South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 361-382

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

FIXED POINT APPROXIMATION OF COUNTABLY INFINITE FAMILY OF NONEXPANSIVE MAPPINGS

Samir Dashputre, Padmavati* and Kavita Sakure**

Department of Mathematics,
Govt. College, Arjunda, Balod, Chhattisgarh - 491225, INDIA

E-mail : samir231973@gmail.com

*Department of Mathematics,
Govt. V.Y.T. College, Durg, Chhattisgarh - 491001, INDIA

E-mail : padmavati.sudha62@gmail.com

**Department of Mathematics,
Govt. Digvijay Auto. P.G. College,
Rajnandgaon, Chhattisgarh - 491441, INDIA

E-mail : kavitaaage@gmail.com

(**Received:** *Mar. 04, 2020* **Accepted:** *Nov. 27, 2020* **Published:** *Apr. 30, 2021)*

Abstract: In this paper, we propose an iterative algorithm and then prove the strong convergence of proposed algorithm in framework of CAT(0) space and Hilbert space for approximating the common fixed point of countably infinite family of nonexpansive mappings and minimizer of proper, convex, lower semicontinuous function. Then we implement the proposed algorithm to solve constrained minimization problem and system of linear equations. Our results genaralize the results of Phuengrattana *et al.* [36] and Suparatulatorn *et al.* [43].

Keywords and Phrases: Proximal point algorithm, CAT(0) spaces, nonexpansive mappings, constrained convex minimization problems, system of linear equations.

2020 Mathematics Subject Classification: 47H09, 47H10.

1. Introduction

Let (X, d) be a metric space. A geodesic path joining $x \in X$ to $y \in X$ (or, more briefly, a geodesic from x to y in X) is a map c from a closed interval $[0, l] \subset \mathbb{R}$ to X such that $c(0) = x, c(l) = y$, and $d(c(t), c(t^{'})) = |t - t^{'}|$ for all $t, t^{'} \in [0, l]$. In particular, c is an isometry and $d(x, y) = l$ The image of c is called a geodesic (or metric) segment joining x and y. In case this geodesic segment is unique, it is denoted by $[x, y]$. The space (X, d) is said to be geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset $Y \subseteq X$ is said to be convex if Y includes every geodesic segment joining any two of its points.

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three points x_1, x_2, x_3 in X (referred as the vertices of Δ) and a geodesic segment between each pair of vertices (referred as the edges of Δ). A comparison triangle for the geodesic triangle $\Delta(x_1, x_2, x_3)$ in (X, d) is a triangle $\Delta(x_1, x_2, x_3) := \Delta(\overline{x_1}, \overline{x_2}, \overline{x_3},)$ in the Euclidean plane E^2 such that $d_{E^2}(\overline{x_i}, \overline{x_j}) = d(x_i, x_j)$ for $i, j \in \{1, 2, 3\}$.

A geodesic space is said to be a CAT(0) space if all geodesic triangles of appropriate size satisfy the following comparison axiom : Let Δ be a geodesic triangle in (X, d) and let $\overline{\Delta}$ be a comparison triangle for Δ. Then Δ is said to satisfy the CAT(0) inequality if for all $x, y \in \Delta$ and all comparison points $\overline{x}, \overline{y} \in \overline{\Delta}$, such that

$$d(x, y) \leq d_{E^2}(\overline{x}, \overline{y}).$$

If x, y_1, y_2 are points in a CAT(0) space and if y_0 is the midpoint of the segment $[y_1, y_2]$, then the CAT(0) inequality implies

$$d(x, y_0)^2 \leq \frac{1}{2}d(x, y_1)^2 + \frac{1}{2}d(x, y_2)^2 - \frac{1}{4}d(y_1, y_2)^2.$$

On the other hand, Martinet [31] introduced the proximal point algorithm (PPA) which is a method for finding a minimizers of convex lower semicontinuous (lsc) function defined on Hilbert space. The proximal point algorithm since become extremely popular among the various researchers inclination in the theory of optimization and also exposed many challenging mathematical problems. The rich literature on PPA has become too extensive (see e.g. [7-9, 12, 21, 24, 37, 38, 40]). In particular, the PPA has been studied in the framework of CAT(0) space [8, 15-17, 36, 43], Riemannian manifold [10, 20] and in Hadamard manifold [1, 3, 4, 11, 29, 44].

In 2017, Suparatulatorn *et al.* [43] has been introduced modified proximal point algorithm in complete CAT(0) space (X, d) for nonexpansive mapping T as follows : Assume that f is convex, proper and lower semi-continuous function on X. The modified proximal point algorithm is given by for $x_1, u \in X$ and $\lambda_n > 0$

$$\begin{cases} y_n = \operatorname{argmin}\left\{f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2\right\} \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) T_n y_n \qquad n \in N. \end{cases} \tag{1.1}$$

Also it was proved that if f has minimizer and $\{\lambda_n\}$, $\{\alpha_n\}$ satisfy some conditions, Then sequence generated by (1.1) strongly converges to its minimizer.

In 2018, Phuengrattana *et al.* [36] has been introduced modified proximal point algorithm in complete CAT(0) space (X, d) for countably infinite family for nonexpansive mapping T as follows : Assume that f is convex, proper and lower semi-continuous function on X. The modified proximal point algorithm is given by for $x_1, u \in X$ and $\lambda_n > 0$

$$\begin{cases} y_n = \operatorname{argmin}\{f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2\} \\ z_n = \alpha_n u \oplus (1 - \alpha_n) T_n y_n \\ x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) T_n z_n \qquad n \in N. \end{cases} \tag{1.2}$$

Also it was proved that if f has minimizer and $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ satisfy some conditions. Then sequence generated by (1.2) strongly converges to its minimizer.

Motivating and inspiring by the research in this direction, the following question arises :

Question 1. *Can we construct an iterative process for obtaining common fixed points of minimizers of proper, convex, lower semi-continuous function and countably infinite family of nonexpansive mappings in complete* CAT(0) *spaces ?*

In this paper, we propose an iterative algorithm and then prove the strong convergence of proposed algorithm in framework of CAT(0) space and Hilbert space for approximating the common fixed point of countably infinite family of nonexpansive mappings and minimizer of proper, convex, lower semicontinuous function. Then we implement the proposed algorithm to solve constrained minimization problem and system of linear equations. Our result generalizes the results of Bacak [8], Bacak *et al.* [9], Cholamijak *et al.* [17], Phuengrattana *et al.* [36] and Suparatulatorn *et al.* [43].

2. Preliminaries

Let (X, d) be a metric space and C a nonempty subset of X. Then a mapping $T : C \longrightarrow C$ is called nonexpansive mapping, if $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in X$. An element $x \in X$ is called fixed point of T, if $Tx = x$. $F(T)$ is the set of all fixed point of T. In 2003, Kirk [26] studied the fixed point theory for CAT(0)

space. Recently many authors worked on fixed point theory (see [2, 30, 33-35]) and particularly in CAT(0) space for nonexpansive theory (see [17, 27, 28, 41]).
Let C be a nonempty convex subset of CAT(0) space. Then the mapping $f : C \longrightarrow C$ is
(i) lower semi-continuous at the point $x \in C$, if for each sequene $\{x_n\}$ such that $\lim_{n\to\infty} x_n = x$, we have

$$f(x) \leq \liminf_{n\to\infty} f(x_n).$$

(ii) lower semi-continuous on C, if f is lower semi-continuous at every point of C.
(iii) convex, if $fo\gamma$ is convex, where $\gamma : [0, 1] \longrightarrow C$ is any geodesic.
The Moreau-Yosida resolvent of function f in the CAT(0) space is given by

$$J_\lambda(x) = \operatorname{argmin}_{y\in C} \Big[f(y) + \frac{1}{2\lambda} d(y, x)^2\Big]$$

for any $\lambda > 0$ and for all $x \in C$.
Remark 1. The resolvent J_λ of function f is nonexpansive for all $\lambda > 0$ (see [23]).
Remark 2. If f is convex, proper and lower semi-continuous function, then the set of fixed point of the resolvent associated with f coincides with the set of minimizers of f (see [7]).

