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# ECCENTRIC CONNECTIVITY POLYNOMIALS AND THEIR TOPOLOGICAL INDICES OF JAHANGIR GRAPHS

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**Abstract:** Let $G = (V, E)$ be a simple and connected graph. The degree of a vertex $u$ and its eccentricity of a graph $G$ is denoted as $d(u)$ and $e(u)$ respectively. The eccentric connectivity polynomial $\xi^c(G, x)$ of a graph $G$ is defined as $\xi^c(G, x) = \sum_{u\in V(G)} d(u)x^{e(u)}$ and the modified eccentric connectivity polynomial $\xi_c(G, x)$ of a graph $G$ is defined as $\xi_c(G, x) = \sum_{u\in V(G)} M(u)x^{e(u)}$, where $M(u) = \sum_{v\in N_G(u)} d(v)$ i.e., sum of the neighbouring vertices of $u \in V(G)$. The first derivative of these polynomials evaluated at $x = 1$ generates eccentric connectivity index $\xi^c(G)$ defined as $\xi^c(G) = \sum_{u\in V(G)} d(u)e(u)$ and modified eccentric connectivity index $\xi_c(G)$ defined as $\xi_c(G) = \sum_{u\in V(G)} M(u)e(u)$ respectively. In this paper, we present the generalized results for eccentric connectivity polynomial, modified eccentric connectivity polynomial and their respective indices for Jahangir graph $J_{n,m}$ with $n \geq 2$ and $m \geq 3$.

**Keywords and Phrases:** Eccentric connectivity indices, eccentric connectivity polynomials, Jahangir graph.

**2020 Mathematics Subject Classification:** 05C07, 05C12, 05C31.

## 1. Introduction

Let $G = (V, E)$ be a simple and connected graph with $V(G)$ as the vertex set and $E(G)$ as the edge set. The degree of a vertex $u$ in a graph $G$ is denoted as $d(u)$ and is defined as the number of edges of a graph $G$ incident with vertex $u$ [5]. The distance $d(u, v)$ between two vertices $u$ and $v$ is the minimum of the lengths of the $u-v$ paths of $G$ [5]. The eccentricity $e(u)$ of a vertex $u$ of a connected graph $G$ is the distance between vertex $u$ and a vertex farthest from $u$ [5]. A topological index is a molecular descriptor which characterizes the topology of a graph through numerical parameters. Amongst the numerous topological indices defined, one interesting distance based topological index is the eccentric connectivity index $\xi^c(G)$ of a graph $G$ defined by Sharma et al. [10] as

$$\xi^c(G) = \sum_{u \in V(G)} d(u)e(u) \tag{1}$$

where $d(u)$ denotes the degree of vertex $u$ and $e(u)$ denotes its eccentricity in a graph $G$.

The eccentric connectivity polynomial $\xi^c(G, x)$ of a graph $G$ was defined by Ghorbani et al. [2] as

$$\xi^c(G, x) = \sum_{u \in V(G)} d(u)x^{e(u)} \tag{2}$$

It is observable that eccentric connectivity index $\xi^c(G)$ is the first derivative of eccentric connectivity polynomial $\xi^c(G, x)$ evaluated at $x = 1$.
For a vertex $u \in V(G)$, the neighbourhood of $u$ in $G$ is denoted as $N_G(u)$ and defined as the set consisting of all vertices that are adjacent to $u$ i.e., $N_G(u) = \{v \in V(G) : uv \in E(G)\}$ and $M(u)$ is the neighbourhood degree sum of a vertex $u \in V(G)$, defined as $M(u) = \sum_{v \in N_G(u)} d(v)$.
The modified eccentric connectivity index $\xi_c(G)$ of a graph $G$ was defined by Ashrafi and Ghorbani [3] as

$$\xi_c(G) = \sum_{u \in V(G)} M(u)e(u) \tag{3}$$

where,

$$M(u) = \sum_{v \in N_G(u)} d(v)$$

i.e., sum of the neighbouring vertices of $u \in V(G)$.
The corresponding modified eccentric connectivity polynomial $\xi_c(G, x)$ is defined as

$$\xi_c(G, x) = \sum_{u \in V(G)} M(u)x^{e(u)} \tag{4}$$

Clearly, the modified eccentric connectivity index $\xi_c(G)$ is the first derivative of modified eccentric connectivity polynomial $\xi_c(G,x)$ evaluated at $x=1$.
For undefined terminologies refer [7].
The Jahangir graph $J_{n,m}$ [1, 9] is a graph on $(nm+1)$ vertices and $m(n+1)$ edges for every $n \geq 2$ and $m \geq 3$ i.e., a graph consisting of a cycle $C_{nm}$ with one additional vertex which is adjacent to $m$ vertices of $C_{nm}$ at a distance $n$ to each other on $C_{nm}$. It is denoted by $J_{n,m}$ for every $n \geq 2$ and $m \geq 3$.

