South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 347-360

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

COMMON FIXED POINT THEOREMS IN BICOMPLEX VALUED METRIC SPACES UNDER BOTH RATIONAL TYPE CONTRACTION MAPPINGS SATISFYING E. A. PROPERTY AND INTIMATE MAPPINGS

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(**Received:** *May 11, 2020* **Accepted:** *Mar. 17, 2021* **Published:** *Apr. 30, 2021)*

Abstract: The main purpose of this paper is to investigate some common fixed point theorems in bicomplex valued metric spaces under both rational type contraction mappings satisfying E. A. property and intimate mappings. Our results generalize some earlier results (Rajput & Singh, 2014; Meena, 2015) and extend some existing theorems (Azam et al., 2011; Rouzkard & Imdad, 2012) regarding common fixed point theorems in complex valued metric spaces. A few examples are provided to justify the results obtained and the course of future prospect of works as carried out is sketched in the paper.

Keywords and Phrases: Bicomplex valued metric space, point of coincidence, E. A. property, common fixed point, intimate mapping.

2020 Mathematics Subject Classification: 47H09, 47H10, 46N99, 54H25.

1. Introduction and Preliminaries

During the last fifty years, fixed point theories in complex valued metric spaces are emerging areas of works in the field of the complex as well as functional analysis.The fixed point theorem, generally known as the Banach's contraction mapping principle [2] appeared in explicit form in Banach's thesis in 1922. The famous theorem states that "Let (X, d) be a complete metric space and T be a mapping of X into itself satisfying $d(Tx, Ty) \leq kd(x, y), \forall x, y \in X$, where k is a constant in $(0, 1)$. Then, T has a unique fixed point $x^* \in X$." Banach's fixed point theorem plays a major role in the fixed point theory. Rajput & Singh [12] and Meena [8] respectively proved some common fixed point theorems under rational type contraction mappings and intimate mappings in the set bicomplex numbers. Many properties on the set bicomplex numbers $\mathbb{C}_2$ are scattered over a number of books and articles {cf. [5], [9]& [10]}. Searching for special algebras, Segre published a paper [13] in which he treated an algebra whose elements were bicomplex numbers. Rochon and Shapiro [10] presented some varieties of algebraic properties of both bicomplex numbers and hyperbolic numbers in a unified manner. Elena *et al.* [5] showed how to introduce elementary functions such as polynomials, exponentials and trigonometric functions in this algebra as well as their inverses which is not possible in the case of quaternions incidentally. They showed how these elementary functions enjoy the properties that are very similar to those enjoyed by their complex counterparts. Some interesting results on fixed point theory are established by Tripathy *et al.*{[15]-[17]} using fuzzy metric. Azam *et al.* [1] made a generalization by introducing a complex valued metric spaces using some contractive type conditions. Although there are several number of generalization such as rectangular metric spaces, pseudo metric spaces, fuzzy metric spaces, quasi metric spaces, quasi semi metric spaces, probabilistic metric spaces, D-metric spaces and cone metric spaces including bicomplex valued metric spaces, yet the area of research in bicomplex valued metric spaces is not expanded to a remarkable stage compared to the other metric spaces till now. Jebril *et al.* {[6] & [7]} and Choi *et al.* [4] respectively investigated some fixed point theorems under rational contractions for a pair of mappings and with two weakly compatible mappings in $\mathbb{C}_2$. Recently Rouzkard & Imdad [11] extended and improved the common fixed point theorems more general than the result of Azam *et al.* [1].

We write regular complex number as $z = x + iy$ where x and y are real and $i^2 = -1$. Let $\mathbb{C}_0$ and $\mathbb{C}_1$ be the set of real and complex numbers respectively and z_1 , $z_2 \in \mathbb{C}_1$. The partial order relation $\precsim$ on $\mathbb{C}_1$ is defined as follows:

$$z_1 \precsim z_2 \text{ if and only if } Re(z_1) \leq Re(z_2) \text{ and } Im(z_1) \leq Im(z_2).$$

Thus $z_1 \precsim z_2$ if one of the following conditions is satisfied

$(i)\, Re\,(z_1) = Re\,(z_2)$ and $Im\,(z_1) = Im\,(z_2)$, $(ii)\, Re\,(z_1) < Re\,(z_2)$ and $Im\,(z_1) = Im\,(z_2)$, $(iii)\, Re\,(z_1) = Re\,(z_2)$ and $Im\,(z_1) < Im\,(z_2)$ and $(iv)\, Re\,(z_1) < Re\,(z_2)$ and $Im\,(z_1) < Im\,(z_2)$.

We write $z_1 \precnsim z_2$ if $z_1 \precsim z_2$ and $z_1 \neq z_2$ i.e., one of $(ii), (iii)$ and (iv) is satisfied and we write $z_1 \prec z_2$ if only (iv) is satisfied.

Azam *et al.* [1] defined the complex valued metric space in the following way.

Definition 1.1. *Let X be a non empty set whereas $\mathbb{C}_1$ be the set of complex numbers. Suppose that the mapping $d : X \times X \to \mathbb{C}_1$, satisfies the following conditions*

(d_1) $0 \precsim d(x,y)$, *for all* $x, y \in X$ *and* $d(x,y) = 0$ *if and only if* $x = y$;

(d_2) $d(x,y) = d(y,x)$ *for all* $x, y \in X$;

(d_3) $d(x,y) \precsim d(x,z) + d(z,y)$, *for all* $x, y, z \in X$.

Then d is called a complex valued metric on X and (X,d) is called a complex valued metric space.

The space $\mathbb{C}_2$ is the first in an infinite sequence of multicomplex spaces which are generalization of $\mathbb{C}_1$.

