South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 285-296

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

THE CONNECTED GEODETIC VERTEX COVERING NUMBER OF A GRAPH

V. M. Arul Flower Mary, J. Anne Mary Leema*, B. Uma Devi and P. Titus*****

Department of Mathematics,
Holy Cross College (Autonomous), Nagercoil, Tamil Nadu - 629004, INDIA
E-mail : arulflowermary@gmail.com

*Department of Mathematics,
Manonmaniam Sundaranar University,
Abishekapatti, Tirunelveli, Tamil Nadu - 627012, INDIA
E-mail : annemary88ma@gmail.com

**Department of Mathematics,
S. T. Hindu College, Nagercoil, Tamil Nadu - 629002, INDIA
E-mail : umasub1968@gmail.com

***Department of Mathematics,
University College of Engineering, Nagercoil
Anna University, Tirunelveli Region,
Tirunelveli, Tamil Nadu - 627007, INDIA
E-mail : titusvino@yahoo.com

(**Received:** *Mar. 18, 2020* **Accepted:** *Jan. 02, 2021* **Published:** *Apr. 30, 2021)*

Abstract: For a connected graph G of order $n \geq 2$, a set $S \subseteq V(G)$ is a *geodetic vertex cover* of G if S is both a geodetic set and a vertex cover of G. The minimum cardinality of a geodetic vertex cover of G is defined as the *geodetic vertex covering number* of G and is denoted by $g_\alpha(G)$. Any geodetic vertex cover of cardinality $g_\alpha(G)$ is a $g_\alpha-$ set of G. A *connected geodetic vertex cover* of G is a geodetic vertex cover S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic vertex cover of G is the *connected geodetic vertex covering number* of G and is denoted by $g_{\alpha c}(G)$. A connected geodetic vertex cover of cardinality $g_{\alpha c}(G)$ is called a $g_{\alpha c}$ - set of G. Some general properties satisfied by connected geodetic vertex covering sets are studied. The connected geodetic vertex covering number of several classes of graphs are determined. Connected graphs of order n with connected geodetic vertex covering number 2, 3, $\frac{n}{2}$ and n are characterized. For any connected graph G of order $n \geq 2$, the necessary and sufficient condition for $g_c(G) = g_{\alpha c}(G)$ is given.

Keywords and Phrases: Geodetic vertex cover, connected geodetic vertex cover, connected geodetic vertex covering number.

2020 Mathematics Subject Classification: 05C12.

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops and multiple edges. The *order* and *size* of G are denoted by n and m, respectively. For basic graph theoretic terminology we refer to [6]. For any two vertices x and y in a connected graph G, *the distance* $d(x, y)$ is the length of a shortest $x - y$ path in G. An $x - y$ path of length $d(x, y)$ is called an $x - y$ *geodesic*. A vertex v is said to lie on an $x - y$ geodesic P if v is a vertex of P including the vertices x and y. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the *radius*, *rad* G and the maximum eccentricity is its *diameter*, *diam* G. The *neighborhood* of a vertex v is the set $N(v)$ consisting of all vertices u which are adjacent with v. A vertex v is an *extreme vertex* of G if the subgraph induced by its neighbors is complete. For a cut vertex v in a connected graph G and a component H of $G - v$, the subgraph H and the vertex v together with all edges joining v and $V(H)$ is called a *branch* of G at v.

A *geodetic set* of G is a set $S \subseteq V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The *geodetic number* $g(G)$ of G is the minimum cardinality of its geodetic sets and any geodetic set of cardinality $g(G)$ is a *minimum geodetic set* or a g - set of G. The geodetic number of a graph was introduced in [2,7] and further studied in [3-5]. It was shown in [7] that determining the geodetic number of a graph is an NP-hard problem. The connected geodetic number (connected geodomination number) was introduced in [8] and further studied in [9]. A *connected geodetic set* of G is a geodetic set S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic set of G is the *connected geodetic number* of G and is denoted by $g_c(G)$. These concepts have many applications in location theory and convexity theory. A subset S of $V(G)$ is said to be a *vertex covering set* of the graph G if every edge of G has at least one end point in S. A vertex covering set with minimum cardinality is called *minimum vertex covering set.* The vertex covering number of the graph G is the cardinality of any minimum vertex covering set of the graph G. It is denoted by $\alpha(G)$. The vertex covering number was studied in [10].

