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SANDIP TRANSFORM: PROPERTIES AND APPLICATIONS

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Abstract: In the present article, new integral transform named Sandip transform is introduced by authors. It yields Laplace, Sumudu, Natural, Stieltjes and Generalized Stieltjes transform as its special cases. The existence theorem, inversion formula, Parseval-type identity and other results of Sandip transform are proved in this paper. H-function and H-transform are used with Mellin transform as useful tool for finding Sandip transform of functions. The relationship between Sandip transform with Mellin, Generalized Stieltjes, H-transform are proved and applications of Sandip transform to partial differential equations are given in this paper.

Keywords and Phrases: Laplace transform, Sumudu Transform, Natural Transform, H-function, Stieltjes transform, Generalized Stieltjes transform, Sandip transform.

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1. Introduction

In the field of Engineering, for the solution of linear ordinary and partial differential equations the method of integral transforms are very efficient. The simplification of given differential equations into initial-boundary value problems can be done with some weight function of two arguments and integration of an equation.

The selections of an integral transform dependants on a weight function and an integration domain. These result into various integral transforms that are available in the literature. The basic integral transforms used more widely are Laplace and Fourier transforms. The recent development in H-function and H-transform shows its applicability in engineering field. Kilbas and Saigo [5] gave applications and representation of all integral transforms into H-transform. The applications of H-function to various fields were discussed by Mathai et.al. [6]. The classical transform defined in book of L. Debnath and D. Bhatta [3], like Laplace, Mellin and many have been used in solution of differential equations and it can be put into the form of a special case of H-transform, also be called as G-transform. In 1993, Sumudu transform was introduced by Watugula [13] and development of Sumudu transform was done by Asiru [2] and Weerakoon [14]. Srivastava and Buschman [10] used Mellin transform for H function. In 2015, Srivastava et.al. [11] defined M-transform and gave its relationship with H-transform.

In present paper, the authors have defined Sandip transform in the first section and used H-function, H-transform, Mellin transform [3] (p. 340), Generalized Stieltjes transform [8], Natural transform [1], Laplace and Sumudu transform to develop concept of Sandip transform and the existence, inversion and theorems of Sandip transform in second section and examples, applications are in the last section.

Definition 1.1. ***Sandip transform***: *Let $y(t)$, be function of $t \geq 0$ then the Sandip transform is,*

$$S_{\lambda,\alpha}[y(t), v_1, v_2] = \int_0^\infty \frac{e^{-v_1 t} y(v_2 t)}{(v_2 t + \alpha)^\lambda} dt = \frac{1}{v_2} \int_0^\infty \frac{e^{-\frac{v_1}{v_2} t} y(t)}{(t + \alpha)^\lambda} dt \tag{1}$$

here $v_1, v_2, \lambda, \alpha \in \mathbb{C}, [Re(v_1), Re(\lambda)] \geq 0, v_2 \in (0, \infty), 0 \leq arg(\alpha) < \pi$.

Laplace transform, Sumudu transform and other Integral transforms can be obtained from Sandip transform by taking particular values of v_1, v_2, λ as follows,

1. Let $v_2 = 1, \lambda = 0$, one can get Laplace transform, [3] (p. 134, 3.2.5)

$$L[y(t), v_1] = \int_0^\infty e^{-v_1 t} y(t) dt, \quad Re(v_1) > 0 \tag{2}$$

2. Let $v_1 = 1, \lambda = 0$, one can get Sumudu transform, [13]
   Consider $y(t) \in A$, here
   $A = \{y | y(t) < M e^{\frac{|t|}{a_j}},\ (t \in (-1)^j \times [0, \infty),\ j = 1, 2;\ M, a_1, a_2 > 0)\}$

$$S[y(t), v_2] = \int_0^\infty e^{-t} y(v_2 t) dt, \quad v_2 \in (-a_1, a_2) \tag{3}$$

3. Let $\lambda = 0$, one can get Natural transform, [1]

$$N[y(t), v_1] = \int_0^\infty e^{-v_1 t} y(v_2 t) dt, \quad (v_1, v_2) > 0 \tag{4}$$

4. Let $v_1 = 0, v_2 = 1, \lambda = 1$, one can get Stieltjes transform, [3], (p. 391, 9.7.3)

$$S_t[y(t), \alpha] = \int_0^\infty \frac{y(t)}{t+\alpha} dt \ , |arg(\alpha)| < \pi \tag{5}$$

5. Let $v_1 = 0, v_2 = 1$, one can get Generalized Stieltjes transform, [3],(p. 401, 9.11.1), [8]

$$S_{tg}[y(t), \alpha] = \int_0^\infty \frac{y(t)}{(t+\alpha)^\lambda} dt \ , |arg(\alpha)| < \pi \tag{6}$$

2. Properties of Sandip Transform

Let us begin the section with existence theorem of Sandip transform defined in (1).