The notion of the space $\mathbb{C}_2$ was defined by Segre [13] as

$$\mathbb{C}_2 = \{w : w = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3, p_k \in \mathbb{C}_0, 0 \leq k \leq 3\}$$

$$\text{i.e., } \mathbb{C}_2 = \{w = z_1 + i_2 z_2 \mid z_1, z_2 \in \mathbb{C}_1\},$$

where $z_1 = p_0 + i_1 p_1, z_2 = p_2 + i_1 p_3$ and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product of all units is commutative and satisfies

$$i_1 i_2 = j, \quad i_1 j = -i_2, \quad i_2 j = -i_1.$$

The inverse of $u = u_1 + i_2\, u_2$ exists if $u_1^2 + u_2^2 \neq 0$ i.e., if $|u_1^2 + u_2^2| \neq 0$ and it is defined as

$$u^{-1} = \frac{1}{u} = \frac{u_1 - i_2\, u_2}{u_1^2 + u_2^2}.$$

If $u = p_0 + i_1 p_1 + i_2 p_2 + i_1 i_2 p_3$ ($p_k \in \mathbb{C}_0$; $k = 0, 1, 2, 3$), then

$$\frac{1}{u} = \left(\frac{p_0\alpha + p_1\beta}{\gamma}\right) + i_1\left(\frac{p_1\alpha - p_0\beta}{\gamma}\right) - i_2\left(\frac{p_2\alpha + p_3\beta}{\gamma}\right) - i_1 i_2\left(\frac{p_3\alpha - p_4\beta}{\gamma}\right),$$

where $\alpha = p_0^2 - p_1^2 + p_2^2 - p_3^2$, $\beta = 2p_0 p_1 + 2p_2 p_3$ and $\gamma = \alpha^2 + \beta^2 = (p_0^2 + p_1^2 + p_2^2 + p_3^2)^2 - 4\,(p_0 p_3 - p_1 p_2)^2$. Obviously $\frac{1}{u}$ exists if $\gamma \neq 0$.

A bicomplex number $u = p_0 + i_1p_1 + i_2p_2 + i_1i_2p_3$ ($p_k \in \mathbb{C}_0$; $k = 0, 1, 2, 3$) is said to be degenerated if the matrix $\begin{pmatrix} p_0 & p_1 \\ p_2 & p_3 \end{pmatrix}$ is degenerated i.e., if the determinant

$$\Delta(u) = \begin{vmatrix} p_0 & p_1 \\ p_2 & p_3 \end{vmatrix} = p_0p_3 - p_1p_2 = 0.$$

The partial order relation $\precsim_{i_2}$ on $\mathbb{C}_2$ was defined by Choi *et al.* [4] as $u \precsim_{i_2} v$ if and only if $u_1 \precsim u_2$ and $v_1 \precsim v_2$, where $u_1, u_2, v_1, v_2 \in \mathbb{C}_1$. The bicomplex valued metric $d : X \times X \to \mathbb{C}_2$ on a non-empty set X and the structure (X, d) on $\mathbb{C}_2$ were defined by Choi *et al.* [4] accordingly.

A norm of a bicomplex number $w = z_1 + i_2 z_2$ is denoted by $||w||$ and is defined [8] by

$$||w|| = \|z_1 + i_2z_2\| = \left(|z_1|^2 + |z_2|^2\right)^{\frac{1}{2}}.$$

Choosing $w = p_0 + i_1p_1 + i_2p_2 + i_1i_2p_3$ ($p_k \in \mathbb{C}_0$; $k = 0, 1, 2, 3$) this can be defined as

$$\|w\| = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^{\frac{1}{2}}. \tag{1.1}$$

One can easily verify that

$$0 \precsim_{i_2} u \precsim_{i_2} v \Rightarrow \|u\| \le \|v\|;\ \|u+v\| \le \|u\| + \|v\|;\ \|\alpha\, u\| = \alpha\, \|u\|; \|u\| \le \|1+u\|, \tag{1.2}$$

for any $u,\ v \in \mathbb{C}_2$ and $\alpha \in \mathbb{C}_0$. If we consider the case that $0 \precnsim_{i_2} u = p_0 + i_1p_1 + i_2p_2 + i_1i_2p_3$ ($p_k \in \mathbb{C}_0$; $k = 0, 1, 2, 3$) (i.e., at least one of p_k 's is positive) and u is degenerated, then $\gamma = \left(p_0^2 + p_1^2 + p_2^2 + p_3^2\right)^2 - 4\left(p_0p_3 - p_1p_2\right)^2$ will be positive and therefore u will be invertible. By the deduction of Rochon & Shapiro [10], we get the following results

(*i*) $\|uv\| \le \sqrt{2}\, \|u\|\, \|v\|$for any $u,\ v \in \mathbb{C}_2$;.

(*ii*) $\|uv\| = \|u\|\, \|v\|$for any $u,\ v \in \mathbb{C}_2$ with at least one of them is degenerated;

(*iii*) $\left\|\frac{1}{u}\right\| = \frac{1}{\|u\|}$ for any degenerated bicomplex number u with $0 \precnsim_{i_2} u$.

Definition 1.2. [1] *Let $\{x_n\}$ be sequence in X and $x \in X$. If for every $c \in \mathbb{C}_2$, with $0 \prec c$ there is $n_0 \in \mathbb{N}$ such that $d(x_n, x) \prec c$ for all $n > n_0$, then x is called the limit of $\{x_n\}$ and we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.*

Definition 1.3. [1] *If every Cauchy sequence is convergent in $\mathbb{C}_2$ then the space is called a complete bicomplex valued metric space.*

Definition 1.4. [3] *Let f and g be two self-maps defined on a set X. Then f and g are said to be weakly compatible if they commute at their coincidence points.*

Definition 1.5. [18] *Let $T, S : X \to X$ be two self mappings of a bicomplex valued metric space (X, d). The pair (T, S) are said to satisfy E. A. property if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$.*

Definition 1.6. [14] *The self mappings T, $S : X \to X$ are said to satisfy the common limit in the range of S property ($CLRs$ property) if $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Sx$ for some $x \in X$.*

Definition 1.7. [1] *Let S and T be self maps on a bicomplex valued metric space (X, d). Then the pair $\{S, T\}$ is said to be $T-$ intimate if and only if $\alpha d(TSz_n, Tz_n) \precsim \alpha d(SSz_n, Sz_n)$, where $\alpha = \limsup\{z_n\}$ or $\liminf\{z_n\}$ is a sequence in X such that $\lim_{n\to\infty} Sz_n = \lim_{n\to\infty} Tz_n = t$ for some t in X.*

Some common fixed point results are established by Rajput & Singh [12] for rational type contraction mapping in $\mathbb{C}_1$ on which they have proved the following theorem.