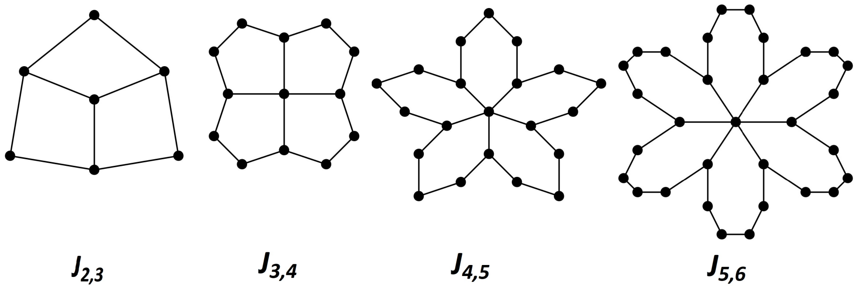


Figure 1: Some examples of Jahangir graphs.

Motivated by [4, 6, 8], we compute eccentric connectivity polynomials, modified eccentric connectivity polynomials and their respective topological indices for Jahangir graph.

## 2. Main Results

**Results for eccentric connectivity polynomial $\xi^c(J_{n,m},x)$ and eccentric connectivity index $\xi^c(J_{n,m})$.**

**Theorem 2.1.** *For the Jahangir graph $J_{2,3}$,*

$$\xi^c(J_{2,3},x) = 15x^3 + 3x^2 \text{ and } \xi^c(J_{2,3}) = 51.$$

**Proof.** Let $J_{2,3}$ be a Jahangir graph. To compute $\xi^c(J_{2,3},x)$ and $\xi^c(J_{2,3})$ entries from Tab. 1 are considered.

| Case | $d(u)$ | $e(u)$ | Number of vertices | Terms of $\xi^c(J_{2,3},x)$ |
|---|---|---|---|---|
| a | 3 | 2 | 1 | $3x^2$ |
| b | 3 | 3 | 3 | $3(3x^3)$ |
| c | 2 | 3 | 3 | $3(2x^3)$ |

Tab. 1: Degrees and eccentricities of vertices in $J_{2,3}$.

**Details of Tab. 1 are discussed as follows:**

**Case a.** The graph $J_{2,3}$ has a central vertex of degree 3 and eccentricity 2. Hence, the first term of $\xi^c(J_{2,3}, x)$ is $3x^2$.

**Case b.** The vertices adjacent to central vertex are 3 in number having degree 3 each and eccentricity 3. Hence, the second term of $\xi^c(J_{2,3}, x)$ is $3(3x^3)$.

**Case c.** The vertices at distance 2 from the central vertex are 3 in number having degree 2 each and eccentricity 3. Hence, the third term of $\xi^c(J_{2,3}, x)$ is $3(2x^3)$.

From equation 2 and Tab. 1, it follows

$$\begin{aligned} \xi^c(J_{2,3}, x) &= \sum_{u \in V(J_{2,3})} d(u) x^{e(u)} \\ &= 15x^3 + 3x^2. \end{aligned}$$

Since, $\xi^c(J_{2,3})$ is the first derivative of $\xi^c(J_{2,3}, x)$ at $x = 1$, it follows

$$\begin{aligned} \xi^c(J_{2,3}) &= \frac{\partial}{\partial x}(\xi^c(J_{2,3}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(15x^3 + 3x^2) \mid_{x=1} \\ &= 51. \end{aligned}$$

**Theorem 2.2.** *If $J_{n,3}$ is a Jahangir graph with $n = 2k$, $k = 2, 3, ...,$ then*

$$\xi^c(J_{n,3}, x) = 3(x^{\frac{n+2}{2}} + 3x^{\frac{n+4}{2}} + 4x^{\frac{3(n^2-2n-8)}{8}} + 6x^{n+1}) \text{ and }$$
$$\xi^c(J_{n,3}) = \tfrac{3}{2}(3n^2 + 10n + 2).$$

**Proof.** Let $J_{n,3}$ be the Jahangir graph with $n = 2k$, $k = 2, 3, ....$ To compute $\xi^c(J_{n,3}, x)$ and $\xi^c(J_{n,3})$ entries from Tab. 2 are considered.

| Case | $d(u)$ | $e(u)$ | Number of vertices | Terms of $\xi^c(J_{n,3}, x)$ |
|---|---|---|---|---|
| a | 3 | $\frac{n+2}{2}$ | 1 | $3x^{\frac{n+2}{2}}$ |
| b | 3 | $\frac{n+4}{2}$ | 3 | $3(3x^{\frac{n+4}{2}})$ |
| c | 2 | $\frac{(n-4)(3n+6)}{8}$ | $3(n-4)$ | $6(2x^{\frac{(n-4)(3n+6)}{8}})$ |
| d | 2 | $n+1$ | 9 | $9(2x^{n+1})$ |

Tab. 2: Degrees and eccentricities of vertices in $J_{n,3}$ with $n = 2k$, $k = 2, 3, ....$