Lemma 2.1. *Suppose that (X, d) is a* CAT(0) *space. Then the following results are true.*
(i) For each $y, z \in X$ and $\lambda > 0$ (see [23, 22])

$$\frac{1}{2\lambda} d(J_\lambda y, z)^2 - \frac{1}{2\lambda} d(y, z)^2 + \frac{1}{2\lambda} d(J_\lambda y, y)^2 \leq f(z) - f(J_\lambda x).$$

(ii) For each $y \in X$ and $\lambda > \mu > 0$ (see [5])

$$J_\lambda y = J_\mu\Big(\frac{\lambda - \mu}{\lambda} J_\lambda y \oplus \frac{\mu}{\lambda} y\Big).$$

Proposition 2.2. [43] *Let (X, d) be a complete* CAT(0) *space. The mapping $f : X \longrightarrow (-\infty, \infty)$ is convex, proper and lower semi-continuous function. Assume that T is nonexpansive mapping on X such that for all λ, $F(T) \cap F(J_\lambda) \neq \phi$. Then $FoJ_\lambda = F(T) \cap F(J_\lambda)$ for any $\lambda > 0$.*
Next, we will use the following lemma for proving our main results (see [39, 41]).

Lemma 2.3. *Suppose that C is nonempty convex subset of a complete* CAT(0) *space (X, d) and $T : C \longrightarrow C$ is nonexpansive mapping. Assume that $u \in C$ is fixed. For each $\tau \in (0, 1)$, the mapping $V_\tau : C \longrightarrow C$ given by $V_\tau x = \tau x \oplus (1-\tau)Tx$*

for $x \in C$ *has a unique fixed point* $y_\tau \in C$, *i.e.,* $y_\tau = V_\tau y_\tau = \tau x \oplus (1-\tau)Ty_\tau$.

Lemma 2.4. *Assume that* T *and* C *are as above lemma. Then* $\{y_\tau\}$ *remains bounded as* $\tau \to 0$ *if and only if* $F(T) \neq \phi$ *. Then the following results are true.*
(i) $\{y_\tau\}$ *converges to unique fixed point* z *of mapping* T*, where* z *is nearest point to* u*.*
(ii) $d(u,y) \leq \mu_n d(u,x_n)^2$ *for all Banach limit* μ *and all bounded sequences* $\{x_n\}$ *with* $\lim_{n\to\infty} d(x_n, Tx_n) = 0$.
Aoyama *et al.* [6] in 2007 introduced the AKTT condition as follows:

Definition 1. *Assume that* C *is nonempty subset of a complete* CAT(0) *space* (X,d) *and* $\{T_n\}$ *is a countably infinite family of mappings from* C *into* C*. Then* $\{T_n\}$ *satisfies AKTT condition, if for each bounded subset* D *of* C*,*

$$\sum_{n=1}^{\infty} \sup_{z\in D} \big\{d(T_{n+1}z, T_n z)\big\} < \infty.$$

Remark 3. If C is closed subset of X and $\{T_n\}$ satisfies the AKTT condition. Define the mapping $T: C \to C$ by $Tx = \lim_{n\to\infty} T_n x$ for all $x \in C$ and we can say that $(\{T_n\}, T)$ satisfies AKTT condition.
Using the above definition, we get the following result from [6, Lemma 3.2].

Lemma 2.5. *Assume that* $(\{T_n\}, T)$ *satisfies AKTT condition, then* $\lim_{n\to\infty}$ $\sup_{z\in D} \{d(Tz, T_n z)\} = 0$ *for all bounded subsets* D *of* C*.*
We know that l^∞ is Banach space of bounded and real sequences. Assume that μ is continuous linear functional on l^∞ and $(c_1, c_2, ...) \in l^\infty$. Write $\mu_n(c_n)$ in place of $\mu((c_1, c_2, ..))$. μ is Banach limit, if μ satisfies $\|\mu\| = \mu(1,1,...) = 1$ and $\mu_n(c_n) = \mu_n(c_{n+1})$ for each $(c_1, c_2, ...) \in l^\infty$. For a Banach limit μ, we have $\liminf_{n\to\infty} c_n \leq \mu_n(c_n \leq) \limsup_{n\to\infty} c_n$ for all $(c_1, c_2, ...) \in l^\infty$. If $(c_1, c_2, ...) \in l^\infty$ with $\lim_{n\to\infty} c_n = c^*$, then $\mu_n(c_n) = c^*$; (see [14]).

Lemma 2.6. [42] *Suppose that* $(c_1, c_2, ...) \in l^\infty$ *with* $\mu_n(c_n) \leq 0$ *for all Banach limit* μ*. If* $\limsup_{n\to\infty}(c_{n+1} - c_n) \leq 0$, *then* $\limsup_{n\to\infty} c_n \leq 0$.

Lemma 2.7. [6] *Suppose that* $\{w_n\}$ *is a sequence of nonnegative real numbers and* $\{\alpha_n\}$ *is sequence of real numbers in* $[0,1]$ *with* $\sum_{n=1}^{\infty} a_n = \infty$, $\{\gamma_n\}$ *is sequence of real numbers with* $\limsup_{n\to\infty} \beta_n \leq 0$*. If*

$$w_{n+1} \leq (1-\alpha_n)w_n + \alpha_n\beta_n + \gamma_n, \quad \forall n \in N,$$

then $\lim_{n\to\infty} w_n = 0$.

The following condition for a geodesic metric space (X, d) to be a CAT(0) space: For any $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$d(x, \alpha y \oplus (1-\alpha)z)^2 \leq \alpha d(x, y)^2 + (1-\alpha)d(x, z)^2 - \alpha(1-\alpha)d(y, z)^2. \quad (2.1)$$

In particular, for any $x, y, z \in X$ and $\alpha \in [0, 1]$,

$$d(x, \alpha y \oplus (1-\alpha)z)^2 \leq \alpha d(x, y)^2 + (1-\alpha)d(x, z)^2. \quad (2.2)$$

The following are basic results of CAT(0) spaces.

Lemma 2.8. *Suppose that (X, d) is a CAT(0) space. Then the following results are true.*
(a) (X, d) is uniquely geodesic (see [13]).
(b) If $h : [0, 1] \longrightarrow [y, z]$ is a function given by $h(\alpha) = \alpha y \oplus (1-\alpha)z$, then h is continuous and bijective (see [19]).
(c) If $p, y, z, \in X$ and $\alpha \in [0, 1]$, then (see [19])

$$d(\alpha p \oplus (1-\alpha)y, \alpha p \oplus (1-\alpha)z) \leq (1-\alpha)d(y, z).$$

(d) Let $y, z \in X$. For each $\alpha \in [0, 1]$, there exists a unique point $x = \alpha y \oplus (1-\alpha)z$ such that

$$d(x, y) = (1-\alpha)d(y, z) \text{ and } d(x, z) = \alpha d(y, z) \text{ (see [25]).}$$

3. Main Results

Theorem 3.1. *Let C be a nonempty closed onvex subset of a complete CAT(0) space (X, d) and $f : C \longrightarrow (-\infty, \infty)$ be a proper, convex and lower semicontinuous function. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of a C generated by*

$$\begin{cases} w_n = \operatorname{argmin}_{w\in C}\{f(w) + \frac{1}{2\lambda_n}d(w, x_n)^2\} \\ y_n = \alpha_n u \oplus (1-\alpha_n)T_n w_n \\ z_n = \beta_n y_n \oplus (1-\beta_n)T_n y_n \\ x_{n+1} = \gamma_n z_n \oplus (1-\gamma_n)T_n z_n \qquad n \in N, \end{cases} \quad (3.1)$$

where $\{T_n\}$ is a countably infinite family of nonexpansive mapping of C into itself with $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap \operatorname{argmin}_{y\in C} f(y) \neq \phi$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\lambda_n\}$ are the sequences which satisfy the following axioms:
(A1) $0 < \alpha_n < 1, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
(A2) $\beta_n \in (b, 1]$ for some $b \in (0, 1)$ and $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.
(A3) $\gamma_n \in (c, 1]$ for some $c \in (0, 1)$ and $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.

(A4) $\lambda_n \geq \lambda > 0$ *for some* $\lambda \in (0, \infty)$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.
Suppose that $\{\{T_n\}, T\}$ *satisfies the AKTT condition and* $F(T) = \cap_{n=1}^{\infty} F(T_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to a point in* Ω *which is nearest to* u.

