha-\psi-$contractive and $\alpha-$admissible mappings and established fixed point results for such mappings. Karapinar & Samet [11] further generalized these notions to obtain extended fixed point results. Salimi et al. [17] modified the notions of $\alpha-\psi-$contractive and $\alpha-$admissible mappings and established fixed point results for such mappings. Recently, in the line of these developments, Cho et al. [5] defined the concept of $\alpha-$Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps. Further, Erdal Karapinar [12] introduced the concept of $\alpha-\psi-$Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al. [5]. Very recently, Popescu [16] also generalized the results of Cho et al. [5] and gave other conditions to prove the existence and uniqueness of a fixed point of $\alpha-$Geraghty contraction type maps. Then K. Anthony Singh [8] defined extended generalized $\alpha-\psi-$ Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Popescu [16].

In this paper, motivated by the results of Salimi et al. [17], Popescu [16] and K. Anthony Singh [8], we define modified generalized $\alpha-\psi-$Geraghty contraction type maps in the setting of metric space and obtain the existence and uniqueness of a fixed point of such maps. Our results extend the fixed point results of K. Anthony Singh [8] and Popescu [16]. We also give some examples to illustrate our results.

2. Preliminaries

In this section, we recall some basic definitions and related results on the topic in the literature.

Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfy the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

Geraghty used such functions to prove the following result.

Theorem 2.1. [6] *Let (X,d) be a complete metric space and let T be a mapping on X. Suppose there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \le \beta(d(x,y))d(x,y).$$

Then T has a unique fixed point $x_* \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.

Definition 2.2. [16] *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function.*

Then T is said to be $\alpha-$orbital admissible if $\alpha(x,Tx)\geq 1$ implies $\alpha(Tx,T^2x)\geq 1$.

Definition 2.3. [16] *Let $T:X\to X$ be a map and $\alpha:X\times X\to\mathbb{R}$ be a function. Then T is said to be triangular $\alpha-$orbital admissible if T is $\alpha-$orbital admissible and $\alpha(x,y)\geq 1$ and $\alpha(y,Ty)\geq 1$ imply $\alpha(x,Ty)\geq 1$.*

Lemma 2.4. [16] *Let $T:X\to X$ be a triangular $\alpha-$orbital admissible map. Assume that there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1}=Tx_n$. Then we have $\alpha(x_n,x_m)\geq 1$ for all $m,n\in\mathbb{N}$ with $n<m$.*

Popescu [16] introduced the following contraction and proved some fixed point results generalising the results of Cho et al. [5].

Definition 2.5. [16] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-$Geraghty contraction type map if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)d(Tx,Ty)\leq\beta\left(M_T(x,y)\right)M_T(x,y),$$

where $M_T(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),[d(x,Ty)+d(y,Tx)]/2\}$.

We recall the following class of auxiliary functions defined in the paper by Erdal Karapinar [12].

Let Ψ denote the class of functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;

(b) ψ is subadditive, that is, $\psi(s+t)\leq\psi(s)+\psi(t)$;

(c) ψ is continuous;

(d) $\psi(t)=0\Leftrightarrow t=0$.

Using this class of auxiliary functions, K. Anthony Singh [8] then introduced the following contraction and proved some fixed point results generalising the results of Popescu [16].

Definition 2.6. [8] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called an extended generalized $\alpha-\psi-$Geraghty contraction type map if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty))\leq\beta(\psi\left(M_T(x,y)\right))\psi(M_T(x,y)),$$

where $M_T(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty),[d(x,Ty)+d(y,Tx)]/2\}$ and $\psi\in\Psi$.

3. Main Results

We now state and prove our main results. First we introduce some new definitions and concepts.

Let Ψ^* denote the class of functions $\psi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:

(a) ψ is nondecreasing;

(b) ψ is continuous;

(c) $\psi(t) = 0 \Leftrightarrow t = 0$.

This class of functions Ψ^* is more general than the class Ψ introduced by Karapinar [12]. Here, we dispense with the subadditivity condition in Ψ which is not needed in the proofs of our results.

