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COINCIDENCE POINTS AND COMMON FIXED POINTS OF EXPANSIVE MAPPINGS IN A_b-METRIC SPACES

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Abstract: In this study, we prove some fixed point theorems for expansive mappings on A_b-metric spaces. Finally, the example is presented to support the new theorem proved. Our results extend/generalize many pre-existing results in literature.

Keywords and Phrases: A_b-metric space, expansive mapping, fixed point.

2020 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

Fixed point theory has great importance in science and mathematics. Since this area has been developed very fast over the past two decades due to huge applications in various fields such as nonlinear analysis, topology and engineering problems, it has attracted considerable attention from researchers. The study of expansive mappings is a very interesting research area in the fixed point theory. Wang et al. [37] proved some fixed point theorems for expansion mappings, which correspond to some contractive mappings in metric spaces. In 1992, Daffer and Kaneko [8] defined an expanding condition for a pair of mappings and proved some common fixed point theorems for two mappings in complete metric spaces. In 1989, Bakhtin [4] introduced the concept of a b-metric space as a generalization of metric spaces, in which many researchers treated the fixed point theory. In 1993, Czerwik [6, 7] extended many results related to the b-metric spaces. In 1994, Matthews [24] introduced the concept of partial metric space in which the self-distance of any point of space may not be zero. The concept of a D-metric space introduced by the first author in [10]. Some specific examples of a D-metric space are presented in Dhage [11]. The details of a D-metric space and its topological properties appear in Dhage [12]. In [13], Dhage has proved some common fixed-point theorems for coincidentally commuting single-valued mappings in D-metric spaces satisfying a condition of generalized contraction. In [20], Jungck introduced more generalized commuting mappings, called compatible mappings, which are more general than commuting and weakly commuting mappings. Jungck and Rhoades [21, 22] defined the concepts of δ-compatible and weakly compatible mappings which extend the concept of compatible mappings in the single-valued setting to set-valued mappings.

Definition 1.1. [32] *Let S and T be self-mappings of a set Y. A point $y \in Y$ is called a coincidence point of S and T iff $Sy = Ty$. In this case, $v = Sy = Ty$ is called a point of coincidence of S and T.*

Definition 1.2. [20] *Two single-valued mappings f and g of a metric space (X, d) into itself are compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X.*

Definition 1.3. [23] *Two self-mappings S and T of a metric space (Y, d) are said to be weakly compatible iff there is a point $y \in Y$ which is a coincidence point of S and T at which S and T commute; that is, $STy = TSy$.*

Proposition 1.1. [2] *Let S and T be weakly compatible selfmaps of a nonempty set X. If S and T have a unique point of coincidence $y = Sx = Tx$, then y is the unique common fixed point of S and T.*

Theorem 1.1. [37] *Let (Y, d) be a complete metric space and T a self mapping on Y. If T is surjective and satisfies*

$$d(Tx, Ty) \geq \gamma d(x, y) \tag{1.1}$$

for all $x, y \in Y$, with $\gamma > 1$ then T has a unique fixed point in Y.

Sedghi et al. [31] introduced a new generalized metric space called an S-metric space. The S-metric space is a space with three dimensions. Sedghi et al. [31]

asserted that every G-metric is an S-metric, see [31, Remarks 1.3 and 2.2]. The Example 2.1 and Example 2.2 of Dung et al. [15] shows that this assertion is not correct. Moreover, the class of all S-metrics and the class of all G-metrics are distinct. Souayah et al. [35] have introduced S_b-metric space and established some fixed point theorems. Abbas et al. [1] introduced the notion of A-metric space, which generalization of the S-metric space. Ughade et al. [36] introduced the notion of A_b-metric space, which generalization of the S_b-metric space and A-metric space.

In this work, we introduce a fixed point theorem for expansive mapping as a new tools in A_b-metric spaces. The obtained results generalize some facts in the literature.