Proof. Let $q \in \Omega$ and $w_n = J_{\lambda_n} x_n$for all $n \in N$. Then using Lemma 2.1, we have

$$\begin{aligned}
d(x_{n+1}, q) &\leq \gamma_n d(z_n, q) + (1-\gamma_n) d(T_n z_n, q) \\
&\leq \gamma_n d(z_n, q) + (1-\gamma_n) d(z_n, q) \\
&= d(z_n, q) \\
&\leq \beta_n d(y_n, q) + (1-\beta_n) d(T_n y_n, q) \\
&\leq \beta_n d(y_n, q) + (1-\beta_n) d(y_n, q) \\
&= d(y_n, q) \\
&\leq \alpha_n d(u, q) + (1-\alpha_n) d(T_n w_n, q) \\
&\leq \alpha_n d(u, q) + (1-\alpha_n) d(w_n, q) \\
&\leq \alpha_n d(u, q) + (1-\alpha_n) d(J_{\lambda_n} x_n, q) \\
&\leq \alpha_n d(u, q) + (1-\alpha_n) d(x_n, q) \\
&\leq \max\{d(u, q), d(x_n, q)\} \\
d(x_{n+1}, q) &\leq \max\{d(u, q), d(x_n, q)\}
\end{aligned}$$

which implies that the sequence $\{x_n\}$ is bounded. Thus $\{y_n\}$, $\{z_n\}$, $\{w_n\}$, $\{x_n\}$, $\{T_n x_n\}$. $\{T_n y_n\}$, $\{T_n z_n\}$, $\{J_{\lambda_n} x_n\}$ and $\{T x_n\}$ are bounded.
Let $\lambda_n > \lambda_{n-1}$, using Proposition 2.2 and Condition (A4), we have

$$\begin{aligned}
d(w_n, w_{n-1}) &\leq d(w_n, J_{\lambda_n} x_{n-1}) + d(J_{\lambda_n} x_{n-1}, w_{n-1}) \\
&= d(J_{\lambda_n} x_n, J_{\lambda_n} x_{n-1}) + d(J_{\lambda_n} x_{n-1}, J_{\lambda_{n-1}} x_{n-1}) \\
&\leq d(x_n, x_{n-1}) + d\Big(J_{\lambda_{n-1}}\big(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_{n-1}\big), J_{\lambda_{n-1}} x_{n-1}\Big) \\
&\leq d(x_n, x_{n-1}) + d\Big(\frac{\lambda_n - \lambda_{n-1}}{\lambda_n} J_{\lambda_n} x_{n-1} \oplus \frac{\lambda_{n-1}}{\lambda_n} x_n, x_{n-1}\Big) \\
&= d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n+1}|}{\lambda_n} d(J_{\lambda_n} x_{n-1}, x_{n-1}) \\
&\leq d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n+1}|}{\lambda} d(J_{\lambda_n} x_{n-1}, x_{n-1}).
\end{aligned}$$

By definition of $\{y_n\}$ and Lemma 2.8 (c), (d), we have

$$\begin{aligned}
d(y_n, y_{n+1}) &= d(\alpha_n u \oplus (1-\alpha_n) T_n w_n, \alpha_{n-1} u \oplus (1-\alpha_{n-1}) T_{n-1} w_{n-1}) \\
&\leq d(\alpha_n u \oplus (1-\alpha_n) T_n w_n, \alpha_n u \oplus (1-\alpha_{n-1}) T_n w_{n-1}) \\
&\quad + d(\alpha_n u \oplus (1-\alpha_n) T_n w_{n-1}, \alpha_n u \oplus (1-\alpha_{n-1}) T_{n-1} w_{n-1})
\end{aligned}$$

$$\begin{aligned}
&+d(\alpha_n u \oplus (1-\alpha_n)T_{n-1}w_{n-1}, \alpha_{n-1}u \oplus (1-\alpha_{n-1})T_{n-1}w_{n-1}) \\
\leq\ & (1-\alpha_n)d(w_n, w_{n-1}) + (1-\alpha_n)d(T_n w_{n-1}, T_{n-1}w_{n-1}) \\
&+|\alpha_n - \alpha_{n-1}|d(u, T_{n-1}w_{n-1}) \\
\leq\ & (1-\alpha_n)(d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda_n} d(J_{\lambda_n}x_n, x_{n-1})) \\
&+(1-\alpha_n)d(T_n w_{n-1}, T_{n-1}w_{n-1}) + +|\alpha_n - \alpha_{n-1}|d(u, T_{n-1}w_{n-1}) \\
\leq\ & (1-\alpha_n)(d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J_{\lambda_n}x_n, x_{n-1})) \\
&+(1-\alpha_n)d(T_n w_{n-1}, T_{n-1}w_{n-1}) + +|\alpha_n - \alpha_{n-1}|d(u, T_{n-1}w_{n-1}).
\end{aligned}$$

Again by definition of $\{z_n\}$ and Lemma 2.8 (c), (d), we have

$$\begin{aligned}
d(z_n, z_{n+1}) &= d(\beta_n y_n \oplus (1-\beta_n)T_n y_n, \beta_{n-1}y_{n-1} \oplus (1-\beta_{n-1})T_{n-1}y_{n-1}) \\
&\leq d(\beta_n y_n \oplus (1-\beta_n)T_n y_n, \beta_n y_n \oplus (1-\beta_{n-1})T_{n-1}y_{n-1}) \\
&\quad +d(\beta_n y_n \oplus (1-\beta_n)T_{n-1}y_{n-1}, \beta_n y_{n-1} \oplus (1-\beta_n)T_{n-1}y_{n-1}) \\
&\quad +d(\beta_n y_n \oplus (1-\beta_n)T_{n-1}y_{n-1}, \beta_{n-1}y_{n-1} \oplus (1-\beta_{n-1})T_{n-1}y_{n-1}) \\
&\leq (1-\beta_n)d(T_n y_n, T_{n-1}y_{n-1}) + \beta_n d(y_n, y_{n-1}) \\
&\quad +|\beta_n - \beta_{n-1}|d(y_{n-1}, T_{n-1}y_{n-1}) \\
&\leq \beta_n d(y_n, y_{n-1}) + (1-\beta_n)(d(T_n y_n, T_n y_{n-1}) + d(T_n y_{n-1}, T_{n-1}y_{n-1})) \\
&\quad +|\beta_n - \beta_{n-1}|d(y_{n-1}, T_{n-1}y_{n-1}) \\
&\leq \beta_n d(y_n, y_{n-1}) + (1-\beta_n)(d(y_n, y_{n-1}) + d(T_n y_{n-1}, T_{n-1}y_{n-1})) \\
&\quad +|\beta_n - \beta_{n-1}|d(y_{n-1}, T_{n-1}y_{n-1}) \\
&\leq d(y_n, y_{n-1}) + (1-\beta_n)d(T_n y_{n-1}, T_{n-1}y_{n-1}) \\
&\quad +|\beta_n - \beta_{n-1}|d(y_{n-1}, T_{n-1}y_{n-1}).
\end{aligned}$$