Theorem 1.1. [12] *Let (X, d) be a complex valued metric space and $A, B, S, T : X \to X$ be four self mappings satisfying the following conditions*

(i) $A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$,

(ii) For all $x, y \in X$ and $0 < \alpha < 1$,

$$d(Ax, By) \precsim \alpha \frac{[d(Ax, Sx)(dAx, Ty) + d(By, Ty)\, d(By, Sx)]}{d(Ax, Ty) + d(By, Sx)} + \beta \frac{\left[\{d(Ax, Ty)\}^2 + \{d(By, Sx)\}^2\right]}{d(Ax, Ty) + d(By, Sx)},$$

(iii) The pairs (A, S) and (B, T) are weakly compatible and

(iv) The pair (A, S) or (B, T) satisfies E. A. property if the range of mappings $S(X)$ or $T(X)$ is closed subspace of X then A, B, S and T have a unique common fixed point in X.

Meena [8] investigated a common fixed point for intimate mappings in $\mathbb{C}_1$. He proved the following theorem in his paper.

Theorem 1.2. [8] *Let A, B, S and T be the four mappings from a complex valued metric space (X, d) into itself, such that*

(a) $A(X) \subset T(X)$ and $B(X) \subset S(X)$,

(b)

$$d(Ax, By) \precsim \alpha d(Sx, Ty) + \beta \frac{d(Ax, Sx) \cdot d(By, Ty)}{d(Ax, Ty) + d(Sx, By) + d(Sx, Ty)},$$

for all $x, y \in X$ *and* $d(Ax, Ty) + d(Sx, By) + d(Sx, Ty) \neq 0$, *where* α, β *are non-negative real numbers with* $\alpha + \beta < 1$,

(*c*) (A, S) *is* $S-$*intimate and* (B, T) *is* $T-$*intimate and*

(*d*) $S(X)$ *is complete.*

Then A, B, S *and* T *have a unique common fixed point in* X.

Our results are the generalizations and extensions of the above theorems which are established by Rajput & Singh [12] and Meena [8]. Here the results of Rochon *et al.* [10] and Elena *et al.* [5] have helped us. Also we have taken some concepts from the papers of Choi *et al.* [4] and Jebril *et al.* [6]. The remaining of the paper is organized as main results containing some theorems, lemma and examples in Section-2, the Section-1 introduces preliminaries and basic definitions.

2. Main Result

In this section we prove some theorems with supporting lemma and examples.

Theorem 2.1. *Let* (X, d) *be a bicomplex valued metric space and* $A, B, S, T : X \to X$ *are four self mappings on* $X \subseteq \mathbb{C}_2$ *satisfying the following conditions*

(*i*) $A(X) \subseteq T(X), B(X) \subseteq S(X)$;

(*ii*) *For all* $z, z' \in X$, $0 < \alpha, \beta, \alpha + \beta < 1$, *and non-singular* $d(Az, Tz') + d(Bz', Sz)$,

$$d(Az, Bz') \precsim_{i_2} \alpha \frac{[d(Az, Sz)(dAz, Tz') + d(Bz', Tz')\, d(Bz', Sz)]}{d(Az, Tz') + d(Bz', Sz)}$$
$$+\beta \frac{\left[\{d(Az, Tz')\}^2 + \{d(Bz', Sz)\}^2\right]}{d(Az, Tz') + d(Bz', Sz)};$$

(*iii*) *The pairs* (A, S) *and* (B, T) *are weakly compatible and*

(*iv*) *The pair* (A, S) *or* (B, T) *satisfies E. A. property.*

If the range of mappings $S(X)$ *or* $T(X)$ *is closed subspace of* $\mathbb{C}_2$ *then* A, B, S *and* T *have a unique common fixed point in* $\mathbb{C}_2$.

Proof. Suppose that the pair (B, T) satisfies E. A. property in $\mathbb{C}_2$. Then there exists a sequence $\{z_n\}$ in $\mathbb{C}_2$ such that $\lim_{n\to\infty} Bz_n = \lim_{n\to\infty} Tz_n = t$ for some $t \in \mathbb{C}_2$. Further since $B(X) \subseteq S(X)$, there exists a sequence $\{z'_n\}$ in $\mathbb{C}_2$ such that $Bz_n = Sz'_n$. Therefore $\lim_{n\to\infty} Sz'_n = t$. Now we claim that $\lim_{n\to\infty} Az'_n = t$. If possible,let $\lim_{n\to\infty} z'_n = r \neq t$. Then by putting $z = z'_n,\ z' = z_n$ in the condition (ii) we have

$$d(Az'_n, Bz_n) \precsim_{i_2} \ \alpha \frac{[d(Az'_n, Sz'_n)\, d(Az'_n, Tz_n) + d(Bz_n, Tz_n)\, d(Bz_n, Sz'_n)]}{d(Az'_n, Tz_n) + d(Bz_n, Sz'_n)}$$
$$+\beta \frac{\left[\{d(Az'_n, Tz_n)\}^2 + \{d(Bz_n, Sz'_n)\}^2\right]}{d(Az'_n, Tz_n) + d(Bz_n, Sz'_n)}.$$