2. Preliminaries

In this part, some useful notions and facts will be given.

Definition 2.1. [36] *Let X be a nonempty set and $s \geq 1$ be a given real number. A mapping $A_b : X^n \to \mathbb{R}$ is called an A_b-metric on X if for all $x_i, a \in X, i = 1, 2, 3, ..n$ and $s \geq 1$, the following conditions hold:*

(Ab1) $A_b(x_1, x_2, x_3, ..., x_{n-1}, x_n) \geq 0$;

(Ab2) $A_b(x_1, x_2, x_3, ..., x_{n-1}, x_n) = 0$ *if and only if* $x_1 = x_2 = ... = x_{n-1} = x_n$;

(Ab3) $A_b(x_1, x_2, x_3, ..., x_{n-1}, x_n) \leq s[A_b(x_1, x_1, x_1, ..., x_1, a)$
$+A_b(x_2, x_2, x_2, ..., x_2, a) + A_b(x_3, x_3, x_3, ..., x_3, a) + ...$
$+A_b(x_{n-1}, x_{n-1}, x_{n-1}, ..., x_{n-1}, a) + A_b(x_n, x_n, x_n, ..., x_n, a)]$.
The pair (X, A_b) is called an A_b-metric space.

Note that the class of A_b-metric spaces is larger than the class of A-metric spaces. Indeed, every A-metric space is an A_b-metric space with $s = 1$. However, the converse is not always true. Also A_b-metric space is an n-dimensional S_b-metric space. Therefore the S_b-metric are special cases of an A_b-metric with $n = 3$.

Example 2.1. [36] Let $X = [1, +\infty)$. Define $A_b : X^n \to [0, +\infty)$ by

$$A_b(x_1, x_2, x_3, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$$

for all $x_i \in X, i = 1, 2, ...n$. Then (X, A_b) is an A_b-metric space with $s = 2 > 1$.

Example 2.2. [36] Let $X = \mathbb{R}$. Define $A_b : X^n \to [0, +\infty)$ by

$$A_b(x_1, x_2, x_3, ..., x_{n-1}, x_n) = |\sum_{i=2}^{n} x_i - (n-1)x_1|^2 + |\sum_{i=3}^{n} x_i - (n-2)x_2|^2 + ... + |\sum_{i=n-3}^{n} x_i - 3x_{n-3}|^2 + |\sum_{i=n-2}^{n} x_i - 2x_{n-2}|^2 + |x_n - x_{n-1}|^2$$

for all $x_i \in X, i = 1, 2, ...n$. Then, (X, A_b) is an A_b-metric space with $s = 2 > 1$.

Lemma 2.1. [36] *Let (X, A_b) be an A_b-metric space with $s \geq 1$. Then for all $x, y \in X$,*

$$A_b(x, x, x, ..., x, y) \leq sA_b(y, y, y, ..., y, x)$$

Lemma 2.2. [36] *Let (X, A_b) be an A_b-metric space with $s \geq 1$. Then for all $x, y, z \in X$,*

$$A_b(x, x, x, ..., x, z) \leq s[(n-1)A_b(x, x, x, ..., x, y) + A_b(z, z, z, ..., z, y)]$$

and

$$A_b(x, x, x, ..., x, z) \leq s[(n-1)A_b(x, x, x, ..., x, y) + sA_b(y, y, y, ..., y, z)]$$

Note also that the following implications hold.