Now,

$$\begin{aligned}
d(x_n, x_{n+1}) &= d(\gamma_n z_n \oplus (1-\gamma_n)T_n z_n, \gamma_{n-1}z_{n-1} \oplus (1-\gamma_{n-1})T_{n-1}z_{n-1}) \\
&\leq d(\gamma_n z_n \oplus (1-\gamma_n)T_n z_n, \gamma_n z_n \oplus (1-\gamma_{n-1})T_{n-1}z_{n-1}) \\
&\quad +d(\gamma_n z_n \oplus (1-\gamma_n)T_{n-1}z_{n-1}, \gamma_n z_{n-1} \oplus (1-\gamma_n)T_{n-1}z_{n-1}) \\
&\quad +d(\gamma_n z_n \oplus (1-\gamma_n)T_{n-1}z_{n-1}, \gamma_{n-1}z_{n-1} \oplus (1-\gamma_{n-1})T_{n-1}z_{n-1}) \\
&\leq (1-\gamma_n)d(T_n z_n, T_{n-1}z_{n-1}) + \gamma_n d(z_n, z_{n-1}) \\
&\quad +|\gamma_n - \gamma_{n-1}|d(z_{n-1}, T_{n-1}z_{n-1}) \\
&\leq \gamma_n d(z_n, z_{n-1}) + (1-\gamma_n)(d(T_n z_n, T_n z_{n-1}) + d(T_n z_{n-1}, T_{n-1}z_{n-1})) \\
&\quad +|\gamma_n - \gamma_{n-1}|d(z_{n-1}, T_{n-1}z_{n-1})
\end{aligned}$$

$$
\begin{aligned}
&\leq \gamma_n d(z_n, z_{n-1}) + (1-\gamma_n)(d(z_n, z_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1})) \\
&\quad + |\gamma_n - \gamma_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\leq d(z_n, z_{n-1}) + (1-\gamma_n) d(T_n z_{n-1}, T_{n-1} z_{n-1}) + |\gamma_n - \gamma_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\leq d(y_n, y_{n-1}) + (1-\beta_n) d(T_n y_{n-1}, T_{n-1} y_{n-1})) + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\quad + (1-\gamma_n) d(T_n z_{n-1}, T_{n-1} z_{n-1}) + |\gamma_n - \gamma_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\leq (1-\alpha_n)\Big(d(x_n, x_{n-1}) + \frac{|\lambda_n - \lambda_{n-1}|}{\lambda} d(J_{\lambda_n} x_n, x_{n-1})\Big) \\
&\quad + (1-\alpha_n) d(T_n w_{n-1}, T_{n-1} w_{n-1}) + |\alpha_n - \alpha_{n-1}| d(u, T_{n-1} w_{n-1}) \\
&\quad + (1-\beta_n) d(T_n y_{n-1}, T_{n-1} y_{n-1}) + |\beta_n - \beta_{n-1}| d(y_{n-1}, T_{n-1} y_{n-1}) \\
&\quad + (1-\gamma_n) d(T_n z_{n-1}, T_{n-1} z_{n-1}) + |\gamma_n - \gamma_{n-1}| d(z_{n-1}, T_{n-1} z_{n-1}) \\
&\leq (1-\alpha_n) d(x_n, x_{n-1}) \Big((1-\alpha_n)\frac{|\lambda_n - \lambda_{n-1}|}{\lambda} + |\alpha_n - \alpha_{n-1}| \\
&\quad + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|\Big)\mathbb{M} \\
&\quad + (1-\alpha_n) d(T_n w_{n-1}, T_{n-1} w_{n-1}) + (1-\beta_n) d(T_n y_{n-1}, T_{n-1} y_{n-1}) \\
&\quad + (1-\gamma_n) d(T_n z_{n-1}, T_{n-1} z_{n-1})
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbb{M} &= \max\Big\{\sup_n d(J_{\lambda_n} x_{n-1}, x_{n-1}), \sup_n d(u, T_{n-1} w_{n-1}), \\
&\qquad \sup_n d(y_{n-1}, T_{n-1} y_{n-1}), \sup_n d(z_{n-1}, T_{n-1} z_{n-1})\Big\}.
\end{aligned}
$$

Assume that

$$
\begin{aligned}
\delta_n &= \Big((1-\alpha_n)\frac{|\lambda_n - \lambda_{n-1}|}{\lambda} + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|\Big)\mathbb{M} \\
&\quad + d(T_n w_{n-1}, T_{n-1} w_{n-1}) + d(T_n y_{n-1}, T_{n-1} y_{n-1}) + d(T_n z_{n-1}, T_{n-1} z_{n-1}),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\sum_{n=2}^{\infty} \delta_n \leq \mathbb{M} \sum_{n=2}^{\infty} \Big((1-\alpha_n)\frac{|\lambda_n - \lambda_{n-1}|}{\lambda} + |\alpha_n - \alpha_{n-1}| + |\beta_n - \beta_{n-1}| + |\gamma_n - \gamma_{n-1}|\Big) \\
&+ \sum_{n=2}^{\infty} \sup\{d(T_n w, T_{n-1} w) : w \in \{z_k\}\} + \sum_{n=2}^{\infty} \sup\{d(T_n w, T_{n-1} w) : w \in \{y_k\}\} \\
&+ \sum_{n=2}^{\infty} \sup\{d(T_n w, T_{n-1} w) : w \in \{x_k\}\}.
\end{aligned}
$$

Now, by conditions $(A_1)-(A_4)$, Lemma 2.5 and AKTT condition, we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{3.2}$$

Using Lemma 2.1, we have

$$\frac{1}{\lambda_n} d(w_n, q)^2 - \frac{1}{\lambda_n} d(x_n, q)^2 + \frac{1}{\lambda_n} d(x_n, w_n)^2 \leq f(q) - f(w_n).$$

Since $f(q) \leq f(w_n)$ for all $n \in N$, therefore

$$d(w_n, q)^2 \leq d(x_n, q)^2 - d(x_n, w_n)^2. \tag{3.3}$$

Now by (3.3) and definition of $\{x_n\}$, we have

$$\begin{aligned}
d(x_{n+1}, q)^2 &\leq d(\gamma_n z_n \oplus (1-\gamma_n) T_n z_n, q)^2 \\
&\leq \gamma_n d(z_n, q)^2 + (1-\gamma_n) d(T_n z_n, q)^2 - \gamma_n (1-\gamma_n) d(z_n, T_n z_n)^2 \\
&\leq d(z_n, q)^2 \\
&= d(\beta_n y_n \oplus (1-\beta_n) T_n y_n, q)^2 \\
&\leq d(y_n, q)^2 \\
&= d(\alpha_n u \oplus (1-\alpha_n) T_n w_n, q)^2 \\
&\leq \alpha_n d(u, q)^2 + (1-\alpha_n) d(T_n w_n, q)^2 - \alpha_n (1-\alpha_n) d(u, T_n w_n)^2 \\
&\leq (1-\alpha_n) d(T_n w_n, q)^2 + \alpha_n \Big(d(u, q)^2 - (1-\alpha_n) d(u, T_n w_n)^2 \Big)
\end{aligned}$$

$$\begin{aligned}
(1-\alpha_n) d(x_n, w_n) &\leq d(x_n, q)^2 - d(x_{n+1}, q)^2 + \alpha_n \Big(d(u, q)^2 - d(x_n, q)^2 \\
&\quad - (1-\alpha_n) d(u, T_n w_n)^2 \Big) \\
&\leq |d(x_n, q) - d(x_{n+1}, q)| d(x_n, q) + d(x_{n+1}, q) \\
&\quad + \alpha_n \Big(d(u, q)^2 - d(x_n, q)^2 - (1-\alpha_n) d(u, T_n w_n)^2 \Big) \\
&\leq d(x_{n+1}, x_n) \Big(d(x_n, q) + d(x_{n+1}, q) \Big) + \alpha_n d(u, q)^2 \\
d(x_n, w_n)^2 &\leq \frac{1}{1-\alpha_n} \Big(d(x_{n+1}, x_n)(d(x_n, q) + d(x_{n+1}, q)) + \alpha_n d(u, q)^2 \Big).
\end{aligned}$$

Thus by (3.3) and condition (A_1), we have

$$\lim_{n\to\infty} d(x_n, w_n) = 0. \tag{3.4}$$

Since $\lambda_n > \lambda > 0$ and by Lemma 2.1, we have

$$\begin{aligned} d(J_\lambda x_n, w_n) &= d\Big(J_\lambda x_n, J_\lambda\Big(\frac{\lambda_n-\lambda}{\lambda_n}J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n\Big)\Big) \\ &\leq d\Big(x_n, \Big(\frac{\lambda_n-\lambda}{\lambda_n}\Big)J_{\lambda_n}x_n \oplus \frac{\lambda}{\lambda_n}x_n\Big) \\ &\leq \Big(1-\frac{\lambda}{\lambda_n}d(x_n,y_n)\Big). \end{aligned}$$

Therefore by (3.4)

$$\lim_{n\to\infty} d(J_{\lambda_n}x_n, w_n) = 0. \tag{3.5}$$

By condition (A_1), we have

$$d(y_n, T_n w_n) = d(\alpha_n u \oplus (1-\alpha_n)T_n w_n, T_n w_n) = \alpha_n d(u, T_n w_n) \to 0. \tag{3.6}$$

