South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 223-232

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

GRAPHIC FUZZY MATROIDS

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(**Received:** *Feb. 03, 2021* **Accepted:** *Mar. 27, 2021* **Published:** *Apr. 30, 2021)*

Abstract: In this paper, the isomorphism between two fuzzy matroids is defined and some of their properties are discussed. Also, the concept of a graphic fuzzy matroid is presented by inducing fuzzy matroid from a fuzzy graph. Some properties of graphic fuzzy matroids are also discussed.

Keywords and Phrases: Fuzzy matroid, fuzzy isomorphism, fuzzy cycle matroid, graphic fuzzy matroid.

2020 Mathematics Subject Classification: 05C72, 05C90, 05C60, 03E99, 05B99.

1. Introduction

The concept of matroids was first proposed by Whitney in 1935 as a generalization of both graphs and linear independence in vector spaces. Matroid theory can be applied in some combinatorial optimization problems as an abstract generalization of a graph and a matrix. Graphic matroids form a fundamental class of matroids, there has been a focus of active research during the last few decades. Matroids were generalized to fuzzy fields by Goetschel and Voxman [7] using the notion of a fuzzy independent set. Their works on the fuzzification of matroids preserve many basic properties of (crisp) matroids. From then on, fuzzy bases, fuzzy circuits, fuzzy rank functions, and fuzzy closure operators are widely studied [[7] - [10]].

Since a graph is one of the motivations and basic examples of crisp matroids, it is interesting to check that: Does there exists any kind of fuzzygraphs which can be

supplied as examples of fuzzy matroids ?
In this paper, we will define the isomorphism of fuzzy matroids and some properties of isomorphic fuzzy matroids will be discussed. Our main aim is to induce fuzzy matroids from fuzzy graphs and to establish their characteristics.

2. Preliminaries

2.1. Matroid Theory

Definition 2.1. [12] *A Matroid M is an ordered pair (E, I) consisting of a finite set E and a collection I of subsets of E satisfying the following three conditions:*

i. $\phi \in I$

ii. If $A \in I$ *and* $A^{'} \subseteq A$*, then* $A^{'} \in I$

iii. If A_1 *and* A_2 *are in* I *and* $|A_1| < |A_2|$*, then there is an element* e *of* $A_2 - A_1$ *such that* $A_1 \cup e \in I$

The members of I are the independent sets of M, the subsets of E which are not members of I are the dependent sets of M, and E is the ground set of M.

Definition 2.2. [12] *Let E_1 and E_2 be two finite sets. Suppose that $M_1 = (E_1, I_1)$ and $M_2 = (E_2, I_2)$ are two matroids. M_1 and M_2 are isomorphic if there exists a mapping $\psi : E_1 \longrightarrow E_2$ such that ψ satisfies the following conditions:*

i. ψ *is a one-to-one correspondence,*

ii. For each $X \subseteq E_1, X \in I_1$ *if and only if* $\psi(X) \in I_2$*,*

denoted by $M_1 \cong M_2$. The mapping ψ s called an isomorphic mapping from M_1 to M_2.

The matroid derived from a graph G is called the *cycle matroid* or *polygon matroid* [12] of G, denoted by $M(G)$. Clearly, a set X of edges is independent in $M(G)$ if and only if X does not contain the edge set of a cycle.
A matroid that is isomorphic to the cycle matroid of a graph is called a *graphic matroid* [12].

2.2. Fuzzy Set Theory

Definition 2.3. [16] *Let X be a set. A fuzzy subset μ on X is a function $\mu : X \longrightarrow [0, 1]$*

We denote the family of fuzzy sets on X by $\mathscr{F}(X)$. If $\mu, \nu \in \mathscr{F}(X)$, then

$$supp\ \mu = \{x \in X \mid \mu(x) > 0\},$$

$$m(\mu) = inf\{\ \mu(x) \mid x \in supp(\mu)\},$$

$$C_r(\mu) = \{x \in X \mid \mu(x) \geq r\}, where\ 0 \leq r \leq 1,$$

$$\mu \vee \nu = max\{\ \mu,\ \nu\},$$

$$\mu \wedge \nu = min\{\ \mu,\ \nu\}.$$

If $\mu, \nu \in \mathscr{F}(X)$, then we write $\mu < \nu$ if

i. $\mu(x) \leq \nu(x)$ for each x in X ,

ii. $\mu(x) < \nu(x)$ for some x in X

2.3. Fuzzy Matroid Theory

Geotschel and Voxman generated the concept of fuzzy matroids as follows.