By taking limit as $n \to \infty$, we get

$$d(r,t) \precsim_{i_2} \alpha \frac{[d(r,t)\, d(r,t) + d(t,t)\, d(t,t)]}{d(r,t) + d(t,t)} + \beta \frac{\left[\{d(r,t)\}^2 + \{d(t,t)\}^2\right]}{d(r,t) + d(t,t)}$$

$$\text{i.e, } d(r,t) \precsim_{i_2} \alpha \frac{\{d(r,t)\}^2}{d(r,t)} + \beta \frac{\{d(r,t)\}^2}{d(r,t)}$$

$$\text{i.e., } d(r,t) \precsim_{i_2} (\alpha + \beta)\, d(r,t),$$

which implies

$$(1 - \alpha - \beta)\, d(r,t) \precsim_{i_2} 0.$$

Therefore we have $\|d(r,t)\| \leq 0$. Hence $r = t$. i.e., $\lim_{n\to\infty} Az'_n = \lim_{n\to\infty} Bz_n = t$. Now suppose that $S(\mathbb{C}_2)$ is a closed subspace of $\mathbb{C}_2$. Then $t = Su$ for some $u \in \mathbb{C}_2$. Subsequently we have $\lim_{n\to\infty} Az'_n = \lim_{n\to\infty} Bz_n = \lim_{n\to\infty} Sz'_n = \lim_{n\to\infty} Tz_n = t = Su$. Now we prove that $Au = Su$ i.e., $Au = t$. By putting $z = u$ and $z' = z_n$ in condition (ii) we get

$$d(Au, Bz_n) \precsim_{i_2} \alpha \frac{[d(Au,Su)\, d(Au,Tz_n) + d(Bz_n,Tz_n)\, d(Bz_n,Su)]}{d(Au,Tz_n) + d(Bz_n,Su)}$$
$$+\beta \frac{\left[\{d(Au,Tz_n)\}^2 + \{d(Bz_n,Su)\}^2\right]}{d(Au,Tz_n) + d(Bz_n,Su)}.$$

Taking limit as $n \to \infty$ in the above, we have

$$d(Au,t) \precsim_{i_2} \alpha \frac{\{d(Au,t)\}^2}{d(Au,t)} + \beta \frac{\{d(Au,t)\}^2}{d(Au,t)},$$

which implies $(1-\alpha-\beta)\, d(Au,t) \precsim_{i_2} 0$. Therefore we get $\|d(Au,t)\| \leq 0$. So $Au = t = Su$. Hence u is a coincidence point of (A,S). Now the weak compatibility of pair (A,S) implies that $ASu = SAu$ or $At = St$. On the other hand since $A(\mathbb{C}_2) \subseteq T(\mathbb{C}_2)$, there exists a point v in $\mathbb{C}_2$ such that $Au = Tv = t$. Now we show that v is a coincidence point of (B,T). i.e., $Bv = Tv = t$. So by putting $z = u, z' = v$ in condition (ii) we have

$$d(Au, Bv) \precsim_{i_2} \alpha \frac{[d(Au,Su)\, d(Au,Tv) + d(Bv,Tv)\, d(Bv,Su)]}{d(Au,Tv) + d(Bv,Su)}$$
$$+\beta \frac{\left[\{d(Au,tv)\}^2 + \{d(Bv,Su)\}^2\right]}{d(Au,Tv) + d(Bv,Su)}.$$

Now by taking limit as $n \to \infty$, we get

$$d(t,Bv) \precsim_{i_2} \alpha \frac{[d(t,t)\,d(t,t) + d(Bv,Tv)\,d(Bv,t)]}{d(t,Tv) + d(Bv,t)} + \beta \frac{\left[\{d(t,Tv)\}^2 + \{d((Bv,t)\}^2\right]}{d(t,Tv) + d(Bv,t)}$$
$$\precsim_{i_2} \alpha \frac{d(Bv,Tv)\,d(Bv,t)}{d(t,t) + d(Bv,Tv)} + \beta \frac{\left[\{d(t,t)\}^2 + \{d((Bv,t)\}^2\right]}{d(t,t) + d(Bv,t)}$$
$$\precsim_{i_2} \alpha \frac{d(Bv,Tv)\,d(Bv,t)}{d(Bv,Tv)} + \beta \frac{\{d(Bv,t)\}^2}{d(Bv,t)},$$

which implies $d(t,Bv) \precsim_{i_2} \alpha d(Bv,t) + \beta d(Bv,t)$ i.e., $(1-\alpha-\beta)\,d(t,Bv) \precsim_{i_2} 0$. Therefore, we have $\|d(t,Bv)\| \leq 0$. Hence $t = Bv$. So $Bv = Tv = t$ and v is the coincidence point of B and T. Also the weak compatibility of pair (B,T) implies that $BTv = TBv$ or $Bt = Tt$. Therefore t is a common coincidence point of A, B, S and T. Now we have to show that t is a common fixed point of A, B, S and T. Putting $z = u, z' = t$ in condition (ii) we have

$$d(t,Bt) = d(Au,Bt) \precsim_{i_2} \alpha \frac{[d(Au,Su)\,d(Au,Tt) + d(Bt,Tt)\,d(Bt,Su)]}{d(Au,Tt) + d(Bt,Su)} + \beta \frac{\left[\{d(Au,Tt)\}^2 + \{d(Bt,Su)\}^2\right]}{d(Au,Tt) + d(Bt,Su)}.$$

By putting $Tt = Bt$ and $Su = Au$ in the above inequality we get

$$d(t,Bt) = d(Au,Bt) \precsim_{i_2} \alpha.0 + \beta \frac{\left[\{d(Au,Bt)\}^2 + \{d(Bt,Au)\}^2\right]}{d(Au,Bt) + d(Bt,Au)} \precsim_{i_2} \beta \frac{2\{d(Bt,Au)\}^2}{2d(Au,Bt)},$$

which implies that $d(t,Bt) = d(Au,Bt) \precsim_{i_2} \beta d(Au,Bt)$. i.e., $(1-\beta)\,d(Au,Bt) \precsim_{i_2} 0$. Therefore $\|d(Au,Bt)\| \leq 0$. Hence $Bt = Au = t$. But $Bt = Tt = t$. Therefore, we get $At = Bt = St = Tt = t$. i.e., t is a common fixed point. If we take $T(\mathbb{C}_2)$ is closed then similar argument arises and if we take E. A. property of the pair (A,S) then also similar result is obtained.