Now

$$\begin{aligned} d(z_n, y_n) &= d(\beta_n y_n + (1-\beta_n)T_n y_n, y_n) \\ &= (1-\beta_n)d(T_n y_n, y_y) \\ d(z_n, y_n) &\to 0. \end{aligned} \tag{3.7}$$

Thus, by (3.6) and (3.7)

$$d(z_n, T_n w_n) \leq d(z_n, y_n) + d(y_n, T_n w_n) \to 0. \tag{3.8}$$

Now, we have

$$\begin{aligned} d(x_{n+1}, z_n) &= d(\gamma_n z_n \oplus (1-\gamma_n)T_n z_n, z_n) \\ &= (1-\gamma_n)d(z_n, T_n z_n) \\ &\leq (1-b)d(z_n, T_n w_n) + d(T_n w_n, T_n x_n) + d(T_n x_n, T_n x_{n+1}) \\ &\quad + d(T_n x_{n+1}, T_n z_n) \\ &\leq (1-b)d(z_n, T_n w_n) + d(w_n, x_n) + d(x_n, x_{n+1}) + d(x_{n+1}, z_n) \\ d(x_{n+1}, z_n) &\leq \frac{1-b}{b}\Big(d(z_n, T_n w_n) + d(w_n, x_n) + d(x_n, x_{n+1})\Big). \end{aligned}$$

Therefore by (3.2), (3.3) and (3.8), we have

$$\lim_{n\to\infty} d(x_{n+1}, z_n) = 0.$$

Now, using (3.2), (3.4), (3.8) and (3.9), we have

$$d(T_n x_n, x_n) \leq d(T_n x_n, T_n w_n) + d(T_n w_n, z_n) + d(z_n, x_{n+1}) + d(x_{n+1}, x_n) \longrightarrow 0. \tag{3.9}$$

Thus, by (3.3), (3.5) and (3.9), we have

$$\begin{aligned} d(T_n o J_\lambda x_n, x_n) &\leq d(T_n o J_\lambda x_n, T_n w_n) + d(T_n w_n, T_n x_n) + d(T_n x_n, x_n) \\ &\leq d(J_\lambda x_n, w_n) + d(w_n, x_n) + d(T_n x_n, x_n) \\ &\longrightarrow 0. \end{aligned} \tag{3.10}$$

Therefore, by Lemma 2.5 and (3.10), we have

$$\begin{aligned} d(ToJ_\lambda x_n, x_n) &\leq d(ToJ_\lambda x_n, T_n o J_\lambda x_n) + d(T_n o J_\lambda x_n, x_n) \\ &\leq \sup\{d(T_q, T_n q) : q \in \{J_\lambda x_k\}\} + d(ToJ_\lambda x_n, x_n) \\ &\longrightarrow 0. \end{aligned} \tag{3.11}$$

Suppose that z_τ, for each $\tau \in [0, 1]$, is a unique point in C such that $z_\tau = \tau u \oplus (1-\tau)Vz_\tau$, where $V = ToJ_\lambda$. Therefore by Proposition 2.2, Lemma 2.4 and (3.11), the sequence $\{z_\tau\}$ converges to a point $z \in F(V) = F(ToJ_\lambda) = F(T) \cap F(J_\lambda) = \cap_{n=0}^{\infty} F(T_n) \cap F(J_\lambda) = \Omega$, which is the nearest to u. For all Banach limit μ's,

$$d(u, z)^2 \leq \mu_n d(u, x_n)^2$$

which implies that

$$\mu_n(d(u, z)^2 - d(u, x_n)) \leq 0.$$

$$\lim_{n\to\infty} \sup \left\{(d(u, z)^2 - d(u, x_{n+1})^2) - (d(u, z)^2 - d(u, x_n)^2\right\} = 0. \tag{3.12}$$

By (3.3) and (3.9), we have

$$\begin{aligned} d(x_n, T_n w_n) &\leq d(x_n, T_n x_n) + d(T_n x_n, T_n w_n) \\ &\leq d(x_n, T_n x_n) + d(x_n, w_n) \\ &\longrightarrow 0. \end{aligned} \tag{3.13}$$

By Lemma 2.6 and (3.13), we have

$$\begin{aligned} &\lim_{n\to\infty} \sup\{d(u, z)^2 - (1 - \alpha_n) d(u, T_n w_n)^2\} \\ &\qquad = \lim_{n\to\infty} \sup\{d(u, z)^2 - d(u, w_n)^2\} \\ &\qquad \leq 0. \end{aligned} \tag{3.14}$$

Now we will show that

$$\lim_{n\to\infty} \sup\{d(x_n, z) = 0.$$

$$\begin{aligned}
d(x_{n+1}, z)^2 &\leq d(\gamma_n z_n \oplus (1-\gamma_n T_n z_n, z)^2) \\
&\leq \gamma_n d(z_n, z)^2 + (1-\gamma_n) d(T_n z_n, z)^2 \\
&\leq d(z_n, z)^2 \\
&= d(\beta_n y_n \oplus (1-\beta_n) T_n y_n, z)^2 \\
&\leq \beta_n d(y_n, z)^2 + (1-\beta_n) d(T_n y_n, z) \\
&\leq d(y_n, z)^2 \\
&\leq d(\alpha_n u \oplus (1-\alpha_n) T_n w_n, z)^2 \\
&\leq \alpha_n d(u, z)^2 + (1-\alpha_n) d(T_n w_n, z)^2 - \alpha_n(1-\alpha_n) d(u, T_n w_n)^2 \\
&\leq \alpha_n d(u, z)^2 + (1-\alpha_n) d(w_n, z)^2 - \alpha_n(1-\alpha_n) d(u, T_n w_n)^2 \\
&\leq \alpha_n d(u, z)^2 + (1-\alpha_n) d(J_{\lambda_n} w_n, z)^2 - \alpha_n(1-\alpha_n) d(u, T_n w_n)^2 \\
&\leq (1-\alpha_n) d(x_n, z)^2 + \alpha_n \big(d(u, z)^2 + (1-\alpha_n) d(u, T_n w_n)^2 \big).
\end{aligned}$$

By Lemma 2.6, (3.14) and $\sum_{n=0}^{\infty} \alpha_n = \infty$, we have

$$\lim_{n\to\infty} d(x_n, z)^2 = 0.$$

Therefore $\{x_n\}$ strongly converges to z of Ω, which is the nearest to u.
The following result can be obtained from Theorem 3.1 because every Hilbert space is CAT(0) space.

Corollary 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $f : C \longrightarrow (-\infty, \infty)$ be a proper, convex and lower semicontinuous function. Suppose that $u, x_1 \in C$ are arbitrarily chosen and $\{x_n\}$ is a sequence of a C generated by*

$$\begin{cases} w_n = \operatorname{argmin}_{w\in C}\{f(w) + \frac{1}{2\lambda_n} d(w, x_n)^2\} \\ y_n = \alpha_n u \oplus (1-\alpha_n) T_n w_n \\ z_n = \beta_n y_n \oplus (1-\beta_n) T_n y_n \\ x_{n+1} = \gamma_n z_n \oplus (1-\gamma_n) T_n z_n \qquad n \in N, \end{cases} \tag{3.15}$$

where $\{T_n\}$ is a countably infinite family of nonexpansive mapping of C into itself with $\Omega = \bigcap_{n=1}^{\infty} F(T_n) \cap \operatorname{argmin}_{y\in C} f(y) \neq \phi$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$, $\{\lambda_n\}$ are the sequences which satisfy the following axioms:
(A1) $0 < \alpha_n < 1, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
(A2) $\beta_n \in (b, 1]$ *for some* $b \in (0, 1)$ *and* $\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$.

(A3) $\gamma_n \in (c, 1]$ *for some* $c \in (0,1)$ *and* $\sum_{n=1}^{\infty} |\gamma_{n+1} - \gamma_n| < \infty$.
(A4) $\lambda_n \geq \lambda > 0$ *for some* $\lambda \in (0, \infty)$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.
Suppose that $\{\{T_n\}, T\}$ *satisfies the AKTT condition and* $F(T) = \cap_{n=1}^{\infty} F(T_n)$. *Then the sequence* $\{x_n\}$ *converges strongly to a point in* Ω *which is nearest to* u.

4. Numerical Results for Proposed Proximal Point Algorithm

In this section, we will discuss the numerical results for proposed proximal point algorithm. The software Scilab is used for solving the numerical results.