**Details of Tab. 2 are discussed as follows:**
**Case a.** The graph $J_{n,3}$ has a central vertex of degree 3 and eccentricity $\frac{n+2}{2}$. Hence, the first term of $\xi^c(J_{n,3}, x)$ is $3x^{\frac{n+2}{2}}$.
**Case b.** The vertices adjacent to the central vertex are 3 in number having degree 3 each and eccentricity $\frac{n+4}{2}$. Hence, the second term of $\xi^c(J_{n,3}, x)$ is $3(3x^{\frac{n+4}{2}})$.
**Case c.** There are 6 vertices each at a distance 2, 3, ..., $(\frac{n-2}{2})$ respectively from the central vertex, total of such vertices are $3(n-4)$ in number having degree 2 each and sum of their eccentricities is $\frac{(n-4)(3n+6)}{8}$. Hence, the third term of $\xi^c(J_{n,3}, x)$ is $6(2x^{\frac{(n-4)(3n+6)}{8}})$.
**Note:** In Case c there are total $3(n-4)$ vertices. Amongst these total vertices every 6 vertices have same eccentricity. For example, in $J_{8,3}$ there are total 12 vertices belonging to Case c each vertex with degree 2. Out of these 12 vertices, 6 vertices have eccentricity 7 and remaining 6 vertices have eccentricity 8. This pattern follows for other $J_{n,3}$ with $n = 2k$, $k = 2, 3, \ldots$ graphs. Since 6 is the common factor for $(3(n-4))$ vertices, by equation 2, the third term of $J_{n,3}$ is $6(2x^{\frac{(n-4)(3n+6)}{8}})$.
**Case d.** There are 3 vertices at a distance $\frac{n+2}{2}$ and 6 vertices at a distance $\frac{n}{2}$ from the central vertex, total number of such vertices are 9 having degree 2 each and eccentricity $n+1$. Hence, the fourth term of $\xi^c(J_{n,3}, x)$ is $9(2x^{n+1})$.

From equation 2 and Tab. 2, it follows

$$\begin{aligned}\xi^c(J_{n,3}, x) &= \sum_{u \in V(J_{n,3})} d(u)x^{e(u)} \\ &= 3(x^{\frac{n+2}{2}} + 3x^{\frac{n+4}{2}} + 4x^{\frac{3(n^2-2n-8)}{8}} + 6x^{n+1}).\end{aligned}$$

Since, $\xi^c(J_{n,3})$ is the first derivative of $\xi^c(J_{n,3}, x)$ at $x = 1$, it follows

$$\begin{aligned}\xi^c(J_{n,3}) &= \frac{\partial}{\partial x}(\xi^c(J_{n,3}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(3(x^{\frac{n+2}{2}} + 3x^{\frac{n+4}{2}} + 4x^{\frac{3(n^2-2n-8)}{8}} + 6x^{n+1})) \mid_{x=1} \\ &= \frac{3}{2}(3n^2 + 10n + 2).\end{aligned}$$

**Theorem 2.3.** *If $J_{n,m}$ is a Jahangir graph with $n$ as even and $m \geq 4$, then*

$$\xi^c(J_{n,m}, x) = m(x^{\frac{n+2}{2}} + 3x^{\frac{n+4}{2}} + 4x^{\frac{(3n^2+2n-16)}{8}} + 2x^{n+2}) \text{ and}$$
$$\xi^c(J_{n,m}) = \tfrac{m}{2}(3n^2 + 10n + 6).$$

**Proof.** Let $J_{n,m}$ be the Jahangir graph with $n$ as even and $m \geq 4$. To compute $\xi^c(J_{n,m}, x)$ and $\xi^c(J_{n,m})$ entries from Tab. 3 are considered.

| Case | $d(u)$ | $e(u)$ | Number of vertices | Terms of $\xi^c(J_{n,m}, x)$ |
|---|---|---|---|---|
| a | $m$ | $\frac{n+2}{2}$ | 1 | $mx^{\frac{n+2}{2}}$ |
| b | 3 | $\frac{n+4}{2}$ | $m$ | $m(3x^{\frac{n+4}{2}})$ |
| c | 2 | $\frac{(n-2)(3n+8)}{8}$ | $2m$ | $2m(2x^{\frac{(n-2)(3n+8)}{8}})$ |
| d | 2 | $n+2$ | $m$ | $m(2x^{n+2})$ |

Tab. 3: Degrees and eccentricities of vertices in $J_{n,m}$ with $n$ as even and $m \geq 4$.

**Details of Tab. 3 are discussed as follows:**

**Case a.** The graph $J_{n,m}$ has a central vertex of degree $m$ and eccentricity $\frac{n+2}{2}$. Hence, the first term of $\xi^c(J_{n,m}, x)$ is $mx^{\frac{n+2}{2}}$.

**Case b.** The vertices adjacent to the central vertex are $m$ in number having degree 3 each and eccentricity $\frac{n+4}{2}$. Hence, the second term of $\xi^c(J_{n,m}, x)$ is $m(3x^{\frac{n+4}{2}})$ .

**Case c.** There are $2m$ vertices each at a distance 2, 3, ..., $(\frac{n}{2})$ respectively from the central vertex having degree 2 each and sum of their eccentricities is $\frac{(n-2)(3n+8)}{8}$. Hence, the third term of $\xi^c(J_{n,m}, x)$ is $2m(2x^{\frac{(n-2)(3n+8)}{8}})$.

**Case d.** The vertices at a distance $\frac{n+2}{2}$ from the central vertex are $m$ in number having degree 2 each and eccentricity $n+2$. Hence, the fourth term of $\xi^c(J_{n,m}, x)$ is $m(2x^{n+2})$.