The geodetic vertex covering number was introduced and studied in [1]. A set $S \subseteq V(G)$ is a *geodetic vertex cover* of G if S is both a geodetic set and a vertex cover of G. The minimum cardinality of a geodetic vertex cover of G is defined as the *geodetic vertex covering number of* G and is denoted by $g_\alpha(G)$. Any geodetic vertex cover of cardinality $g_\alpha(G)$ is a g_α - set of G. A non-trivial connected graph with no cut vertex is called a *non-separable graph.* A *block* of a graph G is a maximal non-separable subgraph of G. A graph G is a *block graph* if every block of G is complete. If a graph G has a spanning cycle Z, then G is called a *hamiltonian graph* and Z a *hamiltonian cycle.* A graph with no induced 3-cycle is called a triangle free graph.

Theorem 1.1. ([4]) *Every extreme vertex of a connected graph G belongs to every geodetic set of G.*

Theorem 1.2. ([9]) *For any non-trivial tree T of order n, $g_c(T) = n$.*

Theorem 1.3. ([8]) *For any connected graph G, $g_c(G) \geq 1 + diam\ G$.*

Theorem 1.4. ([9]) *Let G be a connected graph. Then every vertex of G is either a cut vertex or an extreme vertex if and only if $g_c(G) = n$.*

Theorem 1.5. ([9]) *Let G be a connected graph of order $n \geq 2$. Then $G = K_2$ if and only if $g_c(G) = 2$.*

Theorem 1.6. ([1]) *Let G be a connected graph of order $n \geq 2$. Then $g_\alpha(G) = 2$ if and only if G is either K_2 or $K_{2,n-2}$ $(n \geq 3)$.*

Throughout the following G denotes a connected graph with at least two vertices.

2. The Connected Geodetic Vertex Covering Number

Definition 2.1. *Let G be a connected graph with at least two vertices. A connected geodetic vertex cover of G is a geodetic vertex cover S such that the subgraph $G[S]$ induced by S is connected. The minimum cardinality of a connected geodetic vertex cover of G is the connected geodetic vertex covering number of G and is denoted by $g_{\alpha c}(G)$. A connected geodetic vertex cover of cardinality $g_{\alpha c}(G)$ is called a $g_{\alpha c}$ - set of G.*

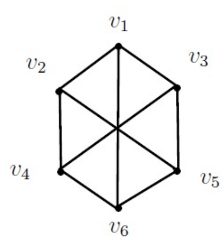

Figure 2.1 : G

Example 2.2. For the graph G given in Figure 2.1, $S = \{v_1, v_4, v_5\}$ is a g_α - set of G so that $g_\alpha(G) = 3$; and $S_1 = \{v_1, v_4, v_5, v_6\}$ is a $g_{\alpha c}$ - set of G so that $g_{\alpha c}(G) = 4$. Thus the geodetic vertex covering number and the connected geodetic vertex covering number of a graph are different.

Remark 2.3. *For the graph G given in Figure 2.1, $S_1 = \{v_1, v_4, v_5, v_6\}$ and $S_2 = \{v_1, v_2, v_3, v_6\}$ are two distinct $g_{\alpha c}$ - sets of G. Thus there can be more than one $g_{\alpha c}$ - set for a graph G.*

Theorem 2.4. *Every extreme vertex of a connected graph G belongs to every connected geodetic vertex cover of G. In particular, every end vertex of G belongs to every connected geodetic vertex cover of G.*

Proof. Since every connected geodetic vertex cover of G is a geodetic set, the result follows from Theorem 1.1.