Definition 2.4. [7] *Suppose that E is a finite set and that $\mathscr{I} \subseteq \mathscr{F}(E)$ is a nonempty family of fuzzy sets satisfying:*

i. (Hereditary property) If $\mu(x) \in \mathscr{I}, \nu \in \mathscr{F}(E)$, and $\nu < \mu$, then $\nu \in \mathscr{I}$

ii. (Exchange property) If $\mu, \nu \in \mathscr{I}$ and $|supp\ \mu| < |supp\ \nu|$, then there exists $\omega \in \mathscr{I}$ such that

a. $\mu < \omega < \mu \vee \nu$

b. $m(\omega) \geq min\{m(\mu),\ m(\nu)\}$.

Then the pair $\mathfrak{M} = (E, \mathscr{I})$ is a fuzzy matroid on E, and $\mathscr{I}$ is the family of independent fuzzy sets of μ.

Theorem 2.1. [7] *Let $\mathfrak{M} = (E, \mathscr{I})$ be a fuzzy matroid, and for each $r, 0 < r \leqslant 1$, let*

$$I_r = \{C_r(\mu) \mid \mu \in \mathscr{I}\}.$$

Then for each $r, 0 < r \leqslant 1$, $M_r = (E, I_r)$ is a (crisp) matroid on E.

3. Isomorphism of fuzzy matroids

In this section, we define the isomorphism of fuzzy matroids and discuss some of their properties.

Definition 3.1. *Let $\mathfrak{M}_1 = (E_1, \mathscr{I}_1)$ and $\mathfrak{M}_2 = (E_2, \mathscr{I}_2)$ be two fuzzy matroids. $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are fuzzy isomorphic if there exists a mapping $\gamma : E_1 \longrightarrow E_2$ such that γ satisfies the following conditions:*

i. γ is a one-to-one correspondence,

ii. Set a fuzzy set mapping Γ from $\mathscr{F}(E_1)$ to $\mathscr{F}(E_2)$ corresponding to γ such that $\forall\ \mu \in \mathscr{F}(E_1)$, $\forall x \in E_2$,

$$\Gamma(\mu)(x) = \mu\left(\gamma^{-1}(x)\right).$$

Now, for $\mu \in \mathscr{F}(E_1),\ \mu \in \mathscr{I}_1$ if and only if $\Gamma(\mu) \in \mathscr{I}_2$,

denoted by $\mathfrak{M}_1 \cong \mathfrak{M}_2$.

Theorem 3.1. *Let $\mathfrak{M}_1 = (E_1, \mathscr{I}_1)$ and $\mathfrak{M}_2 = (E_2, \mathscr{I}_2)$ be two fuzzy matroids such that $\mathfrak{M}_1 \cong \mathfrak{M}_2$. Then*

i. $\forall\ \mu_1,\ \mu_2 \in \mathscr{F}(E_1),\ \mu_1 \leq \mu_2$ if and only if $\Gamma(\mu_1) \leq \Gamma(\mu_2)$.

ii. for each $r,\ 0 < r \leq 1,\ (E_1, (\mathscr{I}_1)_r) \cong (E_2, (\mathscr{I}_2)_r)$.

Proof. Since $\mathfrak{M}_1 \cong \mathfrak{M}_2$, there is an isomorphic map $\gamma : E_1 \longrightarrow E_2$. Also, corresponding to γ we have a map $\Gamma : \mathscr{F}(E_1) \longrightarrow \mathscr{F}(E_1)$ such that $\forall\ \mu \in \mathscr{F}(E_1),\ \forall x \in E_2$,

$$\Gamma(\mu)(x) = \mu\left(\gamma^{-1}(x)\right).$$

Then, for $\mu \in \mathscr{F}(E_1),\ \mu \in \mathscr{I}_1$ if and only if $\Gamma(\mu) \in \mathscr{I}_2$.
Let $\mu_1, \mu_2 \in \mathscr{F}(E_1)$. Then,

i.