Uniqueness:

Let us assume that $\bar{t}$ be another common fixed point of A, B, S and T. i.e., $A\bar{t} = B\bar{t} = S\bar{t} = T\bar{t} = \bar{t}$. Then by putting $z = \bar{t}$ and $z' = t$ in the condition (ii) we have

$$d\left(A\bar{t},Bt\right)\precsim_{i_2}\alpha\frac{\left[d\left(A\bar{t},S\bar{t}\right)\left(dA\bar{t},Tt\right)+d\left(Bt,Tt\right)d\left(Bt,St\right)\right]}{d\left(A\bar{t},Tt\right)+d\left(Bt,S\bar{t}\right)}+\beta\frac{\left[\{d(A\bar{t},Tt)\}^2+\{d(Bt,S\bar{t})\}^2\right]}{d\left(A\bar{t},Tt\right)+d\left(Bt,S\bar{t}\right)}$$

$$\text{i.e., } d\left(A\bar{t},Bt\right)\precsim_{i_2}\alpha.0+\beta\frac{\left[\{d(\bar{t},t)\}^2+\{d(t,\bar{t})\}^2\right]}{d\left(\bar{t},t\right)+d\left(t,\bar{t}\right)}$$

$$\text{i.e., } d\left(\bar{t},t\right)\precsim_{i_2}\beta\frac{2\left\{d\left(t,\bar{t}\right)\right\}^2}{2d\left(\bar{t},t\right)}=\beta.d\left(\bar{t},t\right),$$

which gives that $(1-\beta)\,d\left(\bar{t},t\right)\precsim_{i_2}0$. Therefore we have $\left\|d\left(\bar{t},t\right)\right\|\leq 0$ which implies that $\bar{t}=t$. Hence $At=Bt=St=Tt=t$ is the unique common fixed point of A,B,S and T.

Thus the proof of the theorem is established.

Lemma 2.1. *Let S and T be self maps on a bicomplex valued metric space (X,d). If the pair $\{S,T\}$ is $T-$intimate and $St=Tt=p\in\mathbb{C}_2$ for some t in $\mathbb{C}_2$ then $d\left(Tp,p\right)\precsim_{i_2}d\left(Sp,p\right)$.*

Proof. We consider the sequence $z_n=t$ for all $n\geq 1$. So $\lim\limits_{n\to\infty}Sz_n=\lim\limits_{n\to\infty}Tz_n=St=Tt=p\in\mathbb{C}_2$. Since the pair $\{S,T\}$ is $T-$intimate, we have

$$d\left(TSt,Tt\right)=\lim_{n\to\infty}d\left(TSz_n,Tz_n\right)\precsim_{i_2}\lim_{n\to\infty}d\left(SSz_n,Sz_n\right)=d\left(SSt,St\right),$$

which implies $d\left(Tp,p\right)\precsim_{i_2}d\left(Sp,p\right)$. This completes the proof of the lemma.

Theorem 2.2. *Let (X,d) be a bicomplex valued metric space and A,B,S and T be four self mappings on $X\subseteq\mathbb{C}_2$ such that*

(a) $A\left(X\right)\subset T\left(X\right)$ and $B\left(X\right)\subset S\left(X\right)$,

(b) For all $z,z'\in\mathbb{C}_2$,

$$d\left(Az,Bz'\right)\precsim_{i_2}\alpha d\left(Sz,Tz'\right)+\beta\frac{d\left(Az,Sz\right).d\left(Bz',Tz'\right)}{d\left(Az,Tz'\right)+d\left(Sz,Bz'\right)+d\left(Sz,Tz'\right)},$$

and $d\left(Az,Tz'\right)+d\left(Sz,Bz'\right)+d\left(Sz,Tz'\right)$ is non-singular, where α,β are non-negative real numbers with $\alpha+\sqrt{2}\beta<1$;

(c) (A,S) is $S-$intimate and (B,T) is $T-$intimate and

(d) $S\left(X\right)$ is complete.

Then A,B,S and T have a unique common fixed point in $\mathbb{C}_2$.

Proof. Let z_0 be any arbitrary point in $\mathbb{C}_2$. Then by condition (a), there exists a point $z_1\in\mathbb{C}_2$ such that $Az_0=Tz_1$. Also for $z_1\in\mathbb{C}_2$ we can choose a point $z_2\in\mathbb{C}_2$ such that $Bz_1=Sz_2$ and so on. Inductively we can define a sequence $\{z_n\}$

in $\mathbb{C}_2$ such that $z_{2n} = Az'_{2n} = Tz'_{2n+1}$ and $z_{2n+1} = Bz'_{2n+1} = Sz'_{2n+2}$. Then by (b) we have

$$\begin{aligned}
&d(z_{2n}, z_{2n+1}) = d\left(Az'_{2n}, Bz'_{2n+1}\right)\\
&\precsim_{i_2} \alpha d\left(Sz'_{2n}, Tz'_{2n+1}\right) + \beta \frac{d\left(Az'_{2n}, Sz'_{2n}\right).d\left(Bz'_{2n+1}, Tz'_{2n+1}\right)}{d\left(Az'_{2n}, Tz'_{2n+1}\right) + d\left(Sz'_{2n}, Bz'_{2n+1}\right) + d\left(Sz'_{2n}, Tz'_{2n+1}\right)}\\
&\precsim_{i_2} \alpha d(z_{2n-1}, z_{2n}) + \beta \frac{d(z_{2n}, z_{2n-1}).d(z_{2n+1}, z_{2n})}{d(z_{2n}, z_{2n}) + d(z_{2n-1}, z_{2n+1}) + d(z_{2n-1}, z_{2n})},
\end{aligned}$$

which implies

$$\begin{aligned}
\|d(z_{2n}, z_{2n+1})\| &\leq \alpha \|d(z_{2n-1}, z_{2n})\| + \beta \left\| \frac{d(z_{2n}, z_{2n-1}).d(z_{2n+1}, z_{2n})}{d(z_{2n-1}, z_{2n+1}) + d(z_{2n-1}, z_{2n})} \right\|\\
&\leq \alpha \|d(z_{2n-1}, z_{2n})\| + \beta \frac{\|d(z_{2n}, z_{2n-1}).d(z_{2n+1}, z_{2n})\|}{\|d(z_{2n-1}, z_{2n+1}) + d(z_{2n-1}, z_{2n})\|}\\
&\leq \alpha \|d(z_{2n-1}, z_{2n})\| + \beta \frac{\sqrt{2}\,\|d(z_{2n}, z_{2n-1})\|.\|d(z_{2n+1}, z_{2n})\|}{\|d(z_{2n-1}, z_{2n+1}) + d(z_{2n-1}, z_{2n})\|}.
\end{aligned}$$