Theorem 2.5. *For any connected graph G of order $n \geq 2$, $2 \leq g_\alpha(G) \leq g_{\alpha c}(G) \leq n$.*

Proof. Any geodetic vertex cover of G needs at least two vertices and so $g_\alpha(G) \geq 2$. Since every connected geodetic vertex cover of G is also a geodetic vertex cover, it follows that $g_\alpha(G) \leq g_{\alpha c}(G)$. Also, since $V(G)$ induces a connected geodetic vertex cover of G, it is clear that $g_{\alpha c}(G) \leq n$.

Remark 2.6. *The bounds in Theorem 2.5 are sharp. For the path $G = P_3$, $g_\alpha(G) = 2$ and for the complete graph $G = K_n$ $(n \geq 2)$, $g_{\alpha c}(G) = n$. Also, all the inequalities in Theorem 2.5 are strict. For the graph G given in Figure 2.2, $S = \{v_1, v_3, v_4, v_6\}$ is a g_α - set of G and $S_1 = \{v_1, v_3, v_4, v_5, v_6\}$ is a $g_{\alpha c}$ - set of G. Hence $g_\alpha(G) = 4, g_{\alpha c}(G) = 5, n = 6$ and so $2 < g_\alpha(G) < g_{\alpha c}(G) < n$.*

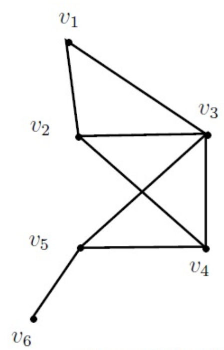

Figure 2.2 : G

Corollary 2.7. *Let G be any connected graph of order $n \geq 2$. If $g_{\alpha c}(G) = 2$, then $g_\alpha(G) = 2$.*

Corollary 2.8. *Let G be any connected graph of order $n \geq 2$. If $g_\alpha(G) = n$, then*

$g_{\alpha c}(G) = n$.

Corollary 2.9. *For any connected graph G with k extreme vertices, $g_{\alpha c}(G) \geq max(2, k)$.*

Proof. This follows from Theorem 2.4 and Theorem 2.5.

Corollary 2.10. *For the complete graph K_n $(n \geq 2)$, $g_{\alpha c}(K_n) = n$.*

Theorem 2.11. *Let G be a connected graph with cut vertices and let S be a connected geodetic vertex cover of G. If v is a cut vertex of G, then every component of $G - v$ contains an element of S.*

Proof. Let v be a cut vertex of G and let S be a connected geodetic vertex cover of G. Suppose that there exists a component, say G_1, of $G - v$ such that G_1 contains no vertex of S. Let $x \in V(G_1)$. Since S is a connected geodetic vertex cover of G, there exists a pair of vertices u and w in S such that x lies in some $u - w$ geodesic $P : u = u_0, u_1, ..., u_i = x, ...u_l = w$ in G. Since v is a cut vertex of G, the $u - x$ subpath of P and the $x - w$ subpath of P both contain v, it follows that P is not a path, contrary to the assumption.

Corollary 2.12 *Let G be a connected graph with cut vertices and let S be a connected geodetic vertex cover of G. Then every branch of G contains an element of S.*

Theorem 2.13. *Every cut vertex of a connected graph G belongs to every connected geodetic vertex cover of G.*

Proof. Let G be a connected graph and let S be a connected geodetic vertex cover of G. Let v be any cut vertex of G and let $G_1, G_2, ..., G_r$ $(r \geq 2)$ be the components of $G - v$. By Theorem 2.11, S contains at least one vertex from each G_i $(1 \leq i \leq r)$. Since $G[S]$ is connected, it follows that $v \in S$.

Corollary 2.14. *For any connected graph G with k extreme vertices and l cut vertices, $g_{\alpha c}(G) \geq max\{2, k + l\}$.*

Proof. This follows from Theorems 2.4 and 2.13.

Corollary 2.15. *For any non-trivial tree T of order n, $g_{\alpha c}(T) = n$.*

Proof. This follows from Corollary 2.14.

Theorem 2.16. *For the cycle C_n $(n \geq 4)$, $g_{\alpha c}(C_n) = n - 1$.*

Proof. Let $C_n : v_1, v_2, ..., v_n, v_1$ be the cycle of order n. It is clear that $S = \{v_1, v_2, v_3, ..., v_{n-1}\}$ is a minimum connected geodetic vertex cover of G and so $g_{\alpha c}(C_n) = n - 1$.