G-metric space $\Rightarrow$ D^*-metric space $\Rightarrow$ S-metric space $\Rightarrow$ A-metric space

$\Downarrow$ $\Downarrow$ $\Downarrow$

G_b-metric space $\Rightarrow$ S_b-metric space $\Rightarrow$ A_b-metric space

Definition 2.2. [36] *Let (X, A_b, s) be an A_b-metric space. Then*

(1) *A sequence $\{x_k\}$ is called convergent to x in (X, A_b) if*

$$\lim_{k \to +\infty} A_b(x_k, x_k, x_k, x_k, ..., x_k, x) = 0.$$

That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k \geq n_0$, we have $A_b(x_k, x_k, x_k, x_k, ..., x_k, x) < \epsilon$ and we write

$$\lim_{k \to +\infty} x_k = x.$$

(2) *A sequence $\{x_k\}$ is called Cauchy in (X, A_b) if*

$$\lim_{k,m \to +\infty} A_b(x_k, x_k, x_k, x_k, ..., x_k, x_m) = 0.$$

That is, for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $k, m \geq n_0$, we have $A_b(x_k, x_k, x_k, x_k, ..., x_k, x_m) < \epsilon$.

(3) (X, A_b) *is said to be complete if every Cauchy sequence in* (X, A_b) *is convergent.*

Lemma 2.3. [36] *Let* (X, A_b) *be an* A_b*-metric space with* $s \geq 1$*. If the sequence* $\{x_k\}$ *in* X *converges to* x*, then* x *is unique.*

Lemma 2.4. [36] *Every convergent sequence in* A_b*-metric space* (X, A_b) *is a Cauchy sequence.*

Definition 2.3. [36] *The* A_b*-metric space* (X, A_b) *is said to be bounded if there exists a constant* $r > 0$ *such that* $A_b(x, x, x, ..., x, y) \leq r$ *for all* $x, y \in X$*. Otherwise,* X *is unbounded.*

Definition 2.4. [36] *Given a point* x_0 *in* A_b*-metric space* (X, A_b) *and a positive real number* r*, the set*

$$B(x_0, r) = \{y \in X : A_b(y, y, y, ..., y, x_0) < r\}$$

is called an open ball centered at x_0 *with radius* r*.*
The set

$$\overline{B(x_0, r)} = \{y \in X : A_b(y, y, y, ..., y, x_0) \leq r\}$$

is called a closed ball centered at x_0 *with radius* r*.*

Definition 2.5. [36] *A subset* G *in* A_b*-metric space* (X, A_b) *is said to be an open set if for each* $x \in G$ *there exists an* $r > 0$ *such that* $B(x, r) \subset G$*. A subset* $F \subset X$ *is called closed if* X/F *is open.*

Definition 2.6. [36] *Let* (X, A_b) *be an* A_b*-metric space with* $s \geq 1$*. A map* $f : X \to X$ *is said to be contraction if there exits a constant* $\lambda \in [0, 1)$ *such that*

$$A_b(fx_1, fx_2, fx_3, ..., fx_n) \leq \lambda A_b(x_1, x_2, x_3, ..., x_n)$$

for all $x_1, x_2, x_3, ..., x_n \in X$*. In case*

$$A_b(fx_1, fx_2, fx_3, ..., fx_n) < A_b(x_1, x_2, x_3, ..., x_n)$$

for all $x_1, x_2, x_3, ..., x_n \in X, x_i \neq x_j$ *for some* $i \neq j$*,*$i, j \in \{1, 2, ..., n\}$*,* f *is called contractive mapping.*

Definition 2.7. [36] *Let* (X, A_b) *be an* A_b*-metric space with* $s \geq 1$*. A map* $f : X \to X$ *is said to be expansion mapping if there exits* $\lambda > 1$ *such that*

$$A_b(fx_1, fx_2, fx_3, ..., fx_n) \geq \lambda A_b(x_1, x_2, x_3, ..., x_n)$$

for all $x_1, x_2, x_3, ..., x_n \in X$*. In case*

$$A_b(fx_1, fx_2, fx_3, ..., fx_n) > A_b(x_1, x_2, x_3, ..., x_n)$$

for all $x_1, x_2, x_3, ..., x_n \in X, x_i \neq x_j$ for some $i \neq j, i, j \in \{1, 2, ..., n\}$, f is called expansive mapping.

3. Main Results

Now, we give some fixed-point results for expansive mappings in a complete A_b-metric space. Our first main result as follows.