Theorem 2.1. *let $y(t)$ be continuous or piecewise continuous function of t in $[0, \infty)$ such that $|y(t)| \leq M t^{Re(\lambda)} e^{\frac{bt}{a}}$, $\forall\ t > T > 0$, $M > 0$, $a > 0$ then Sandip transform of $y(t)$ is exists for $Re(v_1) > Re(b)$, $v_2 \in (0, a)$, $Re(\lambda) \geq 0$, $0 \leq |arg(\alpha)| < \pi$. Also it is defined as in (1)and it is uniformly convergent with respect to v_1 with $Re(v_1) > \frac{bv_2}{a}$.*

Proof. Consider,

$$\begin{aligned} \left|S_{\lambda,\alpha}[y(t), v_1, v_2]\right| &\leq \int_0^\infty \frac{e^{-v_1 t}|y(v_2 t)|}{|(v_2 t+\alpha)|^{Re(\lambda)}} dt \\ &\leq M v_2^{Re(\lambda)} \int_0^\infty \frac{e^{-t(v_1 - \frac{bv_2}{a})} t^{Re(\lambda)}}{(v_2 t+\alpha)^{Re(\lambda)}} dt \end{aligned}$$

Case (1). If $\alpha = 0$ then

$$\begin{aligned} |S_{\lambda,\alpha}[y(t), v_1, v_2]| &\leq M \int_0^\infty e^{-t(v_1 - \frac{bv_2}{a})} dt \\ &= \frac{M}{(v_1 - \frac{bv_2}{a})}, \quad Re(v_1) > \frac{bv_2}{a}, \end{aligned}$$

Case (2). If $0 < |arg(\alpha)| < \pi$ then $\left|\frac{(v_2t)^{Re(\lambda)}}{(v_2t+\alpha)^{Re(\lambda)}}\right| \leq |(v_2t)^{\lambda}|$

$$|S_{\lambda,\alpha}[y(t), v_1, v_2]| \leq Mv_2^{\lambda} \int_0^{\infty} e^{-t(v_1-\frac{bv_2}{a})} t^{\lambda} dt$$
$$= \frac{Ma^{Re(\lambda)}\Gamma(\lambda+1)}{(v_1-\frac{bv_2}{a})^{(\lambda+1)}}, \quad Re(v_1) > \frac{bv_2}{a},\ v_2 < a$$

With the help of these two cases, the existence of Sandip transform defined in (1) is proved and using Weierstrass's test [12], it is uniformly convergent with respect to v_1 with $Re(v_1) > \frac{bv_2}{a}$.

1. Linearity Property:

$$S_{\lambda,\alpha}[(ay_1(t) + by_2(t)), v_1, v_2] = aS_{\lambda,\alpha}[y_1(t), v_1, v_2] + bS_{\lambda,\alpha}[y_2(t), v_1, v_2] \quad (7)$$

2. Scaling Property: Let $0 < a \in \mathbb{R}$

$$S_{\lambda,\alpha}[y(at), v_1, v_2] = a^{(\lambda-1)} S_{\lambda,a\alpha}\left[y(t), \frac{v_1}{a}, v_2\right] \quad (8)$$

3. Let $u(t) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x < 0 \end{cases}$ be unit step function ([3], page 15) then

$$S_{\lambda,\alpha}[u(t), v_1, v_2] = \frac{1}{v_2} S_{tg}[e^{-t\frac{v_1}{v_2}}] \quad (9)$$

   Here S_{tg} is Generalized Stieltjes transform.