From equation 2 and Tab. 3, it follows

$$\begin{aligned}\xi^c(J_{n,m}, x) &= \sum_{u \in V(J_{n,m})} d(u)x^{e(u)} \\ &= m(x^{\frac{n+2}{2}} + 3x^{\frac{n+4}{2}} + 4x^{\frac{(3n^2+2n-16)}{8}} + 2x^{n+2}).\end{aligned}$$

Since, $\xi^c(J_{n,m})$ is the first derivative of $\xi^c(J_{n,m}, x)$ at $x = 1$, it follows

$$\begin{aligned}\xi^c(J_{n,m}) &= \frac{\partial}{\partial x}(\xi^c(J_{n,m}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(m(x^{\frac{n+2}{2}} + 3x^{\frac{n+4}{2}} + 4x^{\frac{(3n^2+2n-16)}{8}} + 2x^{n+2})) \mid_{x=1} \\ &= \frac{m}{2}(3n^2 + 10n + 6).\end{aligned}$$

**Theorem 2.4.** *If $J_{n,m}$ is a Jahangir graph with $n$ as odd and $m \geq 3$, then*

$$\xi^c(J_{n,m}, x) = m(x^{\frac{n+1}{2}} + 3x^{\frac{n+3}{2}} + 4x^{\frac{(3n^2+4n-7)}{8}}) \text{ and } \xi^c(J_{n,m}) = \frac{m}{2}(3n^2 + 8n + 3).$$

**Proof.** Let $J_{n,m}$ be the Jahangir graph with $n$ as odd and $m \geq 3$. To compute $\xi^c(J_{n,m}, x)$ and $\xi^c(J_{n,m})$ entries from Tab. 4 are considered.

| Case | $d(u)$ | $e(u)$ | Number of vertices | Terms of $\xi^c(J_{n,m}, x)$ |
|---|---|---|---|---|
| a | $m$ | $\frac{n+1}{2}$ | 1 | $mx^{\frac{n+1}{2}}$ |
| b | 3 | $\frac{n+3}{2}$ | $m$ | $m(3x^{\frac{n+3}{2}})$ |
| c | 2 | $\frac{(n-1)(3n+7)}{8}$ | $2m$ | $2m(2x^{\frac{(n-1)(3n+7)}{8}})$ |

Tab. 4: Degrees and eccentricities of vertices in $J_{n,m}$ with $n$ as odd and $m \geq 3$.

**Details of Tab. 4 are discussed as follows:**
**Case a.** The graph $J_{n,m}$ has a central vertex of degree $m$ and eccentricity $\frac{n+1}{2}$. Hence, the first term of $\xi^c(J_{n,m}, x)$ is $mx^{\frac{n+1}{2}}$.
**Case b.** The vertices adjacent to the central vertex are $m$ in number having degree 3 each and eccentricity $\frac{n+3}{2}$. Hence, the second term of $\xi^c(J_{n,m}, x)$ is $m(3x^{\frac{n+3}{2}})$ .
**Case c.** There are $2m$ vertices each at a distance 2, 3, ..., $(\frac{n+1}{2})$ respectively from the central vertex having degree 2 each and sum of their eccentricities is $\frac{(n-1)(3n+7)}{8}$. Hence, the third term of $\xi^c(J_{n,m}, x)$ is $2m(2x^{\frac{(n-1)(3n+7)}{8}})$.

From equation 2 and Tab. 4, it follows

$$\begin{aligned}\xi^c(J_{n,m}, x) &= \sum_{u \in V(J_{n,m})} d(u)x^{e(u)} \\ &= m(x^{\frac{n+1}{2}} + 3x^{\frac{n+3}{2}} + 4x^{\frac{(3n^2+4n-7)}{8}}).\end{aligned}$$

Since, $\xi^c(J_{n,m})$ is the first derivative of $\xi^c(J_{n,m}, x)$ at $x = 1$, it follows

$$\begin{aligned}\xi^c(J_{n,m}) &= \frac{\partial}{\partial x}(\xi^c(J_{n,m}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(m(x^{\frac{n+1}{2}} + 3x^{\frac{n+3}{2}} + 4x^{\frac{(3n^2+4n-7)}{8}})) \mid_{x=1} \\ &= \frac{m}{2}(3n^2 + 8n + 3).\end{aligned}$$

**Results for modified eccentric connectivity polynomial $\xi_c(J_{n,m}, x)$ and modified eccentric connectivity index $\xi_c(J_{n,m})$.**
**Theorem 2.5.** *For the Jahangir graph $J_{2,3}$,*

$$\xi_c(J_{2,3}, x) = 39x^3 + 9x^2 \;\; \textit{and} \;\; \xi_c(J_{2,3}) = 135.$$

**Proof.** Let $J_{2,3}$ be a Jahangir graph. To compute $\xi_c(J_{2,3}, x)$ and $\xi_c(J_{2,3})$ entries from Tab. 5 are considered.

| Case | $M(u)$ | $e(u)$ | Number of vertices | Terms of $\xi_c(J_{2,3}, x)$ |
|---|---|---|---|---|
| a | 9 | 2 | 1 | $9x^2$ |
| b | 7 | 3 | 3 | $3(7x^3)$ |
| c | 6 | 3 | 3 | $3(6x^3)$ |

Tab. 5: $M(u)$ and eccentricities of vertices in $J_{2,3}$.