We know that $\|d(z_{2n+1}, z_{2n})\| < \|d(z_{2n-1}, z_{2n+1}) + d(z_{2n-1}, z_{2n})\|$.

$$\text{i.e., } \frac{\|d(z_{2n+1}, z_{2n})\|}{\|d(z_{2n-1}, z_{2n+1}) + d(z_{2n-1}, z_{2n})\|} < 1.$$

Therefore, we get

$$\begin{aligned}
\|d(z_{2n}, z_{2n+1})\| &\leq \alpha \|d(z_{2n-1}, z_{2n})\| + \beta\sqrt{2}\,\|d(z_{2n}, z_{2n-1})\| \leq \left(\alpha + \sqrt{2}\beta\right) \|d(z_{2n-1}, z_{2n})\|\\
&\text{i.e., } \|d(z_{2n}, z_{2n+1})\| \leq \gamma \|d(z_{2n-1}, z_{2n})\|, \text{ where } \gamma = \left(\alpha + \sqrt{2}\beta\right).
\end{aligned}$$

Also, we have

$$\begin{aligned}
&d(z_{2n+2}, z_{2n+1}) = d\left(Az'_{2n+2}, Bz'_{2n+1}\right)\\
&\precsim_{i_2} \alpha d\left(Sz'_{2n+2}, Tz'_{2n+1}\right) + \beta \frac{d\left(Az'_{2n+2}, Sz'_{2n+2}\right).d\left(Bz'_{2n+1}, Tz'_{2n+1}\right)}{d\left(Az'_{2n+2}, Tz'_{2n+1}\right) + d\left(Sz'_{2n+2}, Bz'_{2n+1}\right) + d\left(Sz'_{2n+2}, Tz'_{2n+1}\right)}\\
&\precsim_{i_2} \alpha d(z_{2n+1}, z_{2n}) + \beta \frac{d(z_{2n+2}, z_{2n+1}).d(z_{2n+1}, z_{2n})}{d(z_{2n+2}, z_{2n}) + d(z_{2n+1}, z_{2n+1}) + d(z_{2n+1}, z_{2n})}\\
&= \alpha d(z_{2n+1}, z_{2n}) + \beta \frac{d(z_{2n+2}, z_{2n+1}).d(z_{2n+1}, z_{2n})}{d(z_{2n+2}, z_{2n}) + d(z_{2n+1}, z_{2n})},
\end{aligned}$$

which implies

$$\begin{aligned}
\|d(z_{2n+2}, z_{2n+1})\| &\leq \alpha \|d(z_{2n+1}, z_{2n})\| + \left\| \beta \frac{d(z_{2n+2}, z_{2n+1}).d(z_{2n+1}, z_{2n})}{d(z_{2n+2}, z_{2n}) + d(z_{2n+1}, z_{2n})} \right\|\\
&\leq \alpha \|d(z_{2n+1}, z_{2n})\| + \beta \frac{\|d(z_{2n+2}, z_{2n+1}).d(z_{2n+1}, z_{2n})\|}{\|d(z_{2n+2}, z_{2n}) + d(z_{2n+1}, z_{2n})\|}
\end{aligned}$$

$$\leq \alpha \left\|d\left(z_{2n+1}, z_{2n}\right)\right\| + \beta \frac{\sqrt{2}\left\|d\left(z_{2n+2}, z_{2n+1}\right)\right\| . \left\|d\left(z_{2n+1}, z_{2n}\right)\right\|}{\left\|d\left(z_{2n+2}, z_{2n}\right) + d\left(z_{2n+1}, z_{2n}\right)\right\|}.$$

Again since $\left\|d\left(z_{2n+2}, z_{2n+1}\right)\right\| \leq \left\|d\left(z_{2n+2}, z_{2n}\right) + d\left(z_{2n}, z_{2n+1}\right)\right\|$, therefore we get

$$\begin{aligned} \left\|d\left(z_{2n+2}, z_{2n+1}\right)\right\| &\leq \alpha \left\|d\left(z_{2n+1}, z_{2n}\right)\right\| + \beta\sqrt{2}\left\|d\left(z_{2n+1}, z_{2n}\right)\right\| \leq \left(\alpha + \sqrt{2}\beta\right)\left\|d\left(z_{2n+1}, z_{2n}\right)\right\| \\ \text{i.e.,} \quad & \left\|d\left(z_{2n+2}, z_{2n+1}\right)\right\| \leq \gamma \left\|d\left(z_{2n+1}, z_{2n}\right)\right\|. \end{aligned}$$

Thus we have

$$\left\|d\left(z_{n+1}, z_{n+2}\right)\right\| \leq \gamma \left\|d\left(z_n, z_{n+1}\right)\right\| \leq \gamma^2 \left\|d\left(z_{n-1}, z_n\right)\right\| \leq ... \leq \gamma^{n+1} \left\|d\left(z_0, z_1\right)\right\|.$$