4. First Shifting Property: Let Sandip transform exists for $y(t)$ then

$$S_{\lambda,\alpha}[e^{-at}y(t), v_1, v_2] = S_{\lambda,\alpha}[y(t), v_1 + v_2a, v_2] \quad (10)$$

5. Second Shifting Property: Let Sandip transform exists for $y(t)$ then

$$S_{\lambda,\alpha}[y(t-a)H(t-a), v_1, v_2] = e^{-a\frac{v_1}{v_2}} S_{\lambda,\alpha+a}[y(t), v_1, v_2] \quad (11)$$

6. Convolution Property: Let Sandip transform exists for $y_1(t)$ and $y_2(t)$ then

$$S_{\lambda,\alpha}[(t+\alpha)^{\lambda}(y_1(t) * y_2(t))], v_1, v_2] = \frac{1}{v_2} L\left[y_1(t) * y_2(t), \frac{v_1}{v_2}\right] \quad (12)$$

7. Removal Property: Let $\lambda > l$ and Sandip transform exists for $y(t)$ then

$$S_{\lambda,\alpha}[(t+\alpha)^l y(t), v_1, v_2] = S_{\lambda-l,\alpha}[y(t), v_1, v_2] \tag{13}$$

In particular, if $\lambda = l$ then one can get, relation of Sandip transform with Laplace, Normal and Sumudu transform as,

$$S_{\lambda,\alpha}[(t+\alpha)^\lambda y(t), v_1, v_2] = N[y(t), v_1, v_2] \tag{14}$$

$$= L[y(v_2 t), v_1] \tag{15}$$

$$= \frac{1}{v_1} S\left[y(t), \frac{v_2}{v_1}\right] \tag{16}$$

Theorem 2.2. *let integral defined in (1) converges absolutely, then the inversion of Sandip transform is*

$$y(t) = (t+\alpha)^\lambda L^{-1}\left[S_{\lambda,\alpha}[y(t), v_1, v_2], \frac{t}{v_2}\right], \quad 0 \le \alpha < \pi, v_2 \in (0, a)$$

$$= \frac{(t+\alpha)^\lambda}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} e^{v_1 t}\left[S_{\lambda,\alpha}[y(t), v_1, v_2], \frac{1}{v_2}\right] dv_1, \quad \gamma > 0 \tag{17}$$

Proof. From the definition of Sandip transform,
$[S_{\lambda,\alpha}[y(t), v_1, v_2] = L\left[\frac{y(v_2 t)}{(v_2 t+\alpha)^\lambda}, v_1\right]$
Taking inverse Laplace transform of both sides, one can get
$L^{-1}[S_{\lambda,\alpha}[y(t), v_1, v_2] = \frac{y(v_2 t)}{(v_2 t+\alpha)^\lambda}$
Now changing t by $\frac{t}{v_2}$ and using the formula of inverse Laplace transform given in [3] (p. 159, 3.7.7), we complete the proof.

Theorem 2.3. *Let $y(t)$ be piecewise continuous function having $y^n(t)$ as its continuous n^{th} derivative satisfying conditions of theorem (2.1) such that Sandip transform exists then*

$$S_{\lambda,\alpha}[y'(t), v_1, v_2] = \frac{v_1}{v_2} S_{\lambda,\alpha}[y(t), v_1, v_2] + \lambda S_{\lambda+1,\alpha}[y(t), v_1, v_2] - \frac{1}{\alpha^\lambda v_2} y(0) \tag{18}$$

In general,

$$S_{\lambda,\alpha}[y^n(t), v_1, v_2] = \frac{v_1^n}{v_2^n} S_{\lambda,\alpha}[y(t), v_1, v_2] + \lambda \sum_{k=0}^{n-1} \frac{v_1^{n-k-1}}{v_2^{n-k-1}} S_{\lambda+1,\alpha}[y(t), v_1, v_2] - \frac{1}{\alpha^\lambda} \sum_{k=0}^{n-1} \frac{v_1^{n-k-1}}{v_2^{n-k}} y^k(0) \tag{19}$$

Proof. Consider $y(t)$ and $y'(t)$ having Sandip transform then

$$S_{\lambda,\alpha}[y'(t), v_1, v_2] = \frac{1}{v_2}\left[\frac{e^{-v_1 t} y(v_2 t)}{(v_2 t+\alpha)^\lambda}\right]_0^\infty - \frac{1}{v_2}\int_0^\infty \frac{d}{dt}\left[\frac{e^{-v_1 t}}{(v_2 t+\alpha)^\lambda}\right] y(v_2 t) dt$$
$$= \frac{v_1}{v_2} S_{\lambda,\alpha}[y(t), v_1, v_2] + \lambda S_{\lambda+1,\alpha}[y(t), v_1, v_2] - \frac{1}{\alpha^\lambda v_2} y(0)$$

Here $\lim\limits_{n\to\infty} \frac{e^{-v_1 t} y(v_2 t)}{(v_2 t+\alpha)^\lambda} = 0$, using the conditions given in existence theorem of Sandip transform as, $\left|\frac{e^{-v_1 t} y(v_2 t)}{(v_2 t+\alpha)^\lambda}\right| \leq M a^\lambda e^{-(v_1+b)t} \to 0$ as $t \to \infty$.
To prove the general case, we replace $y'(t)$ by $y''(t)$ and continue the same procedure.