Theorem 3.1. *Let (X, A_b, s) be a complete A_b-metric space with the coefficient $s > 1$. Suppose the mappings $S, T : X \to X$ satisfy the condition*

$$A_b\left(\underbrace{Tx, Tx, ..., Tx}_{(n-1)times}, Ty\right) \geq \alpha A_b\left(\underbrace{Sx, Sx, ..., Sx}_{(n-1)times}, Sy\right) + \beta A_b\left(\underbrace{Tx, Tx, ..., Tx}_{(n-1)times}, Sx\right) + \gamma A_b\left(\underbrace{Ty, Ty, ..., Ty}_{(n-1)times}, Sy\right) \tag{3.1}$$

for all $x, y \in X$, where α, β, γ are nonnegative real numbers with $\alpha + \beta + \gamma > s^2$. Assume the following hypotheses: (i) $\beta < 1$ and $\alpha \neq 0$, (ii) $S(X) \subseteq T(X)$, (iii) $T(X)$ or $S(X)$ is complete. Then T and S have a point of coincidence in X. Moreover, if $\alpha > 1$, then the point of coincidence is unique. If T and S are weakly compatible and $\alpha > 1$, then T and S have a unique common fixed point in X.

Proof. Let $u_0 \in X$ and choose $u_1 \in X$ such that $Su_0 = Tu_1$. This is possible since $S(X) \subseteq T(X)$. Continuing this process, we can construct a sequence $\{u_k\}_{k\in\mathbb{N}}$ in X such that $Tu_k = Su_{k-1}$ for all $k \geq 1$.

By (3.1), we have

$$A_b\left(\underbrace{Su_{k-1}, Su_{k-1}, ...Su_{k-1}}_{(n-1)\text{times}}, Su_k\right) = A_b\left(\underbrace{Tu_k, Tu_k, ..., Tu_k}_{(n-1)\text{times}}, Tu_{k+1}\right)$$

$$\geq \alpha A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) + \beta A_b\left(\underbrace{Tu_k, Tu_k, ..., Tu_k}_{(n-1)\text{times}}, Su_k\right)$$

$$+ \gamma A_b\left(\underbrace{Tu_{k+1}, Tu_{k+1}, ..., Tu_{k+1}}_{(n-1)\text{times}}, Su_{k+1}\right)$$

$$\Rightarrow \quad A_b\left(\underbrace{Su_{k-1}, Su_{k-1}, ..., Su_{k-1}}_{(n-1)\text{times}}, Su_k\right) \geq \alpha A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right)$$

$$+ \beta A_b\left(\underbrace{Su_{k-1}, Su_{k-1}, ..., Su_{k-1}}_{(n-1)\text{times}}, Su_k\right) + \gamma A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right)$$

The last inequality gives

$$A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) \leq \lambda A_b\left(\underbrace{Su_{k-1}, Su_{k-1}, ..., Su_{k-1}}_{(n-1)\text{times}}, Su_k\right)$$

where $\lambda = \frac{1-\beta}{\alpha+\gamma}$. It is easy to see that $\lambda \in \left[0, \frac{1}{s^2}\right)$ By induction, we get that

$$A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) \leq \lambda^k A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right) \tag{3.2}$$

for all $k \geq 0$.
For $m, k \in \mathbb{N}$ with $m > k$, we have by repeated use of $(Ab3)$