4.1. Solution of Constrained Convex Minimization Problem

The following result can be obtained by Theorem 3.2.

Theorem 4.1. *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* $f : C \longrightarrow (-\infty, \infty]$ *be a proper, convex and lower semicontinuous function such that* f *attains minimizer. Suppose that* $u, x_1 \in C$ *are arbitrarilty chosen and* $\{x_n\}$ *is a sequence of a* C *generated by*

$$\begin{cases} w_n = \text{argmin}_{w \in C} \{f(w) + \frac{1}{2\lambda_n} d(w, x_n)^2\} \\ x_{n+1} = \alpha_n z_n + (1 - \alpha_n) w_n \qquad n \in N, \end{cases} \tag{4.1}$$

where $\{\alpha_n\}$ *and* $\{\lambda_n\}$ *are the sequences which satisfy the folllowing axioms:*
(A1) $0 < \alpha_n < 1, \lim_{n \to \infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ *and* $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
(A2) $\lambda_n \geq \lambda > 0$ *for some* $\lambda \in (0, \infty)$ *and* $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.
Then the sequence $\{x_n\}$ *converges strongly to minimizer of* f.

Example 1. Consider the following minimization problem.

$$\min_{\mathbf{x} \in \mathbb{R}^7} \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{x}\|_2 + (4, 5, -6, 7, 6, -5, -4)^t \mathbf{x} + 11 \tag{4.2}$$

where $\mathbf{x} = (v_1, , v_2, v_3, v_4, v_5, v_6, v_7)^t$ and $-200 \leq v_1, v_2, v_3, v_4, v_5, v_6, v_7 \leq 200$. Suppose that

$$f(\mathbf{x}) = \min_{\mathbf{x} \in \mathbb{R}^7} \|\mathbf{x}\|_1 + \frac{1}{2}\|\mathbf{x}\|_2 + (4, 5, -6, 7, 6, -5, -4)^t \mathbf{x} + 11.$$

Then it is clear that f is convex, proper and lower continuous function. Now, by using soft threshholding operator [22] and proximality operator [18], we have

$$\begin{aligned} J_1(\mathbf{x}) &= \text{argmin}_{\mathbf{w} \in C} \left[f(\mathbf{w}) + \frac{1}{2}\|\mathbf{w} - \mathbf{x}\| \right] \\ &= prox_f \mathbf{x} \\ &= prox_{\frac{\|\cdot\|_1}{2}} \Big(\frac{\mathbf{x} - (4, 5, -6, 7, 6, -5, -4)^t}{2} \Big) \\ &= \Big(\max\Big\{ \frac{|v_1 - 4| - 1}{2}, 0 \Big\} \text{sgn}(v_1 - 1), \Big\{ \frac{|v_2 - 5| - 1}{2}, 0 \Big\} \text{sgn}(v_2 - 1), \end{aligned}$$

$$\left\{\frac{|v_3+6|-1}{2},0\right\}\operatorname{sgn}(v_3-1),\left\{\frac{|v_4-7|-1}{2},0\right\}\operatorname{sgn}(v_4-1),$$
$$\left\{\frac{|v_5-6|-1}{2},0\right\}\operatorname{sgn}(v_5-1),\left\{\frac{|v_6+5|-1}{2},0\right\}\operatorname{sgn}(v_6-1),$$
$$\left\{\frac{|v_7+4|-1}{2},0\right\}\operatorname{sgn}(v_7-1)\Big)$$

where sgn(.) is signum function given by for $\delta \in \mathbb{R}$

$$\operatorname{sgn}(\delta) = \begin{cases} 1, & \delta > 0 \\ 0, & \delta = 0 \\ -1 & \delta < 0. \end{cases} \tag{4.3}$$

We select the initial value $\mathbf{x} = (3, , 7-8, 6, 4, -2, 5)^t$, $u = (3, 4, 4, 8, 5, 6, 4)^t$ and $\alpha_n = \frac{1}{950n}$. Then we have numerical results in Table 1. From Table 1, it is clear

Table 1: Numerical result for Example 1 with initial value $x_1 = (3, , 7-8, 6, 4, -2, 5)^t$

n	$\mathbf{x_n} = (\mathbf{v_1^n}, \mathbf{v_2^n}, \mathbf{v_3^n}, \mathbf{v_4^n}, \mathbf{v_5^n}, \mathbf{v_6^n}, \mathbf{v_7^n})^\mathbf{t}$						
1	3.000000000	7.000000000	-8.000000000	6.000000000	4.000000000	-2.000000000	5.000000000
2	0.002210526	0.001894737	4.496631579	1.501473684	1.504421053	0.506631579	0.007473684
3	0.001105263	0.000947368	4.498315789	1.500736842	1.502210526	0.503315789	0.003736842
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
49	0.000046053	0.000039474	4.499929825	1.500030702	1.500092105	0.500138158	0.000155702
50	0.000045113	0.000038668	4.499931257	1.500030075	1.500090226	0.500135338	0.000152524
⋮	⋮	⋮	⋮	⋮	⋮	⋮	
68	0.000032993	0.00002828	4.499949725	1.500021995	1.500065986	0.500098979	0.000111548
69	0.000032508	0.000027864	4.499950464	1.500021672	1.500065015	0.500097523	0.000109907
70	0.000032037	0.00002746	4.499951182	1.500021358	1.500064073	0.50009611	0.000108314

that sequence $\{x_n\}$ strongly converges the point $(0, 0, 5, 1.5, 1.5, 0.5, 0)^t$ which is minimizer of function f.

4.2. Solution of system of linear equations

Example 2. Assume that $X = \mathbb{R}^5$ with the Euclidean norms and $C = \{\mathbf{x} = (v_1, v_2, v_3, v_4, v_5)^t \in \mathbb{R}^5 : 0 \leq v_1, v_2, v_3, v_4, v_5 \leq 100\}$. Define the mapping T_n for each $\mathbf{x} = (v_1, v_2, v_3, v_4, v_5)^t \in C$ as follows:

$$T_n\mathbf{x} = \Big(\frac{v_1+5}{7n}, \frac{v_2-4n+1}{9n}, \frac{v_3+11}{13n}, \frac{v_4}{17n}, \frac{v_5+7n}{19n}\Big)^t \quad \forall n \in \mathbb{N}.$$

Define a mapping $f :\longrightarrow (-\infty, \infty]$ by

$$f(\mathbf{x}) = \frac{1}{2}\|A\mathbf{x} - b\|^2,$$

where $A = \begin{pmatrix} 2 & 5 & -7 & 8 & 9 \\ 1 & 4 & 2 & 3 & -6 \\ 9 & -2 & 4 & 6 & 2 \\ 5 & -3 & 9 & 5 & 7 \\ 5 & 7 & -2 & 4 & 5 \end{pmatrix}$ and $b = \begin{pmatrix} 48 \\ 18 \\ 43 \\ 65 \\ 43 \end{pmatrix}$ It is clear that the mapping f is convex, proper and lower semi-continuous and the mapping T_n is nonexpansive. By [18],

$$\begin{aligned} J_1(\mathbf{x}) &= \operatorname{argmin}_{w \in C} \left[f(\mathbf{w}) + \frac{1}{2}\|\mathbf{w} - \mathbf{x}\|^2 \right] \\ &= prox_f \mathbf{x} \\ &= (I + A^t A)^{-1}(\mathbf{x} + A^t b), \end{aligned}$$

where I is identity matrix. Thus the algorithm (3.15) becomes

$$\begin{cases} \mathbf{w}_n = J_1 \mathbf{x}_n = (I + A^t A)^{-1}(\mathbf{x}_n + A^t b) \\ \mathbf{y}_n = \alpha_n \mathbf{u} \oplus (1 - \alpha_n) T_n \mathbf{w}_n \\ \mathbf{z}_n = \beta_n \mathbf{y}_n \oplus (1 - \beta_n) T_n \mathbf{y}_n \\ \mathbf{x}_{n+1} = \gamma_n \mathbf{x}_n + (1 - \gamma_n) T_n \mathbf{z}_n \qquad n \in N. \end{cases} \tag{4.4}$$

Consider $\alpha_n = \frac{1}{20n}, \beta_n = \frac{11n}{23n+4}, \gamma_n = \frac{n}{45n+7}$, $u = (2, 7, 4, 6, 4)^t$. It is clear that all the assumptions of Theorem 3.1 are satisfied. Take initial value $x_1 = (1, 1, 4, 6, 2)^t$, we have the numerical results in Table 2.