Theorem 2.4. *Let $y(t)$ be piecewise continuous function having $y^n(t)$ as its continuous n^{th} derivative satisfying conditions of theorem (2.1) such that Sandip transform exists then*

$$S_{\lambda,\alpha}[(t+\alpha)^\lambda y^n(t), v_1, v_2] = \frac{v_1^n}{v_2^n} N[y(t), v_1, v_2] - \sum_{k=0}^{n-1} \frac{v_1^{n-k-1}}{v_2^{n-k}} y^k(0) \tag{20}$$

In particular, $\lambda = 0$, one can get

$$S_{0,\alpha}[y^n(t), v_1, v_2] = \frac{v_1^n}{v_2^n} S_{0,\alpha}[y(t), v_1, v_2] - \sum_{k=0}^{n-1} \frac{v_1^{n-k-1}}{v_2^{n-k}} y^k(0) \tag{21}$$

Theorem 2.5. *Let $x(t)$ be function satisfying conditions of theorem (2.1) such that Sandip transform exists and if $a > 0$ then following identities holds true,*

$$S_{\lambda,\alpha}[x(t)L[y(v_1), t], av_2, v_2] = \int_0^\infty y(v_1) S_{\lambda,\alpha}[e^{-at} x(t), v_1 v_2, v_2] dv_1, \tag{22}$$

$$S_{\lambda,\alpha}[x(t)N[y(v_1), t, v_2], av_2, v_2] = \int_0^\infty y(v_1 v_2) S_{\lambda,\alpha}[e^{-at} x(t), v_1 v_2, v_2] dv_1 \tag{23}$$

Proof. Using the definition of Sandip transform,

$$S_{\lambda,\alpha}[x(t)L[y(v_1), t], av_2, v_2] = \int_0^\infty \frac{e^{-av_2 t} x(v_2 t)}{(v_2 t+\alpha)^\lambda}\left[\int_0^\infty e^{-v_1 v_2 t} y(v_1) dv_1\right] dt$$
$$= \int_0^\infty y(v_1)\left[\int_0^\infty \frac{e^{-v_1 v_2 t} e^{-av_2 t} x(v_2 t)}{(v_2 t+\alpha)^\lambda} dt\right] dv_1$$
$$= \int_0^\infty y(v_1) S_{\lambda,\alpha}[e^{-at} x(t), v_1 v_2, v_2] dv_1$$

The proof of second is similar to that of first one.
Following deductions can be obtained from (2.5) by taking particular values as,
Deduction 1. Let, $a = 1$ then
$\int_0^\infty y(v_1)S_{\lambda,\alpha}[e^{-t}x(t), v_1v_2, v_2]dv_1 = \frac{1}{v_2}S_{tg}[e^{-t}x(t)L(y(v_1), t), \alpha, \lambda]$
Here S_{tg} is generalized Stieltjes transform.
Deduction 2. Let, $y(t) = t^{m-1}$ then

$$M[S_{\lambda,\alpha}[e^{-at}x(t), v_1v_2, v_2], m] = \Gamma(m)S_{\lambda,\alpha}\left[\frac{x(t)}{t^m}, av_2, v_2\right]$$

Here $M[x(t), v] = \int_0^\infty t^{v-1}x(t)dt$, is called Mellin transform [1]
Deduction 3. Let,

$$y(t) = H_{u,v}^{m,n}\left[t\middle|\begin{matrix}(a_i,\alpha_i)_{1,u}\\(b_i,\beta_j)_{1,v}\end{matrix}\right]$$

then there exists relationship between H-transform and Sandip transform as,

$$H[S_{\lambda,\alpha}[e^{-at}x(t), v_1v_2, v_2], v_2] = S_{\lambda,\alpha}\left[\frac{x(t)}{t}H_{u+1,v}^{m,n+1}\left[\frac{v_2}{t}\middle|\begin{matrix}(0,1),(a_i,\alpha_i)_{1,u}\\(b_i,\beta_j)_{1,v}\end{matrix}\right], av_2, v_2\right]$$