Definition 3.1. *Let $T : X \to X$ be a map and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Then T is said to be $\alpha-$orbital admissible with respect to η if $\alpha(x, Tx) \geq \eta(x, Tx)$ implies $\alpha(Tx, T^2x) \geq \eta(Tx, T^2x)$.*
Note that if $\eta(x, y) = 1$, then T becomes an $\alpha-$orbital admissible mapping and if $\alpha(x, y) = 1$, then T is called an $\eta-$orbital subadmissible mapping.

Definition 3.2. *Let $T : X \to X$ be a map and $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions. Then T is said to be triangular $\alpha-$orbital admissible with respect to η if T is $\alpha-$orbital admissible with respect to η and $\alpha(x, y) \geq \eta(x, y)$ and $\alpha(y, Ty) \geq \eta(y, Ty)$ imply $\alpha(x, Ty) \geq \eta(x, Ty)$.*
Note that if $\eta(x, y) = 1$, then T becomes a triangular $\alpha-$orbital admissible mapping and if $\alpha(x, y) = 1$, then T is called a triangular $\eta-$orbital subadmissible mapping.

Lemma 3.3. *Let $T : X \to X$ be a triangular α-orbital admissible mapping with respect to η. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.*

Proof. Since T is $\alpha-$orbital admissible mapping with respect to η and $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$, we have $\alpha(Tx_1, T^2x_1) \geq \eta(Tx_1, T^2x_1)$ i.e. $\alpha(x_2, x_3) \geq \eta(x_2, x_3)$. Continuing in this way, we get $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \geq 1$. Let us suppose that $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ and prove that $\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1})$, where $n < m$. Since T is triangular $\alpha-$orbital admissible map with respect to η and $\alpha(x_m, x_{m+1}) \geq \eta(x_m, x_{m+1})$, we get $\alpha(x_n, x_{m+1}) \geq \eta(x_n, x_{m+1})$. Thus we have proved that $\alpha(x_n, x_m) \geq \eta(x_n, x_m)$ for all $m, n \in \mathbb{N}$ with $n < m$.

Theorem 3.4. *Let (X, d) be a complete metric space, $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions and let $T : X \to X$ be a map. Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Rightarrow \psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$ and $\psi \in \Psi^*, \beta \in \mathcal{F}$.
Suppose that the following conditions are satisfied

(1) T is a triangular $\alpha-$orbital admissible mapping with respect to η,

(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$,

(3) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$. We construct a sequence of points $\{x_n\}$ in X such that $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N}$, then x_n is a fixed point of T. Therefore, we assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. By hypothesis, $\alpha(x_1, x_2) \geq \eta(x_1, x_2)$ and the map T is triangular $\alpha-$orbital admissible with respect to η. Therefore by Lemma 3.3., we have
$\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all $n \in \mathbb{N}$.
Then, we have

$$\begin{aligned}\psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \\ &\leq \beta(\psi(M_T(x_n, x_{n+1})))\psi(M_T(x_n, x_{n+1})) \quad \text{for all } n \in \mathbb{N} \end{aligned} \tag{3.1}$$

Here, we have

$$\begin{aligned} M_T(x_n, x_{n+1}) &= \max\{d(x_n, x_{n+1}), d(x_n, Tx_n), d(x_{n+1}, Tx_{n+1}), \\ &\qquad [d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)]/2\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), d(x_n, x_{n+2})/2\} \\ &\leq \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2}), [d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})]/2\} \\ &= \max\{d(x_n, x_{n+1}), d(x_{n+1}, x_{n+2})\}. \end{aligned}$$

Let us suppose that $d(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$. Since $\beta(\psi(M_T(x_n, x_{n+1}))) < 1$, we have from (3.1)
$\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2}))$ which is a contradiction.
Therefore, we must have

$d(x_n, x_{n+1}) > d(x_{n+1}, x_{n+2})$ for all $n \in \mathbb{N}$ and by then $M_T(x_n, x_{n+1}) = d(x_n, x_{n+1})$.