$$\begin{aligned}\mu_1 \leq \mu_2 \Leftrightarrow \mu_1\left(\gamma^{-1}(x)\right) &\leq \mu_2\left(\gamma^{-1}(x)\right),\ \forall x \in E_2\\ &\Leftrightarrow \Gamma(\mu_1)(x) \leq \Gamma(\mu_2)(x),\ \forall x \in E_2\\ &\Leftrightarrow \Gamma(\mu_1) \leq \Gamma(\mu_2)\end{aligned}$$

ii. Let $A \subseteq E_1,\ A \in (\mathscr{I}_1)_r$, for some $r,\ 0 < r \leq 1$.
$A \in (\mathscr{I}_1)_r$ if and only if $A = C_r(\mu)$ for some $\mu \in \mathscr{I}_1$. For this $\mu \in \mathscr{I}_1$, by the fuzzy isomorphism between $\mathfrak{M}_1$ and $\mathfrak{M}_2$, we have $\Gamma(\mu) \in \mathscr{I}_2$ and $\psi(A) = C_r(\Gamma(\mu))$. This is if and only if $\gamma(A) \in (\mathscr{I}_2)_r$.
That is, if $A \in (\mathscr{I}_1)_r$, then $\gamma(A) \in (\mathscr{I}_2)_r$.

4. Graphic fuzzy matroids

In this section, graphic fuzzy matroid is defined and some properties discussed.

A *fuzzy graph* [16] $G = (V, \sigma, \mu)$ is a triple consisting of a nonempty set V together with a pair of functions $\sigma : V \longrightarrow [0, 1]$ and $\mu : E \longrightarrow [0, 1]$ such that for all $x, y \in V$,

$$\mu(xy) \leq \sigma(x) \bigwedge \sigma(y).$$

A fuzzy graph $G = (V, \sigma, \mu)$ is *connected* if the corresponding graph G^* is connected. if G^* is a tree, then the fuzzy graph (V, σ, μ) is also called a *fuzzy tree* [16].
Let G^* be a simple graph and let $\sigma(x) = 1,\ \forall x \in V$, then we denote the fuzzy graph by $G = (V, \mu)$.
Now, we can obtain a fuzzy matroid from a fuzzy graph as follows.

Theorem 4.1. *Let $G = (V, \sigma, \mu)$ be a fuzzy graph with a corresponding graph $G^* = (V, E)$. For each $r,\ 0 < r \leq 1$, let*

$$\begin{aligned} E_r &= \{e \in E \mid \mu(e) \geq r\} \\ \mathscr{F}_r &= \{F \mid F\ is\ a\ forest\ in\ the\ (crisp) graph\ (V, E_r)\} \\ \mathscr{E}_r &= \{\mathcal{E}(F) \mid F\ \in \mathscr{F}_r\}, where\ \mathcal{E}(F)\ is\ the\ edge\ set\ of\ F. \end{aligned}$$

If

$$\mathscr{I} = \{\mu \in \mathscr{F}(E) \mid C_r(\mu) \in \mathscr{E}_r\ for\ each\ r,\ 0 < r \leq 1\ \}$$

then, $(E, \mathscr{I})$ is a fuzzy matroid.

Proof. We prove the properties (i) and (ii) of definition 2.4.

i. Suppose $\mu \in \mathscr{I}, \nu \in \mathscr{F}(E)$, and $\nu \leq \mu$. Then $C_r(\mu) \in \mathscr{E}_r$ for each $0 < r \leq 1$. Also, for each r, $C_r(\nu) \subseteq C_r(\mu)$ and $(E, \mathscr{E}_r)$ is a crisp matroid by Theorem 2.1.

$$\begin{aligned} &\Rightarrow C_r(\nu) \in \mathscr{E}_r\ for\ each\ r. \\ &\Rightarrow \nu \in \mathscr{I} \end{aligned}$$

Thus, $(E, \mathscr{I})$ satisfies the hereditary property.