So for any $m > n$, we get

$$\begin{aligned} \left\|d\left(z_n, z_m\right)\right\| &\leq \left\|d\left(z_n, z_{n+1}\right)\right\| + \left\|d\left(z_n, z_{n+1}\right)\right\| + ... + \left\|d\left(z_{m-1}, z_m\right)\right\| \\ &\leq \gamma^n \left\|d\left(z_0, z_1\right)\right\| + \gamma^{n+1}\left\|d\left(z_0, z_1\right)\right\| + ... + \gamma^{m-1}\left\|d\left(z_0, z_1\right)\right\|, \end{aligned}$$

which implies $\left\|d\left(z_n, z_m\right)\right\| \leq \frac{\gamma^n}{1-\gamma}\left\|d\left(z_0, z_1\right)\right\| \to 0$ as $m, n \to \infty$. So the sequence $\{z_n\} = \{Sz'_{2n}\}$ is a Cauchy sequence in $S(\mathbb{C}_2)$. Again since $S(\mathbb{C}_2)$ is complete, therefore the sequence $\{z_n\}$ converges to a point $p = Su$ for some $u \in \mathbb{C}_2$. Thus $Az'_{2n}, Sz'_{2n}, Bz'_{2n+1}, Tz'_{2n+1} \to p$ as $n \to \infty$. Now

$$d\left(Au, Bz'_{2n+1}\right) \precsim_{i_2} \alpha d\left(Su, Tz'_{2n+1}\right) + \beta \frac{d(Au, Su).d\left(Bz'_{2n+1}, Tz'_{2n+1}\right)}{d\left(Au, Tz'_{2n+1}\right) + d\left(Su, Bz'_{2n+1}\right) + d\left(Su, Tz'_{2n+1}\right)}$$

implies that

$$\begin{aligned} & \left\|d\left(Au, Bz'_{2n+1}\right)\right\| \\ \leq\ & \alpha \left\|d\left(Su, Tz'_{2n+1}\right)\right\| + \left\|\beta \frac{d(Au, Su).d\left(Bz'_{2n+1}, Tz'_{2n+1}\right)}{d\left(Au, Tz'_{2n+1}\right) + d\left(Su, Bz'_{2n+1}\right) + d\left(Su, Tz'_{2n+1}\right)}\right\| \\ \leq\ & \alpha \left\|d\left(Su, Tz'_{2n+1}\right)\right\| + \beta \frac{\left\|d(Au, Su).d\left(Bz'_{2n+1}, Tz'_{2n+1}\right)\right\|}{\left\|d\left(Au, Tz'_{2n+1}\right) + d\left(Su, Bz'_{2n+1}\right) + d\left(Su, Tz'_{2n+1}\right)\right\|} \\ \leq\ & \alpha \left\|d\left(Su, Tz'_{2n+1}\right)\right\| + \beta \frac{\left\|d(Au, Su)\right\|.\left\|d\left(Bz'_{2n+1}, Tz'_{2n+1}\right)\right\|}{\left\|d\left(Au, Tz'_{2n+1}\right) + d\left(Su, Bz'_{2n+1}\right) + d\left(Su, Tz'_{2n+1}\right)\right\|}. \end{aligned}$$

Taking limit as $n \to \infty$, we have $\left\|d(Au, p)\right\| \leq \alpha \left\|d(Su, p)\right\|$, therefore we get $\left\|d(Au, p)\right\| = 0$. i.e., $Au = p = Su$. Again since $A(\mathbb{C}_2) \subset T(\mathbb{C}_2)$, therefore there exists a $v \in \mathbb{C}_2$ such that $Au = Tv = p$. Now applying (b) we have

$$d(p, Bv) = d(Au, Bv) \precsim_{i_2} \alpha d(Su, Tv) + \beta \frac{d(Au, Su).d(Bv, Tv)}{d(Au, Tv) + d(Su, Bv) + d(Su, Tv)},$$

which implies

$$\begin{aligned}\|d(p,Bv)\| &\leq \alpha\|d(Su,Tv)\| + \beta\frac{\|d(Au,Su).d(Bv,Tv)\|}{\|d(Au,Tv)+d(Su,Bv)+d(Su,Tv)\|}\\ &\leq \alpha\|d(Su,Tv)\| + \beta\frac{\sqrt{2}\,\|d(Au,Su)\|.\|d(Bv,Tv)\|}{\|d(Au,Tv)+d(Su,Bv)+d(Su,Tv)\|}.\end{aligned}$$

Thus $\|d(p,Bv)\| = 0$ and this gives that $p = Bv = Tv = Au = Su$. Now since $Au = Su = p$ and (A,S) is $S-$intimate, therefore by Lemma 2.1, we have $\|d(Sp,p)\| \leq \|d(Ap,p)\|$. Also by (b)we have

$$d(Ap,p) = d(Ap,Bv) \precsim_{i_2} \alpha d(Sp,Tv) + \beta\frac{d(Ap,Sp).d(Bv,Tv)}{d(Ap,Tv)+d(Sp,Bv)+d(Sp,Tv)}.$$

i.e., $\|d(Ap,p)\| \leq \alpha\|d(Sp,p)\|$, which yields that $\|d(Ap,p)\| = 0$. Therefore $Ap = p$ and $Sp = p$. Similarly we can show that $Bp = Tp = p$.

Uniqueness

Let us consider that p and q are two common fixed points of A, B, S and T such that $p \neq q$. Then using (b) we get

$$d(p,q) = d(Ap,Bq) \precsim_{i_2} \alpha d(Sp,Tq)+\beta\frac{d(Ap,Sp).d(Bq,Tq)}{d(Ap,Tq)+d(Sp,Bq)+d(Sp,Tq)} \precsim_{i_2} \alpha d(p,q).$$

i.e.,$\|d(p,q)\| \leq \alpha\|d(p,q)\|$, which implies that $p = q$.

This completes the proof of the theorem.

Example 2.1. Let $X = \{z_1, z_2\} \subset \mathbb{C}_1$ with $d : X \times X \to \mathbb{C}_2$ is defined by

$$d(z_1,z_2) = \begin{cases}1, \text{if } z_1 \neq z_2\\ 0, \text{if } z_1 = z_2.\end{cases}$$

Then (X,d) is a complete bicomplex valued metric space. Define $A = B, S, T : X \to X$ by $Az = z_1'$ for all $z \in X$, $Sz_1' = Tz_2' = z_2'$ and $Sz_2' = Tz_2' = z_1'$. Then all the conditions of above theorem are satisfied except intimate condition. We see that $\|d(SAz_2', Sz_2')\| = \|d(z_2', z_1')\| > 0 = \|d(AAz_2', Az_2')\|$, where $\{z_2'\}$ is a constant sequence in $\mathbb{C}_2$ such that $Az_2' = Sz_2' = z_1'$. Thus the pair (A,S) is not $S-$intimate. Therefore A, S and T do not have a common fixed point.