Table 2: Numerical result for modified proximal point algorithm with initial value $x_1 = (1, 1, 4, 6, 2)^t$

n	$\mathbf{x_n} = (\mathbf{v_1^n}, \mathbf{v_2^n}, \mathbf{v_3^n}, \mathbf{v_4^n}, \mathbf{v_5^n})^\mathbf{t}$				
1	1.000000000	1.000000000	4.000000000	6.000000000	2.000000000
2	0.171310107	1.584728790	2.511663762	5.002374445	2.003307657
3	0.677274751	1.809970545	2.779011014	5.000380611	2.000563415
4	0.802556501	1.878141412	2.857522366	5.000141169	2.000218585
5	0.858139232	1.910531053	2.894911652	5.000072935	2.000116979
6	0.889386246	1.929376730	2.916765928	5.000044899	2.000074087
7	0.909379373	1.941685507	2.931098452	5.000030762	2.000051964
8	0.923261869	1.950349551	2.941221310	5.000022637	2.000038999
9	0.933460682	1.956776481	2.948751302	5.000017523	2.000030699
10	0.941268952	1.961732577	2.954571379	5.000014083	2.000025031
⋮	⋮	⋮	⋮	⋮	⋮
50	0.989693099	1.993159073	2.991804811	5.000001136	2.000002331
⋮	⋮	⋮	⋮	⋮	⋮
58	0.991152297	1.994124388	2.992959004	5.000000945	2.000001953
59	0.991306151	1.994226231	2.993080814	5.000000926	2.000001914
60	0.991454746	1.994324603	2.993198482	5.000000907	2.000001876

It is clear that from Table 2 the sequence $\{x_n\}$ strongly converges to unique fixed point $(1, 2, 3, 5, 2)^t$. The point $(1, 2, 3, 5, 2)^t$ is solution of set of common fixed point of countably infinite family of nonexpansive mappings.

Now, consider the system of linear equations whose solution is $(1,2,3,5,2)^t$ as follows.

$$\begin{aligned}
2v_1 + 5v_2 - 7v_3 + 8v_4 + 9v_5 &= 48\\
v_1 + 4v_2 + 2v_3 + 3v_4 - 6v_5 &= 18\\
9v_1 - 2v_2 + 4v_3 + 6v_4 - 2v_5 &= 43\\
5v_1 - 3v_2 + 9v_3 + 5v_4 + 7v_5 &= 65\\
5v_1 + 7v_2 - 2v_3 + 4v_4 + 5v_5 &= 43.
\end{aligned}$$

4.3. Solution of constrained minimization problem

Assume that A is bounded linear operator on a subset C of H and $b \in C$. Consider the constrained linear system

$$Ax = b. \tag{4.5}$$

Define a mapping $f : C \longrightarrow (-\infty, \infty]$ by

$$f(x) = \frac{1}{2}\|Ax - b\|^2, \tag{4.6}$$

and constrained convex minimization problem

$$\min_{x\in C} f(x) = \min_{x\in C} \frac{1}{2}\|Ax - b\|^2. \tag{4.7}$$

Then by Theorem 3.1, x' is the solution of constrained convex minimization problem (4.7) with minimizer equal to 0 if and only if x' is the solution of constrained linear equation (4.5).
The following result can be obtained by using proximality operator [18].

Theorem 4.2. *Let C be a nonempty closed convex subset of a Hilbert space H and $A : C \longrightarrow C$ bounded and linear operator and $b \in C$. Suppose that $u, x_1 \in C$ are arbitrarilty chosen and $\{x_n\}$ is a sequence in C generated by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n)(I + A^t A)(x_n + A^t b), \quad \forall n \in N, \tag{4.8}$$

where $\{\alpha_n\}$ is the sequences satisfies the condition :
(A_1) $0 < \alpha_n < 1, \lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$.
If the equation (4.5) is consistent, then the sequence $\{x_n\}$ strongly converges to solution of linear system.

Example 3. Consider the following linear system:

$$\begin{aligned}
7v_1 + 8v_2 - 3v_3 - 7v_4 + 4v_5 + 9v_6 + 6v_7 =& 126\\
2v_1 + 3v_2 + 4v_3 - 4v_4 2v_5 - 6v_6 + 8v_7 =& 43\\
v_1 - 5v_2 + 4v_3 + 3v_4 - 2v_5 - 2v_6 + 7v_7 =& 31\\
8v_1 - 6v_2 - 5v_3 + 5v_4 + 4v_5 + 2v_6 + v_7 =& 66\\
4v_1 + 3v_2 - 2v_3 - 2v_4 + 5v_5 + 6v_6 + 9v_7 =& 135\\
6v_1 + 2v_2 - 3v_3 - 5v_4 - 8v_5 + 7v_6 + 6v_7 =& 50\\
7v_1 - 7v_2 + 4v_3 + 4v_4 - 2v_5 + v_6 + 5v_7 =& 37
\end{aligned} \tag{4.9}$$

subject to $-250 \leq v_1, v_2, v_3, v_4, v_5, v_6, v_7 \leq 250$.

Take $A = \begin{pmatrix} 7 & 8 & -3 & 7 & 4 & 9 & 6\\ 2 & 3 & 4 & -4 & 2 & -6 & 8\\ 1 & -5 & 4 & 3 & -2 & -2 & 7\\ 8 & -6 & -5 & 5 & 4 & 2 & 1\\ 4 & 3 & -2 & -2 & 5 & 6 & 9\\ 6 & 2 & -3 & -5 & -8 & 7 & 6\\ 7 & -7 & 4 & 4 & -2 & -1 & 5 \end{pmatrix}$, $x = \begin{pmatrix} v_1\\ v_2\\ v_3\\ v_4\\ v_5\\ v_6\\ v_7 \end{pmatrix}$ and $b = \begin{pmatrix} 123\\ 23\\ 35\\ 61\\ 133\\ 47\\ 41 \end{pmatrix}$

Select $\alpha_n = \frac{1}{750n}$, $u = (2.1, 1.8, 1.3, 2.9, 5.7, 6.8, 7.1)^t$ and initial value $x_1 = (3.5, 2.7, 2.8, 2.6, 5.5, 6.5, 6.3)^t$. By algorithm (4.8) and Theorem 4.2, we have the numerical results in Table 3.

Table 3: Numerical result for algorithm 4.8 with initial value $x_1 = (3.7, 2.7, 2.8, 2.6, 5.5, 6.8, 7.1)^t$

n	$\mathbf{x_n = (v_1^n, v_2^n, v_3^n, v_4^n, v_5^n, v_6^n, v_7^n)^t}$						
1	3.7	2.7	2.8	2.6	5.5.	6.8	7.1
2	2.208678658	1.608403386	1.095457942	3.620931366	4.832835438	6.93138019	6.980946888
3	2.208751207	1.608275485	1.095321399	3.621412629	4.832256556	6.931467893	6.980867413
4	2.295036256	1.808332034	1.112068156	3.820040063	4.772230771	6.892318699	6.969331545
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
19	2.208815696	1.608161794	1.095200027	3.621840418	4.831741995	6.931545852	6.980796769
20	2.208816120	1.608161046	1.095199228	3.621843232	4.831738610	6.931546365	6.980796304
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
58	2.208821211	1.608152071	1.095189646	3.621877005	4.831697986	6.980790727	
59	2.208821255	1.608151993	1.095189564	3.621877296	4.831697636	6.931552573	6.980790679
60	2.208821297	1.608151919	1.095189484	3.621877578	4.831697298	6.931552624	6.980790633
61	2.208821338	1.608151846	1.095189407	3.621877849	4.831696971	6.931552674	6.980790588
62	2.208821416	1.608151776	1.095189332	3.621878112	4.831696654	6.931552722	6.980790544
63	2.208821416	1.608151709	1.095189260	3.621878367	4.831696348	6.931552768	6.980790502

From Table 3, it is clear that the point $x_{63} = (2.2088214, 1.60681517,$ 1.095189, 3.621878, 4.831696, 6.9315527, 6.9807905$)^t$ is approximate solution of system (4.9).