Thus the sequence $\{d(x_n, x_{n+1})\}$ is positive and decreasing.
Now, we prove that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.
It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence which is bounded from below. Therefore there exists $r \geq 0$ such that $\lim_{n\to\infty} d(x_n, x_{n+1}) = r$. We show that $r = 0$. We suppose on the contrary that $r > 0$. We have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \beta(\psi(d(x_n, x_{n+1}))) < 1.$$

Now by taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \beta(\psi(d(x_n, x_{n+1}))) = 1.$$

By the property of β, we have

$$\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0,$$

which is a contradiction.
Therefore, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r = 0. \tag{3.2}$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, for all positive integers k , there exist $m_k > n_k > k$ with

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{3.3}$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$d(x_{m_k-1}, x_{n_k}) < \epsilon. \tag{3.4}$$

By (3.3) and (3.4), we have

$$\begin{aligned}\epsilon &\leq d(x_{m_k}, x_{n_k})\\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k})\\ &< d(x_{m_k-1}, x_{m_k}) + \epsilon\end{aligned}$$

that is,

$$\epsilon \leq d(x_{m_k}, x_{n_k}) < \epsilon + d(x_{m_k-1}, x_{m_k}) \text{ for all } k \in \mathbb{N}. \tag{3.5}$$

Then in view of (3.2) and (3.5), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{3.6}$$

Again, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{m_k-1})\\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{n_k-1}) + d(x_{m_k-1}, x_{n_k-1})\\ \text{and } d(x_{m_k-1}, x_{n_k-1}) &\leq d(x_{m_k-1}, x_{m_k}) + d(x_{n_k-1}, x_{n_k}) + d(x_{m_k}, x_{n_k}).\end{aligned}$$

Taking limit as $k \to \infty$ and using (3.2) and (3.6), we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{3.7}$$

Also, we have

$$|d(x_{n_k}, x_{m_k-1}) - d(x_{n_k}, x_{m_k})| \leq d(x_{m_k}, x_{m_k-1}).$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \epsilon.$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \epsilon.$$

By Lemma 3.3., we get $\alpha(x_{n_k-1}, x_{m_k-1}) \geq \eta(x_{n_k-1}, x_{m_k-1})$. Therefore, we have

$$\begin{aligned}\psi(d(x_{m_k}, x_{n_k})) &= \psi(d(Tx_{m_k-1}, Tx_{n_k-1}))\\ &\leq \beta(\psi(M_T(x_{n_k-1}, x_{m_k-1})))\psi(M_T(x_{n_k-1}, x_{m_k-1})).\end{aligned}$$

Here we have

$$\begin{aligned}M_T(x_{n_k-1}, x_{m_k-1}) &= \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, Tx_{n_k-1}), d(x_{m_k-1}, Tx_{m_k-1}),\\ &\qquad [d(x_{n_k-1}, Tx_{m_k-1}) + d(x_{m_k-1}, Tx_{n_k-1})]/2\}\\ &= \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}),\\ &\qquad [d(x_{n_k-1}, x_{m_k}) + d(x_{m_k-1}, x_{n_k})]/2\}.\end{aligned}$$

And we see that

$$\lim_{k\to\infty} M_T(x_{n_k-1}, x_{m_k-1}) = \epsilon.$$

Now we have

$$\frac{\psi(d(x_{n_k}, x_{m_k}))}{\psi(M_T(x_{n_k-1}, x_{m_k-1}))} \leq \beta(\psi(M_T(x_{n_k-1}, x_{m_k-1}))) < 1.$$

By using (3.6) and taking limit as $k \to \infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \beta(\psi(M_T(x_{n_k-1}, x_{m_k-1}))) = 1.$$

So, $\lim_{k\to\infty} \psi(M_T(x_{n_k-1}, x_{m_k-1})) = 0 \Rightarrow \lim_{k\to\infty} M_T(x_{n_k-1}, x_{m_k-1}) = 0 = \epsilon$, which is a contradiction.

Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. As T is continuous, we have $Tx_n \to Tx^*$ i.e. $\lim_{n\to\infty} x_{n+1} = Tx^*$ and so $x^* = Tx^*$. Hence x^* is a fixed point of T.