$$\begin{aligned}
A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_m\right) &\leq s(n-1)A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) \\
&\quad + sA_b\left(\underbrace{Su_m, Su_m, ..., Su_m}_{(n-1)\text{times}}, Su_{k+1}\right) \\
&\leq s(n-1)A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) + s^2A_b\left(\underbrace{Su_{k+1}, Su_{k+1}, ..., Su_{k+1}}_{(n-1)\text{times}}, Su_m\right) \\
&\leq s(n-1)A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) \\
&\quad + s^3(n-1)A_b\left(\underbrace{Su_{k+1}, Su_{k+1}, ..., Su_{k+1}}_{(n-1)\text{times}}, Su_{k+2}\right) + s^3A_b\left(\underbrace{Su_m, Su_m, ..., Su_m}_{(n-1)\text{times}}, Su_{k+2}\right) \\
&\leq s(n-1)A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) \\
&\quad + s^3(n-1)A_b\left(\underbrace{Su_{k+1}, Su_{k+1}, ..., Su_{k+1}}_{(n-1)\text{times}}, Su_{k+2}\right) + s^4A_b\left(\underbrace{Su_{k+2}, Su_{k+2}, ..., Su_{k+2}}_{(n-1)\text{times}}, Su_m\right) \\
&\leq s(n-1)A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right) + s^3(n-1)A_b\left(\underbrace{Su_{k+1}, Su_{k+1}, ..., Su_{k+1}}_{(n-1)\text{times}}, Su_{k+2}\right) \\
&\quad + s^5(n-1)A_b\left(\underbrace{Su_{k+2}, Su_{k+2}, ..., Su_{k+2}}_{(n-1)\text{times}}, Su_{k+3}\right) + s^6A_b\left(\underbrace{Su_m, Su_m, ..., Su_m}_{(n-1)\text{times}}, Su_{k+3}\right)
\end{aligned}$$

$$\leq s(n-1)A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_{k+1}\right)$$
$$+ s^3(n-1)A_b\left(\underbrace{Su_{k+1}, Su_{k+1}, ..., Su_{k+1}}_{(n-1)\text{times}}, Su_{k+2}\right)$$
$$+ s^5(n-1)A_b\left(\underbrace{Su_{k+2}, Su_{k+2}, ..., Su_{k+2}}_{(n-1)\text{times}}, Su_{k+3}\right)$$
$$+ s^7(n-1)A_b\left(\underbrace{Su_{k+3}, Su_{k+3}, ..., Su_{k+3}}_{(n-1)\text{times}}, Su_{k+4}\right) + ...$$
$$+ s^{2m-2k-3}(n-1)A_b\left(\underbrace{Su_{m-2}, Su_{m-2}, ..., Su_{m-2}}_{(n-1)\text{times}}, Su_{m-1}\right)$$
$$+ s^{2m-2k-2}(n-1)A_b\left(\underbrace{Su_{m-1}, Su_{m-1}, ..., Su_{m-1}}_{(n-1)\text{times}}, Su_m\right)$$
$$\leq (n-1)\left[s\lambda^k + s^3\lambda^{k+1} + ... + s^{2m-2k-3}\lambda^{m-2}\right]A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right)$$
$$+ s^{2m-2k-2}\lambda^{m-1}A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right)$$
$$= (n-1)s\lambda^k\left[1 + s^2\lambda + s^4\lambda^2 + ... + s^{2m-2k-4}\lambda^{m-k-2}\right]A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right)$$
$$+ s^{2m-2k-3}\lambda^{m-k-1}A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right)$$
$$\leq (n-1)s\lambda^k\left[1 + s^2\lambda + s^4\lambda^2 + s^6\lambda^3 + ...\right]A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right)$$
$$\leq (n-1)\frac{s\lambda^k}{1-\lambda s^2}A_b\left(\underbrace{Su_0, Su_0, ..., Su_0}_{(n-1)\text{times}}, Su_1\right) \tag{3.3}$$

Since $\lambda s^2 < 1$, $\lambda \in \left[0, \frac{1}{s^2}\right) \subseteq [0,1)$. By taking limit as $k, m \to +\infty$ in above inequality, we get

$$\lim_{k,m\to\infty} A_b\left(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, Su_m\right) = 0.$$

for all $m > k$. Therefore, $\{Su_k\}$ is a Cauchy sequence in $S(X)$. So $\{Su_k\}$ is a