Here we use the define H-function and H-transform as given in [5] (p. 1, 1.1.1, 1.1.2) and (p. 71, 3.1.1)) also [10], [11].
Proof. Using the definition of H-transform, one can get

$$\begin{aligned}H[S_{\lambda,\alpha}[e^{-at}x(t), v_1v_2, v_2], v_2] &= \int_0^\infty H_{u,v}^{m,n}\left[v_1v_2\middle|\begin{matrix}(a_i,\alpha_i)_{1,u}\\(b_i,\beta_j)_{1,v}\end{matrix}\right]S_{\lambda,\alpha}[e^{-at}x(t), v_1v_2, v_2]dv_1\\ &= S_{\lambda,\alpha}[x(t)N\left(H_{u,v}^{m,n}\left[v_1\middle|\begin{matrix}(a_i,\alpha_i)_{1,u}\\(b_i,\beta_j)_{1,v}\end{matrix}\right], t, v_2\right), av_2, v_2]\end{aligned}$$

One can get the required equality using Natural transform of H-function [11] (p. 1396, 3.15)

$$N\left(H_{u,v}^{m,n}\left[v_1\middle|\begin{matrix}(a_i,\alpha_i)_{1,u}\\(b_i,\beta_j)_{1,v}\end{matrix}\right], t, v_2\right) = \frac{1}{t}H_{u+1,v}^{m,n+1}\left[\frac{v_2}{t}\middle|\begin{matrix}(0,1),(a_i,\alpha_i)_{1,u}\\(b_i,\beta_j)_{1,v}\end{matrix}\right]$$

Theorem 2.6. *(Parseval-Type theorem) Let $x(t)$ and $y(t)$ be function satisfying conditions of theorem (2.1) such that Sandip transform exists then following identities hold true*

$$\int_0^\infty \frac{y(v_1v_2)}{(v_1v_2+\alpha_1)^{\lambda_1}}S_{\lambda_2,\alpha_2}[x(t), v_1, v_2]dv_1 = \int_0^\infty \frac{x(tv_2)}{(tv_2+\alpha_2)^{\lambda_2}}S_{\lambda_1,\alpha_1}[y(v_1), t, v_2]dt \tag{24}$$

If $\lambda_1 = \lambda_2 = \lambda$ *and* $\alpha_1 = \alpha_2 = \alpha$ *then,*

$$\int_0^\infty \frac{y(v_1v_2)}{(v_1v_2+\alpha)^\lambda} S_{\lambda,\alpha}[x(t),v_1,v_2]dv_1 = \int_0^\infty \frac{x(tv_2)}{(tv_2+\alpha)^\lambda} S_{\lambda,\alpha}[y(v_1),t,v_2]dt \quad (25)$$

Proof. Consider,

$$\begin{aligned}\int_0^\infty \frac{y(v_1v_2)}{(v_1v_2+\alpha_1)^{\lambda_1}} S_{\lambda_2,\alpha_2}[x(t),v_1,v_2]dv_1 &= \int_0^\infty \frac{y(v_1v_2)}{(v_1v_2+\alpha_1)^{\lambda_1}} \left[\int_0^\infty \frac{e^{-v_1t}x(v_2t)}{(v_2t+\alpha_2)^{\lambda_2}}dt\right]dv_1 \\ &= \int_0^\infty \frac{x(v_2t)}{(v_2t+\alpha_2)^{\lambda_2}} \left[\int_0^\infty \frac{e^{-v_1t}y(v_1v_2)}{(v_1v_2+\alpha_1)^{\lambda_1}}dv_1\right]dt \\ &= \int_0^\infty \frac{x(tv_2)}{(tv_2+\alpha_2)^{\lambda_2}} S_{\lambda_1,\alpha_1}[y(v_1),t,v_2]dt\end{aligned}$$

2.1. Examples of Sandip Transform

Theorem 2.7. *Following results hold true for Sandip transform*

$$S_{\lambda,\alpha}[e^{at},v_1,v_2] = \frac{1}{(v_1-av_2)\Gamma(\lambda)\alpha^\lambda} H^{2,1}_{1,2}\left[\left(\frac{v_1-av_2}{v_2}\right)\alpha \middle| \begin{matrix}(1,1)\\(1,1),(\lambda,1)\end{matrix}\right] \quad (26)$$