Theorem 3.5. *Let (X, d) be a complete metric space, $\alpha, \eta : X \times X \to \mathbb{R}$ be two functions and let $T : X \to X$ be a map. Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq \eta(x, y) \Rightarrow \psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$ and $\psi \in \Psi^*, \beta \in \mathcal{F}$.
Suppose that the following conditions are satisfied
(1) T is a triangular $\alpha-$orbital admissible mapping with respect to η,
(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq \eta(x_1, Tx_1)$,
(3) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq \eta(x_n, x_{n+1})$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq \eta(x_{n(k)}, x)$ for all k.
Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.
Proof. The proof goes along similar lines of the proof of Theorem 3.4. We conclude that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \in \mathbb{N}$, converges to a point say $x^* \in X$. By hypothesis (3), there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x^*) \geq \eta(x_{n_k}, x^*)$ for all k. Now for all k, we have

$$\begin{aligned}\psi(d(x_{n_k+1}, Tx^*)) &= \psi(d(Tx_{n_k}, Tx^*)) \\ &\leq \beta(\psi(M_T(x_{n_k}, x^*)))\psi(M_T(x_{n_k}, x^*))\end{aligned}$$

so that

$$\psi(d(x_{n_k+1}, Tx^*)) \leq \beta(\psi(M_T(x_{n_k}, x^*)))\psi(M_T(x_{n_k}, x^*)) \tag{3.8}$$

On the other hand, we have

$$\begin{aligned}M_T(x_{n_k}, x^*) &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, Tx_{n_k}), d(x^*, Tx^*), \\ &\qquad [d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k})]/2\} \\ &= \max\{d(x_{n_k}, x^*), d(x_{n_k}, x_{n_k+1}), d(x^*, Tx^*), \\ &\qquad [d(x_{n_k}, Tx^*) + d(x^*, Tx_{n_k+1})]/2\}\end{aligned}$$

We suppose that $x^* \neq Tx^*$ so that $d(x^*, Tx^*) > 0$. Taking limit $k \to \infty$, we get

$$\lim_{k\to\infty} M_T(x_{n_k}, x^*) = d(x^*, Tx^*).$$

Now we have
$$\frac{\psi(d(x_{n_k+1},Tx^*))}{\psi(M_T(x_{n_k},x^*))} \leq \beta(\psi(M_T(x_{n_k},x^*))) < 1.$$
And taking limit $k \to \infty$ we get
$$\lim_{k\to\infty} \beta(\psi(M_T(x_{n_k},x^*))) = 1.$$
So we have $\lim_{k\to\infty} \psi(M_T(x_{n_k},x^*)) = 0$ which implies that $\lim_{k\to\infty} M_T(x_{n_k},x^*) = 0$ i.e. $d(x^*,Tx^*) = 0$.
This is a contradiction. Therefore we must have $x^* = Tx^*$.

For the uniqueness of a fixed point of the mapping T, we consider the following hypothesis.
(K) For all $x \neq y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq \eta(x,z)$, $\alpha(y,z) \geq \eta(y,z)$ and $\alpha(z,Tz) \geq \eta(z,Tz)$.

Theorem 3.6. *Adding condition (K) in the hypotheses of Theorem 3.4. (resp. Theorem 3.5.), we obtain that x^* is the unique fixed point of T.*
Proof. Due to Theorem 3.4. (or Theorem 3.5.), we obtain that $x^* \in X$ is a fixed point of T. Let $y^* \in X$ be another fixed point of T such that $x^* \neq y^*$. Then by hypothesis (K), there exists $z \in X$ such that $\alpha(x^*,z) \geq \eta(x^*,z), \alpha(y^*,z) \geq \eta(y^*,z)$ and $\alpha(z,Tz) \geq \eta(z,Tz)$.
Since T is triangular $\alpha-$orbital admissible with respect to η, we get
$$\alpha(x^*,T^nz) \geq \eta(x^*,T^nz) \text{ and } \alpha(y^*,T^nz) \geq \eta(y^*,T^nz) \text{ for all } n \in \mathbb{N}.$$
Then we have
$$\begin{aligned}\psi(d(x^*,T^{n+1}z)) &= \psi(d(Tx^*,TT^nz)) \\ &\leq \beta(\psi(M_T(x^*,T^nz)))\psi(M_T(x^*,T^nz)), \quad \forall n \in \mathbb{N}.\end{aligned}$$
Here we have
$$\begin{aligned}M_T(x^*,T^nz) &= \max\{d(x^*,T^nz), d(x^*,Tx^*), d(T^nz,TT^nz), \\ &\qquad [d(x^*,TT^nz) + d(T^nz,Tx^*)]/2\} \\ &= \max\left\{d(x^*,T^nz), d(T^nz,T^{n+1}z), \left[d(x^*,T^{n+1}z) + d(x^*,T^nz)\right]/2\right\}\end{aligned}$$
By Theorem 3.4. (or Theorem 3.5.) we deduce that the sequence $\{T^nz\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$, we get $\lim_{n\to\infty} M_T(x^*,T^nz) = d(x^*,z^*)$. Let us suppose that $z^* \neq x^*$. Then we have
$$\frac{\psi(d(x^*,T^{n+1}z))}{\psi(M_T(x^*,T^nz))} \leq \beta(\psi(M_T(x^*,T^nz))) < 1.$$