Theorem 2.17. *For the complete bipartite graph $G = K_{s,t} (2 \leq s \leq t)$, $g_{\alpha c}(G) = s + 1$.*

Proof. Let $U = \{u_1, u_2, ..., u_s\}$ and $W = \{w_1, w_2, ..., w_t\}$ be the partite sets of G.

Let $S = U \cup \{w_1\}$. We claim that S is a minimum connected geodetic vertex cover of G. Clearly every vertex w_j $(2 \leq j \leq t)$ of W lies on the geodesic u_i, w_j, u_k for any $i \neq k$ and S covers all the edges of G. Hence S is a geodetic vertex cover of G. Since $G[S]$ is connected, S is a connected geodetic vertex cover of G. Let T be any set of vertices such that $|T| < |S|$. If $T \subsetneq U$, then $G[T]$ is not connected and so T is not a connected geodetic vertex cover of G. If $T \subsetneq W$, again T is not a connected geodetic vertex cover of G by a similar argument. If $T \supseteq U$, then since $|T| < |S|$, we have $T = U$, which is not a connected geodetic vertex cover of G. If $T \supseteq W$, then since $|T| < |S|$, we have $T = W$, which is not a connected geodetic vertex cover of G. Thus $T \subsetneq U \cup W$ such that T contains at least one vertex from each of U and W. Then since $|T| < |S|$, there exist vertices $u_i \in U$ and $w_j \in W$ such that $u_i \notin T$ and $w_j \notin T$. Then clearly the edge $u_i w_j$ cannot be covered by any of the vertices of T so that T is not a connected geodetic vertex cover of G. Thus in any case T is not a connected geodetic vertex cover of G. Hence S is a minimum connected geodetic vertex cover of G so that $g_{\alpha c}(G) = |S| = s + 1$.

Theorem 2.18. *For the wheel $W_n = K_1 + C_{n-1} (n \geq 5)$, $g_{\alpha c}(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.*

Proof. Let $C_n : v_1, v_2, ..., v_{n-1}, v_1$ be the cycle of W_n and let x be the vertex of K_1 in W_n. Then $S = \{x, v_1, v_3, ..., v_{2\lceil \frac{n-1}{2} \rceil - 1}\}$ is a minimum connected geodetic vertex cover of G and so $g_{\alpha c}(W_n) = \lceil \frac{n-1}{2} \rceil + 1$.

Now we proceed to characterize graphs for which $g_{\alpha c}(G) = 2$, $g_{\alpha c}(G) = 3$, $g_{\alpha c}(G) = \frac{n}{2}$ and $g_{\alpha c}(G) = n$.

Theorem 2.19. *For any connected graph G, $g_{\alpha c}(G) = 2$ if and only if $G = K_2$.*

Proof. If $G = K_2$, then $g_{\alpha c}(G) = 2$. Conversely, let $g_{\alpha c}(G) = 2$. Let $S = \{u, v\}$ be a minimum connected geodetic vertex cover of G. Then uv is an edge. If $G \neq K_2$, then there exists an edge xy different from uv. Then at least one of the vertices x and y cannot lie on the $u - v$ geodesic so that S is not a $g_{\alpha c}$ - set, which is a contradiction.Thus $G = K_2$.

Theorem 2.20. *Let G be a connected graph of order $n \geq 3$. Then $g_{\alpha c}(G) = 3$ if and only if there exists a connected geodetic set S of G on 3 vertices such that $V(G) - S$ is either empty or an independent set.*

Proof. Let $g_{\alpha c}(G) = 3$. Let $S = \{x, y, z\}$ be a minimum connected geodetic vertex cover of G. Then S is a connected geodetic set on 3 vertices. If $n = 3$, then $V(G) - S$ is empty. Let $n \geq 4$. Now claim that $V(G) - S$ is an independent set. If not, there exist two vertices $u, v \in V(G) - S$ such that $uv \in E(G)$. Then uv is not covered by any of the vertices in S, which is a contradiction.