Example 2.2. Let $X = \mathbb{C}_1$ be the set of complex numbers. Define $d : X \times X \to \mathbb{C}_2$ by $d(z_1,z_2) = i_2\|z_1 - z_2\|$ where $z_1 = x_1 + i_1y_1$ and $z_2 = x_2 + i_1y_2$. Then (X,d) is a complete bicomplex valued metric space. Define $A, B, S, T : \mathbb{C}_2 \to \mathbb{C}_2$ as $Az = 0, Bz = 0, Sz = z$ and $Tz = \frac{z}{2}$. Clearly $A(\mathbb{C}_2) \subset T(\mathbb{C}_2)$ and $B(\mathbb{C}_2) \subset S(\mathbb{C}_2)$. Now considering the sequence $\left\{z_n = \frac{1}{n}, n \in \mathbb{N}\right\}$ in $\mathbb{C}_2$ we get $\lim_{n\to\infty} Az_n = \lim_{n\to\infty} Sz_n = 0$. Also we have $\lim_{n\to\infty} d\left(SAz_n, Sz_n\right) \precsim_{i_2} \lim_{n\to\infty} d\left(AAz_n, Az_n\right)$. Thus the pair (A, S) is $S-$intimate. Again since $\lim_{n\to\infty} d\left(TBz_n, Tz_n\right) \precsim_{i_2} \lim_{n\to\infty} d\left(BBz_n, Bz_n\right)$, therefore the pair (B, T) is $T-$intimate. Further the mappings satisfies all the conditions of above theorem. Hence A, B, S and T have a unique common fixed point in $\mathbb{C}_2$.

3. Future Prospect

In the line of the works as carried out in the paper one may think of the deduction of fixed point theorems using common limit in the range (CLR) property, expansive metric and other different types of metrices under the flavour of bicomplex analysis. This may be an active area of research to the future workers in this branch.

References

[1] Azam, A., Fisher, B. and Khan, M., Common Fixed Point Theorems in Complex Valued Metric Spaces, Num. Func. Anal. Opt., 32 (2011), pp. 243-253.

[2] Banach, S., Sur les operations dans les ensembles abs traits et leur application aux equations integrales, Fund. Math., 3 (1922), pp. 133-181.

[3] Bhatt, S., Chaukiyal, S., Dimri, R. C., A common fixed point theorem for weakly compatible maps in complex-valued metric spaces, Int. J. Math. Sci. Appl., 1(3) (2011), pp. 1385-1389.

[4] Choi, J., Datta, S. K., Biswas, T., Islam, N., Some fixed point theorems in connection with two weakly compatible mappings in bicomplex valued metric spaces, Honam Mathematical J., 39(1) (2017), pp. 115-126.

[5] Elena, M., Shapiro, M. and Strupra, D. C., Bicomplex Number and Their Elementary Functions, CUBO, A Mathematics Journal 14(2) (2012), pp. 61-80.

[6] Jebril, I. H., Datta, S. K., Sarkar, R. and Biswas, N., Common fixed point theorems under rational contractions for a pair of mappings in bicomplex valued metric spaces, Journal of Interdisciplinary Mathematics, 22 (7) (2019), pp. 1071-1082.

[7] Jebril, I. H., Datta, S. K., Sarkar, R. and Biswas, N., Common fixed point theorems under rational contractions using two mappings and six mappings

and coupled fixed point theorem in bicomlex valued b-metric space, TWMS J. of Apl. & Eng. Math. (Accepted for publication and to appear).

[8] Meena, G., Common Fixed point for Intimate Mappings in Complex valued Metric Spaces, Gen. Math. Notes, 26 (2) (2015), pp. 97-103.

[9] Price G. B., An Introduction to Multicomplex Spaces and Functions, MARCEL DEKKER, INC., (1990), ISBN 0-8247-8345-X.

[10] Rochon, D. & Shapiro, M., On algebraic properties of bicomplex and hyperbolic numbers, Anal. Univ. Oradea, fasc. math., 11 (2004), pp. 71-110.

[11] Rouzkard, F. and Imdad, M., Some common fixed point theorems on complex valued metric spaces, Computer and Mathematics with Applications, 64(6) (2012), pp. 1866-1874.

[12] Rajput, N., Singh, N., Some common fixed point results for rational type contraction mapping in complex valued metric space, International Journal of Mathematical Archive, 5(6) (2014), pp. 125-128.

[13] Segre, C., Le Rappresentazioni Reali delle Forme Complesse a Gli Enti Iperalgebrici, Math. Ann., 40 (1892), pp. 413-467.

[14] Sintunavarat, W., Kumam, P., Common fixed point theorem for a pair of weakly compatible mappings in fuzzy metric space, J. Appl. Math., (2011), Article ID 637958, 14 pages.

[15] Tripathy, B. C., Paul, S. and Das N. R., Banach's and Kannan's fixed point results in fuzzy 2-metric spaces: Proyecciones J. Math, 32(4) (2013), 359-375.

[16] Tripathy, B. C., Paul, S. and Das N. R., A fixed point theorem in a generalized fuzzy metric space, Boletim da Sociedade Paranaense de Matematica, 32(2) (2014), 221-227.

[17] Tripathy, B. C., Paul, S. and Das N. R., Fixed point and periodic pint theorems in fuzzy metric space, Songklanakarin Journal of Science and Technology, 37(1) (2015), 89-92.

[18] Verma, R. K., Pathak, H. K., Common Fixed Point Theorems for a pair of Mappings in Complex-Valued Metric Spaces, JMCS, 6 (2013), pp.18-26.