Taking limit $n \to \infty$, we get $\lim_{n\to\infty} \beta(\psi(M_T(x^*, T^n z))) = 1$. Therefore we have $\lim_{n\to\infty} \psi(M_T(x^*, T^n z)) = 0$. This implies $\lim_{n\to\infty} M_T(x^*, T^n z) = 0$ i.e. $d(x^*, z^*) = 0$, which is a contradiction. Therefore, we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus we have $y^* = x^*$. Hence x^* is the unique fixed point of T.
If we take $\eta(x, y) = 1$ in Theorem 3.4. and Theorem 3.5., then we get the following theorem.

Theorem 3.7. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Assume that*

$$x, y \in X, \quad \alpha(x, y) \geq 1 \Rightarrow \psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$ and $\psi \in \Psi^*, \beta \in \mathcal{F}$.
Suppose that the following conditions are satisfied
(1) T is a triangular $\alpha-$orbital admissible mapping,
(2) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,
(3) either T is continuous or if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.
Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.
For the uniqueness of a fixed point of the mapping T, we consider the following hypothesis.
(K) For all $x \neq y \in X$, there exists $z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$ and $\alpha(z, Tz) \geq 1$.

Theorem 3.8. *Adding condition (K) in the hypotheses of Theorem 3.7., we obtain that x^* is the unique fixed point of T.*
If we take $\alpha(x, y) = 1$ in Theorem 3.4. and Theorem 3.5., we get the following theorem.

Theorem 3.9. *Let (X, d) be a complete metric space, $\eta : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Assume that*

$$x, y \in X, \quad \eta(x, y) \leq 1 \Rightarrow \psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where $M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}$ and $\psi \in \Psi^*, \beta \in \mathcal{F}$.
Suppose that the following conditions are satisfied
(1) T is a triangular $\eta-$orbital subadmissible mapping,
(2) there exists $x_1 \in X$ such that $\eta(x_1, Tx_1) \leq 1$,

(3) either T is continuous or if $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\eta(x_{n(k)}, x) \leq 1$ for all k.
Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

For the uniqueness of a fixed point of the mapping T, we consider the following hypothesis.
(K) For all $x \neq y \in X$, there exists $z \in X$ such that $\eta(x, z) \leq 1$, $\eta(y, z) \leq 1$ and $\eta(z, Tz) \leq 1$.

Theorem 3.10. *Adding condition (K) in the hypotheses of Theorem 3.9., we obtain that x^* is the unique fixed point of T.*
Clearly Theorem 3.7. implies the following results.

Definition 3.11. [8] *Let (X, d) be a metric space and let $\alpha : X \times X \to \mathbb{R}$ be a function. Then a map $T : X \to X$ is called an extended generalized $\alpha-\psi-$ Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x, y)\psi(d(Tx, Ty)) \leq \beta(\psi(M_T(x, y)))\psi(M_T(x, y)),$$

where

$$M_T(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\} \quad \text{and} \quad \psi \in \Psi.$$

Theorem 3.12. [8] *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied*

(1) T is an extended generalized $\alpha-\psi-$ Geraghty contraction type map,
(2) T is a triangular $\alpha-$ orbital admissible mapping,
(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,
(4) T is continuous.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.

Theorem 3.13. [8] *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied*

(1) T is an extended generalized $\alpha-\psi-$ Geraghty contraction type map,
(2) T is a triangular $\alpha-$ orbital admissible mapping,
(3) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,
(4) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq 1$ for all n and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $\alpha(x_{n(k)}, x) \geq 1$ for all k.

Then T has a fixed point $x^* \in X$, and $\{T^n x_1\}$ converges to x^*.
(K) For all $x \neq y \in X$, there exists $z \in X$ such that $\alpha(x,z) \geq 1, \alpha(y,z) \geq 1$ and $\alpha(z,Tz) \geq 1$.