Cauchy sequence in $S(X)$. Suppose that $S(X)$ is a complete subspace of X. Then there exists $v \in S(X) \subseteq T(X)$ such that $Tu_k = Su_{k-1} \to v$. In case, $T(X)$ is complete, this holds also with $v \in T(X)$. Let $z \in X$ be such that $Tz = v$. By (3.1), we have

$$\begin{aligned}
A_b\Big(\underbrace{Su_{k-1}, Su_{k-1}, ..., Su_{k-1}}_{(n-1)\text{times}}, T_z\Big) &= A_b\Big(\underbrace{Tu_k, Tu_k, ..., Tu_k}_{(n-1)\text{times}}, T_z\Big) \\
\geq \alpha A_b\Big(\underbrace{Su_k, Su_k, ..., Su_k}_{(n-1)\text{times}}, S_z\Big) &+ \beta A_b\Big(\underbrace{Tu_k, Tu_k, ..., Tu_k}_{(n-1)\text{times}}, Su_k\Big) \\
&+ \gamma A_b\Big(\underbrace{T_z, T_z, ..., T_z}_{(n-1)\text{times}}, S_z\Big)
\end{aligned}$$

Taking $k \to \infty$ in the above inequality, we get

$$0 \geq (\alpha + \gamma) A_b\Big(\underbrace{v, v, ..., v}_{(n-1)\text{times}}, Sz\Big)$$

This implies that $A_b\Big(\underbrace{v, v, ..., v}_{(n-1)\text{times}}, Sz\Big) = 0$, since $\alpha + \gamma > 0$. Thus $Sz = v$ and then $Tz = Sz = v$. Therefore, v is a point of coincidence of S and T.

Now we suppose that $\alpha > 1$. Let y be another point of coincidence of S and T. So $Tx = Sx = y$ for some $x \in X$. From (3.1), we have

$$\begin{aligned}
A_b\Big(\underbrace{v, v, ..., v}_{(n-1)\text{times}}, y\Big) &= A_b\Big(\underbrace{Tz, Tz, ..., Tz}_{(n-1)\text{times}}, Tx\Big) \\
&\geq \alpha A_b\Big(\underbrace{Sz, Sz, ..., Sz}_{(n-1)\text{times}}, Sx\Big) + \beta A_b\Big(\underbrace{Tz, Tz, ..., Tz}_{(n-1)\text{times}}, Sz\Big) \\
&+ \gamma A_b\Big(\underbrace{Tx, Tx, ..., Tx}_{(n-1)\text{times}}, Sx\Big) \\
&= \alpha A_b\Big(\underbrace{v, v, ..., v}_{(n-1)\text{times}}, y\Big)
\end{aligned}$$

which implies that

$$(\alpha - 1) A_b\Big(\underbrace{v, v, ..., v}_{(n-1)\text{times}}, y\Big) \leq 0$$

Since $\alpha > 1$, we have $A_b\left(\underbrace{v, v, ..., v}_{(n-1)\text{times}}, y\right) = 0$ and hence $v = y$. Therefore, S and T have a unique point of coincidence in X.
If S and T are weakly compatible, then by Proposition 1.1, S and T have a unique common fixed point in X.

Corollary 3.1. *Let (X, A_b, s) be a complete A_b-metric space with the coefficient $s > 1$. Suppose the mappings $S, T : X \to X$ satisfy the condition*

$$A_b\left(\underbrace{Tx, Tx, ..., Tx}_{(n-1)times}, Ty\right) \geq \alpha A_b\left(\underbrace{Sx, Sx, ..., Sx}_{(n-1)times}, Sy\right) \tag{3.4}$$

for all $x, y \in X$, where $\alpha > s^2$ is a constant. If $S(X) \subseteq T(X)$ and $T(X)$ or $S(X)$ is complete, then T and S have a unique point of coincidence in X. Moreover, if T and S are weakly compatible, then T and S have a unique common fixed point in X.
Proof. It follows by taking $\beta = \gamma = 0$ in Theorem 3.1.
The following Corollary is the b-metric version of Banach's contraction principle.