**Details of Tab. 5 are discussed as follows:**

**Case a.** The graph $J_{2,3}$ has a central vertex with $M(u) = 9$ and eccentricity 2. Hence, the first term of $\xi_c(J_{2,3}, x)$ is $9x^2$.

**Case b.** The vertices adjacent to central vertex are 3 in number with $M(u) = 7$ each and eccentricity 3. Hence, the second term of $\xi_c(J_{2,3}, x)$ is $3(7x^3)$.

**Case c.** The vertices at distance 2 from the central vertex are 3 in number with $M(u) = 6$ each and eccentricity 3. Hence, the third term of $\xi_c(J_{2,3}, x)$ is $3(6x^3)$.

From equation 4 and Tab. 5, it follows

$$\xi_c(J_{2,3}, x) = \sum_{u \in V(J_{2,3})} M(u) x^{e(u)} = 39x^3 + 9x^2.$$

Since, $\xi_c(J_{2,3})$ is the first derivative of $\xi_c(J_{2,3}, x)$ at $x = 1$, it follows

$$\begin{aligned} \xi_c(J_{2,3}) &= \frac{\partial}{\partial x}(\xi_c(J_{2,3}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(39x^3 + 9x^2) \mid_{x=1} = 135. \end{aligned}$$

**Theorem 2.6.** *If $J_{2,m}$ is the Jahangir graph with $m \geq 4$, then*

$$\xi_c(J_{2,m}, x) = m(3x^2 + (4+m)x^3 + 6x^4) \ \ and \ \ \xi_c(J_{2,m}) = m(3m + 42).$$

**Proof.** Let $J_{2,m}$ be the Jahangir graph with $m \geq 4$. To compute $\xi_c(J_{2,m}, x)$ and $\xi_c(J_{2,m})$ entries from Tab. 6 are considered.

| Case | $M(u)$ | $e(u)$ | Number of vertices | Terms of $\xi_c(J_{2,m}, x)$ |
|---|---|---|---|---|
| a | $3m$ | 2 | 1 | $3mx^2$ |
| b | $4+m$ | 3 | $m$ | $m((4+m)x^3)$ |
| c | 6 | 4 | $m$ | $m(6x^4)$ |

Tab. 6: $M(u)$ and eccentricities of vertices in $J_{2,m}$ with $m \geq 4$ .

**Details of Tab. 6 are discussed as follows:**
**Case a.** The graph $J_{2,m}$ has a central vertex with $M(u) = 3m$ and eccentricity 2. Hence, the first term of $\xi_c(J_{2,m}, x)$ is $3mx^2$.
**Case b.** The vertices adjacent to central vertex are $m$ in number with $M(u) = (4+m)$ each and eccentricity 3. Hence, the second term of $\xi_c(J_{2,m}, x)$ is $m((4+m)x^3)$.
**Case c.** The vertices at distance 2 from the central vertex are $m$ in number with $M(u) = 6$ each and eccentricity 4. Hence, the third term of $\xi_c(J_{2,m}, x)$ is $m(6x^4)$.

From equation 4 and Tab. 6, it follows

$$\begin{aligned}\xi_c(J_{2,m}, x) &= \sum_{u \in V(J_{2,m})} M(u)x^{e(u)} \\ &= m(3x^2 + (4+m)x^3 + 6x^4).\end{aligned}$$

Since, $\xi_c(J_{2,m})$ is the first derivative of $\xi_c(J_{2,m}, x)$ at $x = 1$, it follows

$$\begin{aligned}\xi_c(J_{2,m}) &= \frac{\partial}{\partial x}(\xi_c(J_{2,m}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(m(3x^2 + (4+m)x^3 + 6x^4)) \mid_{x=1} \\ &= m(3m + 42).\end{aligned}$$

**Theorem 2.7.** *For the Jahangir graph $J_{4,3}$,*

$$\xi_c(J_{4,3}, x) = 42x^5 + 21x^4 + 9x^3 \;\; \textit{and} \;\; \xi_c(J_{4,3}) = 321.$$

**Proof.** Let $J_{4,3}$ be a Jahangir graph. To compute $\xi_c(J_{4,3}, x)$ and $\xi_c(J_{4,3})$ entries from Tab. 7 are considered.

| Case | $M(u)$ | $e(u)$ | Number of vertices | Terms of $\xi_c(J_{4,3}, x)$ |
|---|---|---|---|---|
| a | 9 | 3 | 1 | $9x^3$ |
| b | 7 | 4 | 3 | $3(7x^4)$ |
| c | 5 | 5 | 6 | $6(5x^5)$ |
| d | 4 | 5 | 3 | $3(4x^5)$ |

Tab. 7: $M(u)$ and eccentricities of vertices in $J_{4,3}$.