References

[1] Ahmadi, P. and Khatibzadeh, H., On the convergence of inexact proximal point algorithm on Hadamard manifolds, Taiwanese J. Math., 18 (2014), 419-433.

[2] Ahmad, I., Mishra, V. N. and Ahmad, R., An iterative algorithm for a system of generalized implicit variational inclusions, Springer Plus, 5 (2016), 2016 : 1283.

[3] Ansari, Q. H. and Babu, F., Proximal point algorithm for inclusion problems in Hadamard manifolds with applications, Optim. Lett., (2019) 1:21.

[4] Ansari, Q. H., Babu, F. and Yao, J., Regularization of proximal point algorithm in Hadamard manifolds, Journal of Fixed Point Theory, (2019) 21:25.

[5] Ambrosio, L., Gigli, N. and Savare, G., Gradient flows in metric spaces and in the space of probability measures, 2nd Edn. Lectures in Mathematics ETH Zrich, Birkhuser, Basel (2008).

[6] Aoyama, K., Kimura, Y., Takahashi, W. and Toyoda, M., Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space, Nonlinear Anal., 67 (2007), 2350-2360.

[7] Ariza-Ruiz, D., Leustean, L. and Lopez, G., Firmly nonexpansive mappings in classes of geodesic spaces, Trans. Am. Math. Soc., 366 (2014), 4299-4322.

[8] Bacak, M., The proximal point algorithm in metric spaces, Isr. J. Math., 194 (2013), 689-701.

[9] Bacak, M. and Reich, S., The asymptotic behavior of a class of nonlinear semigroups in Hadamard spaces, J. Fixed Point Theory Appl., 16 (2014), 189-202.

[10] Bento, G. C., CruzNeto, J. X. and Oliveira, P. R., A new approach to the proximal point method: convergence on general Riemannian manifolds, J. Optim. Theory Appl., 168 (2016), 743-755.

[11] Bento, G. C., Ferreira, O. P. and Pereira, Y. R. L., Proximal point method for vector optimization on Hadamard manifold, Operation Research Letters, 46(1) (2018), 13-18.

[12] Boikanyo, O. A. and Morosanu, G., A proximal point algorithm converging strongly for general errors, Optim. Lett., 4 (2010), 635-641.

[13] Bridson, M. and Haefliger, A., Metric spaces of non-positive curvature, Springer, Berlin (1999).

[14] Bruck, E. and Reich, S., Accretive operators, Banach limits and dual ergodic theorems, Bull. Acad. Pol. Sci., 29 (1981), 585-589.

[15] Chang, S. S., Yao, J. C., Wang, L. and Qin, L. J., Some convergence theorems involving proximal point and common fixed points for asymptotically nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl., (2016) 68 : 2016.

[16] Cholamjiak, P., The modified proximal point algorithm in CAT(0) spaces, Optim. Lett., 9 (2015), 1401-1410.

[17] Cholamjiak, P., Abdou, A. A. and Cho, Y. J., Proximal point algorithms involving fixed points of nonexpansive mappings in CAT(0) spaces, Fixed Point Theory Appl., 227 (2015), 1-13.

[18] Combettes, P. L. and Pesquet, J. C., Proximal splitting methods in signal processing. In: Bauschke, H. H., Burachik, R., Combettes, P. L., Elser, V., Luke, D. R., Wolkowicz, H. (eds.), Fixed-Point Algorithms for Inverse Problems in Science and Engineering, Springer, New York, (2011), pp. 185-212.

[19] Dhompongsa, S. and Panyanak, B., On Δ-convergence theorems in CAT(0) spaces, Comput. Math. Appl., 56 (2008), 2572-2579.

[20] Ferreira, O. P. and Oliveira, P. R., Proximal point algorithm on Riemannian manifolds, Optimization, 51 (2020), 257-270.

[21] Guler, O., On the convergence of the proximal point algorithm for convex minimization, SIAM J. Control Optim., 29 (1991), 403-419.

[22] Hale, E. T., Yin, W. and Zhang Y., A fixed-point continuation method for l_1- regularized minimization with applications to compressed sensing, Tech. Rep., CAAM TR07–07 (2007). MJOM Modified Proximal Point Algorithms Page 19 of 20-97.

[23] Jost, J., Convex functionals and generalized harmonic maps into spaces of nonpositive curvature, Comment. Math. Helv., 70 (1995), 659-673.

[24] Kamimura, S. and Takahashi, W., Approximating solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theory, 106 (2000), 226-240.

[25] Kirk, W. A., Geodesic geometry and fixed point theory II. In: International Conference on Fixed Point Theory and Applications, Yokohama Publications, Yokohama, (2004) pp. 113-142.

[26] Kirk, W. A., Geodesic geometry and fixed point theory. In: Seminar of Mathematical Analysis (Malaga/Seville, 2002/2003), pp. 195-225, Univ. Sevilla Secr. Publ., Seville (2002/2003).

[27] Kirk, W. A. and Panyanak, B., A concept of convergence in geodesic spaces, Nonlinear Anal., 68 (2008), 3689-3696.

[28] Kopecka, E. and Reich, S., Approximating fixed points in the Hilbert ball, J. Nonlinear Convex Anal., 15 (2014), 819-829.

[29] Li, C., Lopez, G. and Martin-Marquez, V., Monotone vector fields and the proximal point algorithm on Hadamard manifolds, J. London Math. Soc., 79 (2009), 663-683.

[30] Liu, X., Zhou, M., Mishra, L. N., Mishra, V. N. and Damjanović, B., Common fixed point theorem of six self-mappings in Menger spaces using (CLR_{ST}) property, Open Mathematics, 16 (2018), 1423-1434.

[31] Martinet, B., Regularisation d'inquations variationnelles par approximations successives, Rev. Fr. Inf. Rech. Oper., 4 (1970), 154-158.

[32] Mayer, U. F., Gradient flows on nonpositively curved metric spaces and harmonic maps, Commun. Anal. Geom., 6 (1998), 199-253.

[33] Mishra, L. N., Tiwari, S. K. and Mishra, V. N., Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, Journal of Applied Analysis and Computation, 5(4) (2015), 600-612.

[34] Mishra, L. N., Tiwari, S. K., Mishra, V. N. and Khan, I. A, Unique fixed point theorems for generalized contractive mappings in partial metric spaces, Journal of Function Spaces, 2015 (2015), Article ID 960827, 8 pages.

[35] Patir, B., Goswami, N. and Mishra, V. N., Some results on fixed point theory for a class of generalized nonexpansive mappings, Fixed Point Theory Appl, (2018), 19.

[36] Phuengrattana, W., Onjai-uea, N. and Cholamjiak, P., Modified proximal point algorithms for solving constrained minimization and fixed point problems in complete CAT(0) Spaces, Meediterr. J. Math. (2018) 15:97.

[37] Reich, S. and Saback, S., Two strong convergence theorems for a proximal method in reflexive Banach spaces, Numer. Funct. Anal. Optim., 31 (2010), 22-44.

[38] Reich, S. and Salinas, Z., Weak convergence of infinite products of operators in Hadamard spaces, Rend. Circ. Mat. Palermo, 65 (2016), 55-71.

[39] Reich, S. and Shemen, L., A note on Halpern's algorithm in the Hilbert ball, J. Nonlinear Convex Anal., 14 (2013), 853-862.

[40] Rockafeller, R. T., Monotone operators and the proximal point algorithm, SIAM J. Control Optim., 14 (1976), 877-898.

[41] Saejung, S., Halpern's iteration in CAT(0) spaces, Fixed Point Theory Appl., 2010, Article ID 471781 (2010).

[42] Shioji, N. and Takahashi, W., Strong convergence of approximated sequences for nonexpansive mapping in Banach space, Proc. Am. Math. Soc., 125 (1997), 3641-3645.

[43] Suparatulatorn, R., Cholamjiak, P. and Suantai, S., On solving the minimization problem and the fixed-point problem for nonexpansive mappings in CAT(0) spaces, Optim. Methods Softw., 32 (2017), 182-192.

[44] Wang, J., Li, C., Lopez, G. and Yao J., Convergence analysis of inexact proximal point algorithms on Hadamard manifolds, J. Glob. Optim., (2015) 61: 553.