Conversely, assume that there exists a connected geodetic set S on 3 vertices such that $V(G) - S$ is either empty or an independent set. Let $S = \{x, y, z\}$.

Suppose $V(G) - S$ is empty. Then $S = V(G)$ and so S is a connected geodetic vertex cover of G. We claim that S is a minimum connected geodetic vertex cover of G. Suppose that S' is a connected geodetic vertex cover of G of order 2. Then by Theorem 2.19, $g_{\alpha c}(G) = 2$ if and only if $G = K_2$. But G is a connected graph of order $n \geq 3$. Hence no two vertex subset of S is a connected geodetic vertex cover of G. Thus $g_{\alpha c}(G) = 3$. If not, let $V(G) - S$ be independent. Then every edge of G has at least one end in S and hence S is a vertex cover of G. Hence S is a connected geodetic vertex cover of G. Since $n \geq 3$ and by Theorem 2.19, S is a minimum connected geodetic vertex cover of G. Thus $g_{\alpha c}(G) = 3$.

Theorem 2.21. *Let G be a connected non-complete hamiltonian graph of even order n. Then $g_{\alpha c}(G) = \frac{n}{2}$ if and only if there exists a connected geodetic set S on $\frac{n}{2}$ vertices such that $V(G) - S$ is an independent set.*

Proof. Let $g_{\alpha c}(G) = \frac{n}{2}$. Let S be a minimum connected geodetic vertex cover of G. Then S is a connected geodetic set on $\frac{n}{2}$ vertices. Suppose $V(G) - S$ is not an independent set. Then there exist two vertices $u, v \in V(G) - S$ such that $uv \in E(G)$. Then uv is not covered by any of the vertices in S, which is a contradiction.

Conversely, assume that there exists a connected geodetic set S on $\frac{n}{2}$ vertices such that $V(G) - S$ is an independent set. Since $V(G) - S$ is an independent set of vertices, every edge of G has at least one end in S and hence S is a vertex cover of G. Hence S is a connected geodetic vertex cover of G. Let C be a hamiltonian cycle of G. To cover the edges of C, at least $\frac{n}{2}$ vertices are needed so that G cannot have a vertex cover of order less than $\frac{n}{2}$. Hence S is a minimum connected geodetic vertex cover of G. Thus $g_{\alpha c}(G) = \frac{n}{2}$.

Theorem 2.22. *Let G be a connected graph. Then every vertex of G is either an extreme vertex or a cut vertex if and only if $g_{\alpha c}(G) = n$.*

Proof. Let G be a connected graph with every vertex is either an extreme vertex or a cut vertex. Then by Theorems 2.4 and 2.13, $g_{\alpha c}(G) = n$.

Conversely, assume that $g_{\alpha c}(G) = n$. Suppose that there is a vertex x in G which is neither a cut vertex nor an extreme vertex. Since x is not an extreme vertex, $< N(x) >$ is not a complete subgraph and hence there exist u and v in $N(x)$ such that $d(u, v) = 2$. Clearly x lies on a $u - v$ geodesic in G. Also, since x is not a cut vertex of G, $G - x$ is connected and hence $V - \{x\}$ is a vertex cover of G. Thus $V - \{x\}$ is a connected geodetic vertex cover of G and so $g_{\alpha c}(G) \leq |V - \{x\}| = n - 1$, which is a contradiction.

Corollary 2.23. *Let G be a connected block graph of order $n \geq 2$. Then $g_{\alpha c}(G) = n$.*

Proof. Since every block of G is complete, every vertex of G is either an extreme vertex or a cut vertex. Hence $g_{\alpha c}(G) = n$.

Corollary 2.24. *If $G = K_1 + \cup m_j K_j$, then $g_{\alpha c}(G) = n$ where m_j denotes the number of copies of K_j.*

Theorem 2.25. *Let G be a triangle free graph of order n with $\delta(G) \geq 2$. Then $g_{\alpha c}(G) \leq n - 1$.*

Proof. Let G be a triangle free graph of order n with $\delta(G) \geq 2$. Let v be a vertex in G.