**Details of Tab. 7 are discussed as follows:**
**Case a.** The graph $J_{4,3}$ has a central vertex with $M(u) = 9$ and eccentricity 3. Hence, the first term of $\xi_c(J_{4,3}, x)$ is $9x^3$.
**Case b.** The vertices adjacent to central vertex are 3 in number with $M(u) = 7$ each and eccentricity 4. Hence, the second term of $\xi_c(J_{4,3}, x)$ is $3(7x^4)$.

**Case c.** The vertices at distance 2 from the central vertex are 6 in number with $M(u) = 5$ each and eccentricity 5. Hence, the third term of $\xi_c(J_{4,3}, x)$ is $6(5x^5)$.
**Case d.** The vertices at distance 3 from the central vertex are 3 in number with $M(u) = 4$ each and eccentricity 5. Hence, the fourth term of $\xi_c(J_{4,3}, x)$ is $3(4x^5)$.

From equation 4 and Tab. 7, it follows

$$\begin{aligned}\xi_c(J_{4,3}, x) &= \sum_{u \in V(J_{4,3})} M(u)x^{e(u)} \\ &= 42x^5 + 21x^4 + 9x^3.\end{aligned}$$

Since, $\xi_c(J_{4,3})$ is the first derivative of $\xi_c(J_{4,3}, x)$ at $x = 1$, it follows

$$\begin{aligned}\xi_c(J_{4,3}) &= \frac{\partial}{\partial x}(\xi_c(J_{4,3}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(42x^5 + 21x^4 + 9x^3) \mid_{x=1} \\ &= 321.\end{aligned}$$

**Theorem 2.8.** *If $J_{n,3}$ is the Jahangir graph with $n = 2k$, $k = 3, 4, \ldots$, then*

$$\begin{aligned}\xi_c(J_{n,3}, x) &= 3(3x^{\frac{n+2}{2}} + 7x^{\frac{n+4}{2}} + 10x^{\frac{n+6}{2}} + 8x^{\frac{(3n^2-10n-48)}{8}} + 12x^{n+1}) \ \textit{and} \\ \xi_c(J_{n,3}) &= 9n^2 + 36n + 33.\end{aligned}$$

**Proof.** Let $J_{n,3}$ be the Jahangir graph with $n = 2k$, $k = 3, 4, \ldots$. To compute $\xi_c(J_{n,3}, x)$ and $\xi_c(J_{n,3})$ entries from Tab. 8 are considered.

| Case | $M(u)$ | $e(u)$ | Number of vertices | Terms of $\xi_c(J_{n,3}, x)$ |
|---|---|---|---|---|
| a | 9 | $\frac{n+2}{2}$ | 1 | $9x^{\frac{n+2}{2}}$ |
| b | 7 | $\frac{n+4}{2}$ | 3 | $3(7x^{\frac{n+4}{2}})$ |
| c | 5 | $\frac{n+6}{2}$ | 6 | $6(5x^{\frac{n+6}{2}})$ |
| d | 4 | $\frac{(n-6)(3n+8)}{8}$ | $3(n-4)$ | $6(4x^{\frac{(n-6)(3n+8)}{8}})$ |
| e | 4 | $n+1$ | 9 | $9(4x^{n+1})$ |

Tab. 8: $M(u)$ and eccentricities of vertices in $J_{n,3}$ with $n = 2k$, $k = 3, 4, \ldots$.

**Details of Tab. 8 are discussed as follows:**
**Case a.** The graph $J_{n,3}$ has a central vertex with $M(u) = 9$ and eccentricity $\frac{n+2}{2}$. Hence, the first term of $\xi_c(J_{n,3}, x)$ is $9x^{\frac{n+2}{2}}$.

**Case b.** The vertices adjacent to central vertex are 3 in number with $M(u) = 7$ each and eccentricity $\frac{n+4}{2}$. Hence, the second term of $\xi_c(J_{n,3}, x)$ is $3(7x^{\frac{n+4}{2}})$.
**Case c.** The vertices at distance 2 from the central vertex are 6 in number with $M(u) = 5$ each and eccentricity $\frac{n+6}{2}$. Hence, the third term of $\xi_c(J_{n,3}, x)$ is $6(5x^{\frac{n+6}{2}})$.
**Case d.** There are 6 vertices each at a distance $3, 4, ..., (\frac{n-2}{2})$ respectively from the central vertex, total of such vertices are $3(n-4)$ in number having $M(u) = 4$ each and sum of their eccentricities is $\frac{(n-6)(3n+8)}{8}$. Hence, the fourth term of $\xi_c(J_{n,3}, x)$ is $6(4x^{\frac{(n-6)(3n+8)}{8}})$.
**Note:** In Case d there are total $3(n-4)$ vertices. Amongst these total vertices every 6 vertices have same eccentricity. For example, in $J_{10,3}$ there are total 12 vertices belonging to Case d, each vertex with $M(u) = 4$. Out of these 12 vertices, 6 vertices have eccentricity 9 and remaining 6 vertices have eccentricity 10. This pattern follows for other $J_{n,3}$ with $n = 2k$, $k = 3, 4, ...$ graphs. Since 6 is the common factor for $(3(n-4))$ vertices, by equation 4, the fourth term of $J_{n,3}$ is $6(4x^{\frac{(n-6)(3n+8)}{8}})$.
**Case e.** There are 3 vertices at a distance $\frac{n+2}{2}$ and 6 vertices at a distance $\frac{n}{2}$ from the central vertex having $M(u) = 4$ each and eccentricity $n + 1$. Hence, the fifth term of $\xi_c(J_{n,3}, x)$ is $9(4x^{n+1})$.