Corollary 3.2. *Let (X, A_b, s) be a complete A_b-metric space with the coefficient $s > 1$. Suppose the mappings $S : X \to X$ satisfies the contractive condition*

$$A_b\left(\underbrace{Sx, Sx, ..., Sx}_{(n-1)times}, Sy\right) \leq \lambda A_b\left(\underbrace{x, x, ..., x}_{(n-1)times}, y\right) \tag{3.5}$$

for all $x, y \in X$, where $\lambda \in \left(0, \frac{1}{s^2}\right)$ is a constant. Then S has a unique fixed point in X. Furthermore, the iterative sequence $\{S^n x\}$ converges to the fixed point.
Proof. It follows by taking $\beta = \gamma = 0$ and $T = I$, the identity mapping on X, in Theorem 3.1.

Corollary 3.3. *Let (X, A_b, s) be a complete A_b-metric space with the coefficient $s > 1$. Suppose the mappings $T : X \to X$ is onto and satisfies*

$$A_b\left(\underbrace{Tx, Tx, ..., Tx}_{(n-1)times}, Ty\right) \geq \alpha A_b\left(\underbrace{x, x, ..., x}_{(n-1)times}, y\right) \tag{3.6}$$

for all $x, y \in X$, where $\alpha > s^2$ is a constant. Then T has a unique fixed point in X.
Proof. Taking $S = I$ and $\beta = \gamma = 0$ in Theorem 3.1, we obtain the desired result.

Corollary 3.4. *Let (X, A_b, s) be a complete A_b-metric space with the coefficient $s > 1$. Suppose the mappings $T : X \to X$ is onto and satisfies the condition*

$$A_b\Big(\underbrace{Tx, Tx, ..., Tx}_{(n-1)times}, Ty\Big) \geq \alpha A_b\Big(\underbrace{x, x, ..., x}_{(n-1)times}, y\Big) + \beta A_b\Big(\underbrace{Tx, Tx, ..., Tx}_{(n-1)times}, x\Big) + \gamma A_b\Big(\underbrace{Ty, Ty, ..., Ty}_{(n-1)times}, y\Big) \tag{3.7}$$

for all $x, y \in X$, where α, β, γ are nonnegative real numbers with $\beta < 1$, $\alpha \neq 0$ $\alpha + \beta + \gamma > s^2$. Then T has a fixed point in X. Moreover, if $\alpha > 1$, then the fixed point of T is unique.

Proof. It follows by taking $S = I$ in Theorem 3.1.

We conclude with a example.

Example 3.1. Let $X = [0, 1]$. Define $A_b : X^n \to [0, +\infty)$ by

$$A_b(x_1, x_2, x_3, ..., x_{n-1}, x_n) = \sum_{i=1}^{n} \sum_{i<j} |x_i - x_j|^2$$

for all $x_i \in X, i = 1, 2, ..., n$. Therefore, (X, A_b) is an A_b-metric space with $s = 2 > 1$.

Let us define $S, T : X \to X$ as

$$Tu = \frac{u}{3} \quad \text{and} \quad Su = \frac{u}{9} - \frac{u^2}{27}$$

for all $u \in X$. Then, for every $x, y \in X$, the condition (3.1) holds for $\alpha = 9, \beta = \gamma = 0$. Thus, we have all the conditions of Theorem 3.1 and $0 \in X$ is the unique common fixed point of T and S.

4. Conclusions

Wang et al. [37], proved some fixed point theorems for expansive mappings, which correspond to some contractive mappings in metric spaces. In the present article, we introduce a new approach to expansive mappings in fixed point theory by combining the ideas of Wang and establish a fixed point theorem for expansive mappings as a new tool in A_b-metric spaces. The obtained results generalize some facts in the literature.

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