Case (i) v is not a cut vertex. Since G is triangle free and $\delta(G) \geq 2$, v has at least two non adjacent neighbors x and y in G. Then v lies on the geodesic x,v,y and the edges vx, vy are incident with the vertices x and y. Clearly the subgraph induced by $V(G) - \{v\}$ is connected. Hence $S = V(G) - \{v\}$ is a connected geodetic vertex cover of G. Hence $g_{\alpha c}(G) \leq n - 1$.

Case (ii) v is a cut vertex. Let $G_1, G_2, ..., G_r$ $(r \geq 2)$ be the components of $G - v$. Since G is triangle free with $\delta(G) \geq 2$, every component of $G - v$ must contain at least a path on 3 vertices. Let G_1 contain a path $P : x, u, y$. Then clearly the vertex u lies on the geodesic x, u, y and the edges ux, uy are covered by the vertices x and y. Hence $S = V(G) - \{u\}$ is a connected geodetic vertex cover of G and so $g_{\alpha c}(G) \leq n - 1$.

Theorem 2.26. *Let G be a connected non-complete graph. If G has a minimum cut set consisting of i independent vertices, then $g_{\alpha c}(G) \leq n - i + 1$.*

Proof. Let $U = \{v_1, v_2, ..., v_i\}$ be a minimum independent cut set of vertices of G. Since G is non-complete, it is clear that $1 \leq i \leq n-2$. Let $G_1, G_2, ..., G_r$ $(r \geq 2)$ be the components of $G - U$ and let $S = V(G) - U$. Then every vertex v_j $(1 \leq j \leq i)$ is adjacent to at least one vertex of G_t for every t $(1 \leq t \leq r)$. It is clear that S is a geodetic set of G and $G[S]$ is not connected. Also, it is clear that $G[S \cup \{x\}]$ is a connected geodetic set for any vertex x in U. Since U is an independent set of vertices, every edge of G has at least one end in S so that S is a vertex cover of G. Hence $S \cup \{x\}$ is a connected geodetic vertex cover of G so that $g_{\alpha c}(G) \leq n - i + 1$.

Theorem 2.27. *If G is a connected graph such that $g_\alpha(G) = 2$, then $g_{\alpha c}(G) = 1 + diam\ G$.*

Proof. Let $g_\alpha(G) = 2$. Then by Theorem 1.6, $G = K_2$ or $G = K_{2,n-2} (n \geq 3)$. If $G = K_2$, clearly $diam\ G = 1$ and $g_{\alpha c}(G) = 2$ by Theorem 2.19. Thus $g_{\alpha c}(G) = 1 + diam\ G$. If $G = K_{2,n-2}$, then clearly $diam\ G = 2$ and $g_{\alpha c}(G) = 3$ by Theorem 2.17. Hence $g_{\alpha c}(G) = 3 = 1 + diam\ G$.

3. Connected geodetic number and connected geodetic vertex covering number of a graph

Theorem 3.1. *For any connected graph G, $2 \le g_c(G) \le g_{\alpha c}(G) \le n$.*

Proof. Any connected geodetic set needs at least two vertices and so $g_c(G) \ge 2$. Since every connected geodetic vertex cover of G is a connected geodetic set of G, $g_c(G) \le g_{\alpha c}(G)$. Also, since $V(G)$ induces a connected geodetic vertex cover of G, it is clear that $g_{\alpha c}(G) \le n$. Hence $2 \le g_c(G) \le g_{\alpha c}(G) \le n$.