From equation 4 and Tab. 8, it follows

$$\begin{aligned}
\xi_c(J_{n,3}, x) &= \sum_{u \in V(J_{n,3})} M(u) x^{e(u)} \\
&= 3(3x^{\frac{n+2}{2}} + 7x^{\frac{n+4}{2}} + 10x^{\frac{n+6}{2}} + 8x^{\frac{(3n^2-10n-48)}{8}} + 12x^{n+1}).
\end{aligned}$$

Since, $\xi_c(J_{n,3})$ is the first derivative of $\xi_c(J_{n,3}, x)$ at $x = 1$, it follows

$$\begin{aligned}
\xi_c(J_{n,3}) &= \frac{\partial}{\partial x}(\xi_c(J_{n,3}, x)) \mid_{x=1} \\
&= \frac{\partial}{\partial x}(3(3x^{\frac{n+2}{2}} + 7x^{\frac{n+4}{2}} + 10x^{\frac{n+6}{2}} + 8x^{\frac{(3n^2-10n-48)}{8}} + 12x^{n+1})) \mid_{x=1} \\
&= 9n^2 + 36n + 33.
\end{aligned}$$

**Theorem 2.9.** *If $J_{n,m}$ is the Jahangir graph with $n = 2k$, $k = 2, 3, ...$ and $m \geq 4$,* then

$$\begin{aligned}
\xi_c(J_{n,m}, x) &= m(3x^{\frac{n+2}{2}} + (4+m)x^{\frac{n+4}{2}} + 10x^{\frac{n+6}{2}} + 8x^{\frac{(3n^2-2n-40)}{8}} + 4x^{n+2}) \textit{ and} \\
\xi_c(J_{n,m}) &= \frac{m}{2}(6n^2 + 21n + 4m + mn + 18).
\end{aligned}$$

**Proof.** Let $J_{n,m}$ be the Jahangir graph with $n = 2k$, $k = 2, 3, ...$ and $m \geq 4$. To compute $\xi_c(J_{n,m}, x)$ and $\xi_c(J_{n,m})$ entries from Tab. 9 are considered.

| Case | $M(u)$ | $e(u)$ | Number of vertices | Terms of $\xi_c(J_{n,m}, x)$ |
|---|---|---|---|---|
| a | $3m$ | $\frac{n+2}{2}$ | 1 | $3mx^{\frac{n+2}{2}}$ |
| b | $4+m$ | $\frac{n+4}{2}$ | $m$ | $m((4+m)x^{\frac{n+4}{2}})$ |
| c | 5 | $\frac{n+6}{2}$ | $2m$ | $2m(5x^{\frac{n+6}{2}})$ |
| d | 4 | $\frac{(n-4)(3n+10)}{8}$ | $2m$ | $2m(4x^{\frac{(n-4)(3n+10)}{8}})$ |
| e | 4 | $n+2$ | $m$ | $m(4x^{n+2})$ |

Tab. 9: $M(u)$ and eccentricities of vertices in $J_{n,m}$ with $n = 2k$, $k = 2, 3, ...$ and $m \geq 4$.

**Details of Tab. 9 are discussed as follows:**
**Case a.** The graph $J_{n,m}$ has a central vertex with $M(u) = 3m$ and eccentricity $\frac{n+2}{2}$. Hence, the first term of $\xi_c(J_{n,m}, x)$ is $3mx^{\frac{n+2}{2}}$.
**Case b.** The vertices adjacent to central vertex are $m$ in number with $M(u) = (4+m)$ each and eccentricity $\frac{n+4}{2}$. Hence, the second term of $\xi_c(J_{n,m}, x)$ is $m((4+m)x^{\frac{n+4}{2}})$.
**Case c.** The vertices at distance 2 from the central vertex are $2m$ in number with $M(u) = 5$ each and eccentricity $\frac{n+6}{2}$. Hence, the third term of $\xi_c(J_{n,m}, x)$ is $2m(5x^{\frac{n+6}{2}})$.
**Case d.** There are $2m$ vertices each at a distance $3, 4, ..., (\frac{n}{2})$ respectively from the central vertex with $M(u) = 4$ each and sum of their eccentricities is $\frac{(n-4)(3n+10)}{8}$. Hence, the fourth term of $\xi_c(J_{n,m}, x)$ is $2m(4x^{\frac{(n-4)(3n+10)}{8}})$.
**Case e.** The vertices at a distance $\frac{n+2}{2}$ from the central vertex are $m$ in number with $M(u) = 4$ each and eccentricity $n+2$. Hence, the fifth term of $\xi_c(J_{n,m}, x)$ is $m(4x^{n+2})$.