Theorem 3.14. [8] *Adding condition (K) in the hypotheses of Theorem 3.12.(resp. Theorem 3.13.), we obtain that x^* is the unique fixed point of T.*
If we further take $\psi(t) = t$ in the above theorems 3.12., 3.13., 3.14., then we get the fixed point results of Popescu [16].
Now, we give two examples to illustrate Theorem 3.7. and Theorem 3.9.

Example 3.15. Let $X = [-2,-1] \cup \{0\} \cup [1,2]$ and let $d(x,y) = |x-y|$ for all $x, y \in X$. Then (X,d) is a complete metric space. And let $\beta(t) = \frac{1}{2}$ for all $t \geq 0$. Then $\beta \in \mathcal{F}$. Also let the function $\psi : [0,\infty) \to [0,\infty)$ be defined as $\psi(t) = \frac{t}{2}$. Then we have $\psi \in \Psi^*$.
Let a map $T : X \to X$ be defined by

$$Tx = \begin{cases} -x & \text{if } x \in [-2,-1) \cup (1,2], \\ 0 & \text{if } x \in \{-1,0,1\}. \end{cases}$$

And let a function $\alpha : X \times X \to \mathbb{R}$ be defined by

$$\alpha(x,y) = \begin{cases} 1 & \text{if } xy \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

If $\alpha(x,Tx) \geq 1$, then $xTx \geq 0$. This implies that $Tx = 0$ and so $\alpha(Tx,T^2x) \geq 1$. Also if $\alpha(x,y) \geq 1$ and $\alpha(y,Ty) \geq 1$, then $Ty = 0$. Thus $xTy = 0$ and so $\alpha(x,Ty) \geq 1$. Therefore, T is triangular α-orbital admissible. Condition (2) of Theorem 3.7. is satisfied with $x_1 = 1$. If $\{x_n\}$ is a sequence in X such that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $x_n x \geq 0$ and so $\alpha(x_n,x) \geq 1$ for all n.
We finally show that

$$x, y \in X,\ \alpha(x,y) \geq 1 \Rightarrow \psi(d(Tx,Ty)) \leq \beta(\psi(M_T(x,y)))\psi(M_T(x,y)).$$

Let $\alpha(x,y) \geq 1$. Then $xy \geq 0$ and we have the following possible cases.
Case 1. $x = 0, y \in \{-1,0,1\}$. Then $Tx = 0$ and $Ty = 0$. Therefore $d(Tx,Ty) = 0$ and $\psi(d(Tx,Ty)) = 0$. Thus we have

$$\psi(d(Tx,Ty)) \leq \beta(\psi(M_T(x,y)))\psi(M_T(x,y)).$$

Case 2. $y=0$, $x\in\{-1,0,1\}$. This case is similar to case 1.
Case 3. $x=0$, $y\in[-2,-1)\cup(1,2]$. Then $d(Tx,Ty)=|y|$, $M_T(x,y)\geq 2|y|$. So,

$$\begin{aligned} d(Tx,Ty)\leq M_T(x,y)/2 &\Rightarrow \frac{d(Tx,Ty)}{2}\leq\frac{1}{2}\frac{M_T(x,y)}{2}\\ &\Rightarrow \psi(d(Tx,Ty))\leq\beta(\psi(M_T(x,y)))\psi(M_T(x,y)).\end{aligned}$$

Case 4. $y=0$, $x\in[-2,-1)\cup(1,2]$. This case is similar to case 3.
Case 5. $x=1$, $y\in(1,2]$. Then $d(Tx,Ty)=|y|$, $M_T(x,y)\geq 2|y|$. So,

$$\begin{aligned} d(Tx,Ty)\leq M_T(x,y)/2 &\Rightarrow \frac{d(Tx,Ty)}{2}\leq\frac{1}{2}\frac{M_T(x,y)}{2}\\ &\Rightarrow \psi(d(Tx,Ty))\leq\beta(\psi(M_T(x,y)))\psi(M_T(x,y)).\end{aligned}$$