Remark 3.2. *The bounds in Theorem 3.1 are sharp. For the graph $G = K_2$, $g_{\alpha c}(G) = 2$. For any non-trivial tree T, $g_c(T) = g_{\alpha c}(T)$, by Theorem 1.2 and Corollary 2.15. Also all the inequalities in Theorem 3.1 are strict. For the graph G given in Figure 3.1, $S = \{v_3, v_5, v_6\}$ is a minimum connected geodetic set of G so that $g_c(G) = 3$ and $M = \{v_2, v_3, v_5, v_6\}$ is a minimum connected geodetic vertex cover of G so that $g_{\alpha c}(G) = 4$ and so $2 < g_c(G) < g_{\alpha c}(G) < n$.*

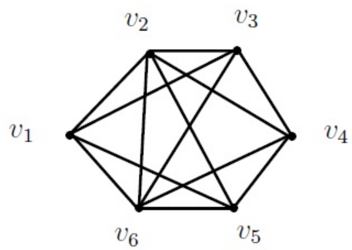

Figure 3.1: G

Corollary 3.3. *Let G be any connected graph. If $g_{\alpha c}(G) = 2$, then $g_c(G) = 2$.*

Corollary 3.4. *Let G be any connected graph. If $g_c(G) = n$, then $g_{\alpha c}(G) = n$.*

Theorem 3.5. *For a connected graph G, $g_{\alpha c}(G) \ge 1 + diam\ G$.*

Proof. This follows from Theorem 1.3 and Theorem 3.1.

Theorem 3.6. *Let G be a connected graph of order n. Then $g_c(G) = g_{\alpha c}(G)$ if and only if every vertex of G is either a cut vertex or an extreme vertex, or there exists a minimum connected geodetic set S of G such that $V(G) - S$ is independent.*

Proof. Let $g_c(G) = g_{\alpha c}(G)$. Suppose that $g_c(G) = g_{\alpha c}(G) = n$. Then by Theorem 2.22, every vertex of G is either a cut vertex or an extreme vertex. Otherwise, $g_c(G) = g_{\alpha c}(G) < n$. Let S be a minimum connected geodetic set of G. Since $g_c(G) = g_{\alpha c}(G)$, S is also a vertex cover of G. Hence no edge of G has its two ends in $V(G) - S$. Thus $V(G) - S$ is independent.

Conversely, let G be a connected graph with every vertex of G is either a cut vertex or an extreme vertex, or there exists a minimum connected geodetic set S of G such that $V(G) - S$ is independent. If every vertex of G is either a cut vertex or an extreme vertex, then by Theorem 1.4 and Theorem 2.22, $g_c(G) = g_{\alpha c}(G) = n$. If S is a minimum connected geodetic set of G such that $V(G) - S$ is independent, then every edge of G has at least one end in S. Thus S is a minimum connected geodetic vertex cover of G. Hence $g_{\alpha c}(G) = |S| = g_c(G)$.

Theorem 3.7. *Let G be a connected graph with $g_c(G) \geq n - 1$. Then $g_{\alpha c}(G) = g_c(G)$.*

Proof. Let G be a connected graph with $g_c(G) \geq n - 1$. If $g_c(G) = n$, then $g_{\alpha c}(G) = n$ and so $g_{\alpha c}(G) = g_c(G)$. If $g_c(G) = n-1$, let S be a minimum connected geodetic set on $n - 1$ vertices. Then $V(G) - S$ is a singleton set and so $V(G) - S$ is independent. Then by Theorem 3.6, $g_c(G) = n - 1 = g_{\alpha c}(G)$.

Remark 3.8. *The converse of Theorem 3.7 is not true. For the complete bipartite graph $G = K_{2,t} (t \geq 3)$, by Theorem 2.17, $g_c(G) = g_{\alpha c}(G) = 3 < n = t + 2$.*

We proved in Theorem 2.5 that $2 \leq g_{\alpha c}(G) \leq n$. The following theorem gives a realization result for these parameters.

Theorem 3.9. *For any two positive integers a and n with $3 \leq a \leq n$, there exists a connected graph G with $g_{\alpha c}(G) = a$ and $|V(G)| = n$.*

Proof. We prove this theorem by considering two cases.

Case(i) $3 \leq a = n$. Take $G = K_n$, the complete graph on n vertices. Then by Corollary 2.10, $g_{\alpha c}(G) = n = a$.