From equation 4 and Tab. 9, it follows

$$\begin{aligned} \xi_c(J_{n,m}, x) &= \sum_{u \in V(J_{n,m})} M(u) x^{e(u)} \\ &= m(3x^{\frac{n+2}{2}} + (4+m)x^{\frac{n+4}{2}} + 10x^{\frac{n+6}{2}} + 8x^{\frac{(3n^2-2n-40)}{8}} + 4x^{n+2}). \end{aligned}$$

Since, $\xi_c(J_{n,m})$ is the first derivative of $\xi_c(J_{n,m}, x)$ at $x = 1$, it follows

$$\begin{aligned}
\xi_c(J_{n,m}) &= \frac{\partial}{\partial x}(\xi_c(J_{n,m}, x)) \mid_{x=1} \\
&= \frac{\partial}{\partial x}(m(3x^{\frac{n+2}{2}} + (4+m)x^{\frac{n+4}{2}} + 10x^{\frac{n+6}{2}} + 8x^{\frac{(3n^2-2n-40)}{8}} + 4x^{n+2})) \mid_{x=1} \\
&= \frac{m}{2}(6n^2 + 21n + 4m + mn + 18).
\end{aligned}$$

**Theorem 2.10.** *If $J_{n,m}$ is the Jahangir graph with $n$ as odd and $m \geq 3$, then*

$$\begin{aligned}
\xi_c(J_{n,m}, x) &= m(3x^{\frac{n+1}{2}} + (4+m)x^{\frac{n+3}{2}} + 10x^{\frac{n+5}{2}} + 8x^{\frac{(3n^2-27)}{8}}) \text{ and} \\
\xi_c(J_{n,m}) &= \frac{m}{2}(6n^2 + 17n + 3m + mn + 11).
\end{aligned}$$

**Proof.** Let $J_{n,m}$ be the Jahangir graph with $n$ as odd and $m \geq 3$. To compute $\xi_c(J_{n,m}, x)$ and $\xi_c(J_{n,m})$ entries from Tab. 10 are considered.

| Case | $M(u)$ | $e(u)$ | Number of vertices | Terms of $\xi_c(J_{n,m}, x)$ |
|---|---|---|---|---|
| a | $3m$ | $\frac{n+1}{2}$ | 1 | $3mx^{\frac{n+1}{2}}$ |
| b | $4+m$ | $\frac{n+3}{2}$ | $m$ | $m((4+m)x^{\frac{n+3}{2}})$ |
| c | 5 | $\frac{n+5}{2}$ | $2m$ | $2m(5x^{\frac{n+5}{2}})$ |
| d | 4 | $\frac{(n-3)(3n+9)}{8}$ | $2m$ | $2m(4x^{\frac{(n-3)(3n+9)}{8}})$ |

Tab. 10: $M(u)$ and eccentricities of vertices in $J_{n,m}$ with $n$ as odd and $m \geq 3$.

**Details of Tab. 10 are discussed as follows:**

**Case a.** The graph $J_{n,m}$ has a central vertex with $M(u) = 3m$ and eccentricity $\frac{n+1}{2}$. Hence, the first term of $\xi_c(J_{n,m}, x)$ is $3mx^{\frac{n+1}{2}}$.

**Case b.** The vertices adjacent to central vertex are $m$ in number with $M(u) = (4+m)$ each and eccentricity $\frac{n+3}{2}$. Hence, the second term of $\xi_c(J_{n,m}, x)$ is $m((4+m)x^{\frac{n+3}{2}})$.

**Case c.** The vertices at distance 2 from the central vertex are $2m$ in number with $M(u) = 5$ each and eccentricity $\frac{n+5}{2}$. Hence, the third term of $\xi_c(J_{n,m}, x)$ is $2m(5x^{\frac{n+5}{2}})$.

**Case d.** There are $2m$ vertices each at a distance $3, 4, ..., (\frac{n+1}{2})$ respectively from the central vertex with $M(u) = 4$ each and sum of their eccentricities is $\frac{(n-3)(3n+9)}{8}$. Hence, the fourth term of $\xi_c(J_{n,m}, x)$ is $2m(4x^{\frac{(n-3)(3n+9)}{8}})$.

From equation 4 and Tab. 10, it follows

$$\begin{aligned}\xi_c(J_{n,m}, x) &= \sum_{u \in V(J_{n,m})} M(u) x^{e(u)} \\ &= m(3x^{\frac{n+1}{2}} + (4+m)x^{\frac{n+3}{2}} + 10x^{\frac{n+5}{2}} + 8x^{\frac{(3n^2-27)}{8}}).\end{aligned}$$

Since, $\xi_c(J_{n,m})$ is the first derivative of $\xi_c(J_{n,m}, x)$ at $x = 1$, it follows

$$\begin{aligned}\xi_c(J_{n,m}) &= \frac{\partial}{\partial x}(\xi_c(J_{n,m}, x)) \mid_{x=1} \\ &= \frac{\partial}{\partial x}(m(3x^{\frac{n+1}{2}} + (4+m)x^{\frac{n+3}{2}} + 10x^{\frac{n+5}{2}} + 8x^{\frac{(3n^2-27)}{8}})) \mid_{x=1} \\ &= \frac{m}{2}(6n^2 + 17n + 3m + mn + 11).\end{aligned}$$

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