ii. Suppose $\mu, \nu \in \mathscr{I}$ and $|supp\ \mu| < |supp\ \nu|$. Let $\delta = min\{m(\mu),\ m(\nu)\}$. Then we have, $supp\ \mu \in I_\delta$ and $supp\ \nu \in I_\delta$. I_δ is a crisp matroid, therefore $\exists$ a set $C \in I_\delta$ such that $supp\ \mu \subseteq C \subseteq supp\ \mu \cup supp\ \nu$. Let

$$\omega(x) = \begin{cases} \mu(x), & \text{if } x \in supp\ \mu \\ \delta, & \text{if } x \in C\ supp\ \mu \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

Then clearly $\omega \in \mathscr{I}$, $\mu < \omega \leq \mu \vee \nu$ and $m(\omega) \geq min\{m(\mu),\ m(\nu)\}$.

Thus, $(E, \mathscr{I})$ is a fuzzy matroid.

The fuzzy matroid $(E, \mathscr{I})$ constructed in the above Theorem is called the *fuzzy cycle matroid* of G, denoted by $\mathfrak{M}_F(G)$.

Distinct fuzzy graphs may induce the same fuzzy cycle matroid and their corresponding graphs may not be isomorphic. This statement is justified below through an example.

Example 4.1. Consider the following fuzzy graphs $G_1 = (V_1, \mu_1)$ and $G_2 = (V_2, \mu_2)$, with G_1^* and G_2^* be corresponding underlying graphs, where $\mu_1(e_i) = \mu_2(e_i') = 1, i = 1, ..., 6$. Then, easily we can get $\mathfrak{M}_F(G_1) = \mathfrak{M}_F(G_2)$. But, clearly, G_1 is not isomorphic to G_2.

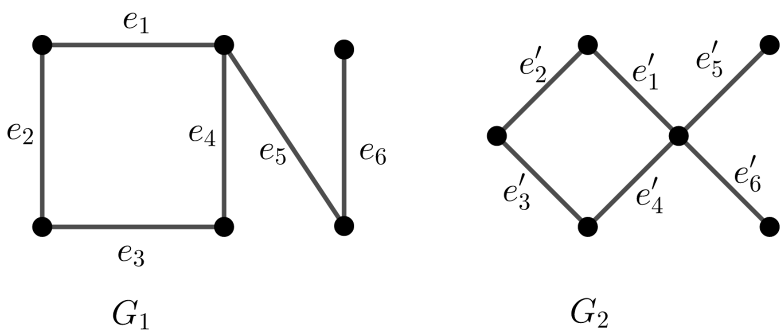

Figure 1: Non-isomorphic fuzzy graphs with same fuzzy cycle matroid

Definition 4.1. *Let $\mathfrak{M} = (E, \mathscr{I})$ be a fuzzy matroid. If there exists a fuzzy graph $G = (V, \mu)$ such that $\mathfrak{M} \cong \mathfrak{M}_F(G)$, then $\mathfrak{M}$ is called a graphic fuzzy matroid.*

Theorem 4.2. *Let $\mathfrak{M}$ be a graphic fuzzy matroid. Then $\mathfrak{M} \cong \mathfrak{M}_F(G)$ for some connected fuzzy graph G.*

Proof. As $\mathfrak{M}$ is a graphic fuzzy matroid, $\mathfrak{M} \cong \mathfrak{M}_F(G)$ for some fuzzy graph $G = (V, \mu)$.

If G is connected, the result is proved.

If G is not a connected fuzzy graph, G^* is also not connected.

Now, we construct a connected graph from G^* as follows.