Case (ii) $3 \leq a < n$. Take $H = K_{a-2}$, the complete graph on $a - 2$ vertices $u_1, u_2, ..., u_{a-2}$. Add $n - a + 2$ new vertices $v_1, v_2, ..., v_{n-a+1}, x$ to H and join v_i $(1 \leq i \leq n - a + 1)$ to both u_{a-2} and x, there by producing the graph G. The graph G is shown in Figure 3.2 and its order is n.

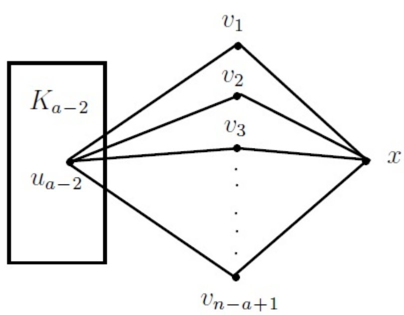

Figure 3.2 : G

Let $S = \{u_1, u_2, ..., u_{a-3}\}$ be the set of all extreme vertices of G. Then by Theorem 1.1, S is a subset of every geodetic set of G. Observe that $S' = S \cup \{x\}$ is a minimum geodetic set of G and the edges of K_{a-2} and the edge $v_i x$ $(1 \leq i \leq n-a+1)$ are covered by the vertices of S'. Now, to cover the edges $u_{a-2}v_i$ $(1 \leq i \leq n-a+1)$, we must include the vertex u_{a-2} to S'. Let $S'' = S' \cup \{u_{a-2}\}$. Clearly S'' is a minimum geodetic vertex cover of G. But $G[S'']$ is not connected. To get a minimum connected geodetic vertex cover of G, we must include one of the vertices from $\{v_i\}$, where $1 \leq i \leq n-a+1$, to S''. Hence a minimum connected geodetic vertex cover of G is $S''' = \{u_1, u_2, ..., u_{a-3}, u_{a-2}, v_1, x\}$ with $|S'''| = a < n$.

Theorem 3.10. *For any two positive integers a and b with $2 \leq a \leq b$, there exists a connected graph G with $g(G) = a$ and $g_{\alpha c}(G) = b$.*

Proof. We prove this theorem by considering two cases.

Case(i) $2 \leq a = b$. Take $G = K_n$, the complete graph on n vertices. Then by Theorem 1.1 and Corollary 2.10, $g(G) = g_{\alpha c}(G) = a$.

Case(ii) $2 \leq a < b$. Let $H = K_a$ be the complete graph on a vertices $u_1, u_2, ..., u_a$. Let $P : v_1, v_2, ..., v_{b-a}$ be a path of order $b - a$. Let G be the graph obtained from H and P by joining the vertices u_a in K_a and v_1 in P as in Figure 3.3.

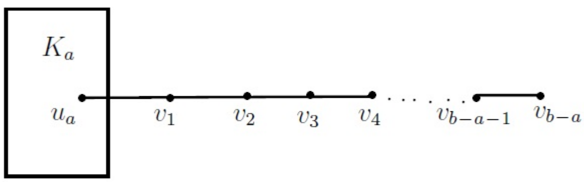

Figure 3.3 : G

Let $S = \{u_1, u_2, ..., u_{a-1}, v_{b-a}\}$ be the set of all simplicial vertices of G so that they must belong to every geodetic set of G. It is clear that S is a minimum geodetic set of G so that $g(G) = a$. Also it is clear that $S' = S \cup \{u_a, v_1, v_2, ..., v_{b-a-1}\}$ is a minimum connected geodetic vertex cover of G so that $g_{\alpha c}(G) = b$.

4. Conclusion

In this paper, we defined a new graph theoretic parameter "connected geodetic vertex covering number" of a graph and established some general properties satisfied by this parameter. The connected geodetic vertex covering number of some standard graphs were determined. Also, we established the relation between the connected geodetic number and the connected geodetic vertex covering number of a graph. We hope that the results presented in this paper will be useful in the study of upper connected geodetic vertex covering number of a graph, forcing connected geodetic vertex covering number of a graph and so on.

5. Acknowledgement

The authors are thankful to the unknown reviewers for their valuable suggestions.

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