Case 6. $x=-1$, $y\in[-2,-1)$. This case is similar to case 5.
Case 7. $y=1$, $x\in(1,2]$. Then $d(Tx,Ty)=|x|$, $M_T(x,y)\geq 2|x|$. So,

$$\begin{aligned} d(Tx,Ty)\leq M_T(x,y)/2 &\Rightarrow \frac{d(Tx,Ty)}{2}\leq\frac{1}{2}\frac{M_T(x,y)}{2}\\ &\Rightarrow \psi(d(Tx,Ty))\leq\beta(\psi(M_T(x,y)))\psi(M_T(x,y)).\end{aligned}$$

Case 8. $y=-1$, $x\in[-2,-1)$. This case is similar to case 6.
Case 9. $x,y\in[-2,-1)$, then $d(Tx,Ty)=|x-y|\leq 1$ and $M_T(x,y)\geq -2x\geq 2$. Therefore $\psi(d(Tx,Ty))\leq\frac{1}{2}$ and $\beta(\psi(M_T(x,y)))\psi(M_T(x,y))\geq\frac{1}{2}$. Thus we have

$$\psi(d(Tx,Ty))\leq\beta(\psi(M_T(x,y)))\psi(M_T(x,y)).$$

Case 10. $x,y\in(1,2]$. This case is similar to case 9.
Case 11. $(x,y)=(1,1)=(-1,-1)$. Then $d(Tx,Ty)=0$ and $\psi(d(Tx,Ty))=0$. Thus we have

$$\psi(d(Tx,Ty))\leq\beta(\psi(M_T(x,y)))\psi(M_T(x,y)).$$

Thus all the conditions of Theorem 3.7. are satisfied and T has a unique fixed point $x^*=0$.

Example 3.16. Let $X=[0,\infty)$ and let $d(x,y)=|x-y|$ for all $x,y\in X$. Then (X,d) is a complete metric space. And let $\beta(t)=\frac{1}{2}$ for all $t\geq 0$. Then $\beta\in\mathcal{F}$. Also let the function $\psi:[0,\infty)\to[0,\infty)$ be defined as $\psi(t)=\frac{t}{3}$. Then we have $\psi\in\Psi^*$.
Let a map $T:X\to X$ be defined by

$$Tx=\begin{cases}\frac{1}{6}x^2 & \text{if } x\in[0,1],\\ x^3+1 & \text{if } x\in(1,\infty).\end{cases}$$

And let a function $\eta : X \times X \to \mathbb{R}$ be defined by

$$\eta(x,y) = \begin{cases} 1 & \text{if } x, y \in [0,1], \\ 2 & \text{otherwise.} \end{cases}$$

We show that T is a triangular η-orbital subadmissible mapping. If $\eta(x, Tx) \leq 1$, then $Tx \in [0,1]$ and also $T^2x \in [0,1]$. Therefore $\eta(Tx, T^2x) \leq 1$. Again if $\eta(x,y) \leq 1$ and $\eta(y, Ty) \leq 1$, then $x, Ty \in [0,1]$. Therefore $\eta(x, Ty) \leq 1$. Thus T is a triangular η-orbital subadmissible mapping.
Condition (2) of Theorem 3.9. is satisfied with $x_1 = 1$.
If $\{x_n\}$ is a sequence in X such that $\eta(x_n, x_{n+1}) \leq 1$ for all $n \in \mathbb{N}$ and $x_n \to x \in X$ as $n \to \infty$, then $x_n \in [0,1]$, $\forall n \in \mathbb{N}$ and hence $x \in [0,1]$. This implies that $\eta(x_n, x) \leq 1$ for all n.
Let $\eta(x,y) \leq 1$. Then $x, y \in [0,1]$. Therefore, we have

$$\begin{aligned} d(Tx,Ty) &= \frac{1}{6}|x^2 - y^2| = \frac{1}{6}|x-y||x+y| \leq \frac{1}{3}|x-y| \leq \frac{1}{2}|x-y| \leq \frac{1}{2}M_T(x,y) \\ &\Rightarrow \frac{1}{3}d(Tx,Ty) \leq \frac{1}{2} \times \frac{1}{3}M_T(x,y) \\ &\Rightarrow \psi(d(Tx,Ty)) \leq \beta(\psi(M_T(x,y)))\psi(M_T(x,y)). \end{aligned}$$

Thus all the conditions of Theorem 3.9. are satisfied and T has a unique fixed point $x^* = 0$.

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