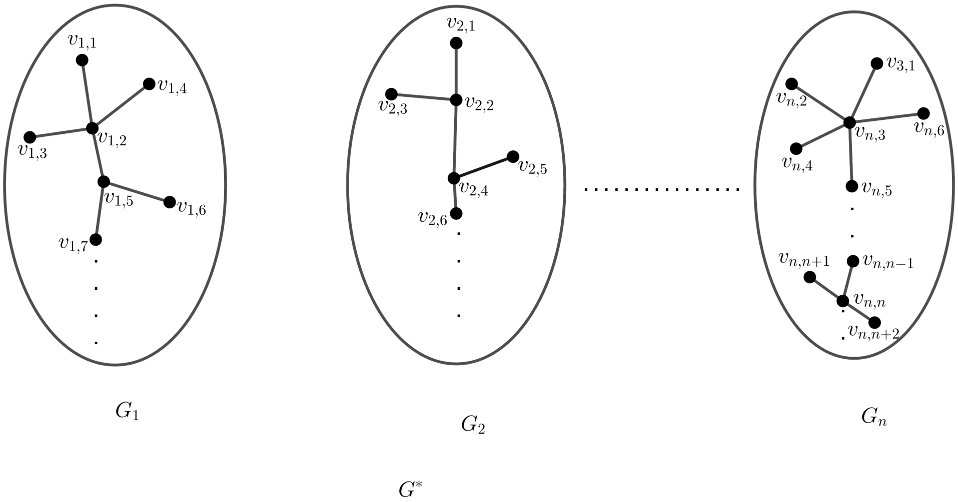

Figure 2: Underlying graph of the fuzzy graph G

We suppose that $G_1, G_2, ...G_n$ are the connected components of G^*. From each connected component G_i, we choose a vertex $v_{i,i}$. Form a new graph H^* by identifying $v_{1,1}, v_{2,2}, ...v_{n,n}$ as a single vertex v.

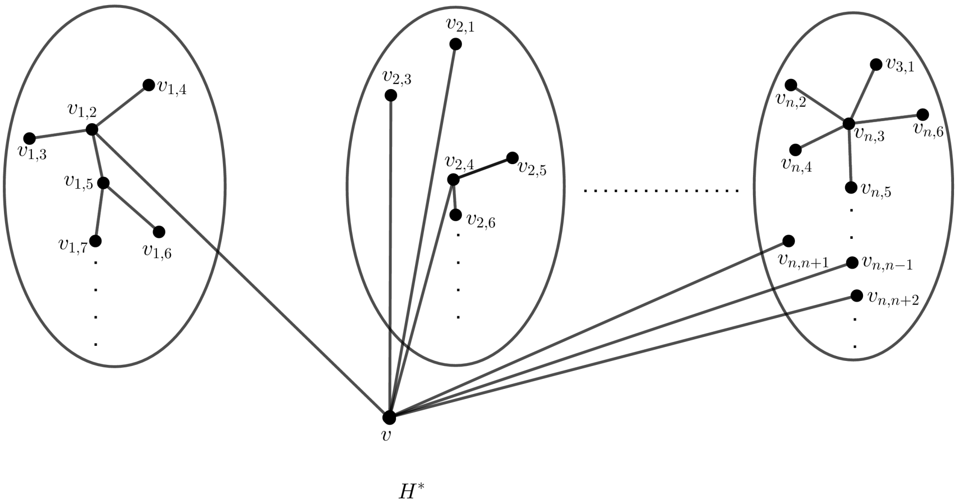

Figure 3: Underlying graph of the fuzzy graph graph H

Obviously, $E(H^*) = E(G^*)$, and H^* is connected. Thus, $H = (V, \mu)$ is a new connected fuzzy graph with H^* as the underlying graph. Moreover, $\forall X \subseteq E(G^*)$, If X does not contain any cycles of G^* then X does not contain any cycles of H^*

and vice versa. Thus, we have the two fuzzy cycle matroids $\mathfrak{M}_F(G)$ and $\mathfrak{M}_F(H)$ are isomorphic. Then by transitivity, we have $\mathfrak{M} \cong \mathfrak{M}_F(H)$.

5. Conclusion

An isomorphism is established between two fuzzy matroid and obtained that if two fuzzy matroids are isomorphic, then for $0 < r \leq 1$, the corresponding crisp matroids are also isomorphic. Then, the construction of a new class of fuzzy matroids from fuzzy graphs is given and hence introduced the concept of graphic fuzzy matroids. Distinct fuzzy graphs may induce the same fuzzy cycle matroid and their corresponding graphs may not be isomorphic. Also, for a graphic fuzzy matroid $\mathfrak{M}$, there exists some connected fuzzy graph G such that $\mathfrak{M}_F(G)$ is isomorphic to $\mathfrak{M}$. Such results will be useful for further investigations in the future.

6. Acknowledgments

The work of the first author is supported by the CSIR, Human Resource and Development Group, India, under the grant number 08/706(0002)/2018-EMR-I Dated 24/04/2019.

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