$(Re(\lambda) > 0,\ a > 0,\ \alpha > 0,\ Re(v_1) > av_2)$

$$S_{\lambda,\alpha}[e^{-at},v_1,v_2] = \frac{1}{(v_1+av_2)\Gamma(\lambda)\alpha^\lambda} H^{2,1}_{1,2}\left[\left(\frac{v_1+av_2}{v_2}\right)\alpha \middle| \begin{matrix}(1,1)\\(1,1),(\lambda,1)\end{matrix}\right] \quad (27)$$

$(Re(\lambda) > 0,\ a > 0,\ \alpha > 0)$

$$S_{\lambda,\alpha}[t^{n-1},v_1,v_2] = \frac{v_2^{n-1}}{v_1^n\Gamma(\lambda)\alpha^\lambda} H^{2,1}_{1,2}\left[\frac{v_1\alpha}{v_2} \middle| \begin{matrix}(1,1)\\(n,1),(\lambda,1)\end{matrix}\right] \quad (28)$$

$(Re(\lambda) > 0,\ a > 0,\ \alpha > 0,\ Re(v_1) > 0)$

$$S_{\lambda,\alpha}[t^{n-1}e^{at},v_1,v_2] = \frac{v_2^{n-1}}{(v_1-av_2)^n\Gamma(\lambda)\alpha^\lambda} H^{2,1}_{1,2}\left[\left(\frac{v_1-av_2}{v_2}\right)\alpha \middle| \begin{matrix}(1,1)\\(n,1),(\lambda,1)\end{matrix}\right] \quad (29)$$

$(Re(\lambda) > 0,\ a > 0,\ \alpha > 0,\ Re(v_1) > av_2)$

Let $(t+\alpha)^\lambda {}_2F_1(a,b;c:-t)$ *be hyper geometric function then*

$$S_{\beta,\alpha}[(t+\alpha)^\lambda {}_2F_1(a,b;c:-t),v_1,v_2]$$

$$= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)v_2} H^{2,2}_{3,2}\left[\frac{v_1}{v_2} \middle| \begin{matrix}(1-b,-1),(1,1)\\(1,1),(a,1),(1-c,-1)\end{matrix}\right] \quad (30)$$

Proof. Lets prove first identity, remaining can be proved with similar procedure. To use procedure we follow [11]. Now consider,

$$S_{\lambda,\alpha}[e^{at}, v_1, v_2] = \int_0^\infty \frac{e^{-(v_1-av_2)t}}{(v_2t+\alpha)^\lambda} dt$$

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty e^{-(v_1-av_2)t} \int_0^\infty e^{-(v_2t+\alpha)z} z^{\lambda-1} dz dt$$

Here we use the result, $\int_0^\infty e^{kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$ (more detail see [3], p. 43)

$$= \frac{1}{\Gamma(\lambda)} \int_0^\infty \left[\int_0^\infty e^{-(v_1-av_2+v_2z)t} dt \right] e^{-\alpha z} z^{\lambda-1} dz \tag{31}$$

Let, $I(z) = \int_0^\infty e^{-(v_1-av_2+v_2z)t} dt$ take its Mellin transform interchanging the order of integration,

$$M[I(z), x] = \int_0^\infty e^{-(v_1-av_2)t} \left[\int_0^\infty z^{x-1} e^{-v_2tz} dz \right] dt$$
$$= \frac{\Gamma(x)}{v_2^x} \int_0^\infty e^{-(v_1-av_2)t} t^{1-x-1} dt$$

$$M[I(z), x] = \frac{\Gamma(x)\Gamma(1-x)}{v_1-av_2} \left(\frac{v_1-av_2}{v_2} \right)^x \tag{32}$$

To obtain $I(z)$ takes inverse Mellin transform of (32),

$$I(z) = \frac{1}{2\pi i(v_1-av_2)} \int_{-i\infty}^{i\infty} \Gamma(x)\Gamma(1-x) \left(\frac{v_1-av_2}{v_2} \right)^x z^{-x} dx \tag{33}$$

To get the required answer, substitute (33) in to (31) and interchanging the order of integration,

$$\begin{aligned} S_{\lambda,\alpha}[e^{at}, v_1, v_2] &= \frac{1}{2\pi i(v_1-av_2)\Gamma(\lambda)} \int_{-i\infty}^{i\infty} \Gamma(x)\Gamma(1-x) \left(\frac{v_1-av_2}{v_2} \right)^x \left[\int_0^\infty e^{-\alpha z} z^{\lambda-x-1} dz \right] dx \\ &= \frac{1}{2\pi i(v_1-av_2)\Gamma(\lambda)\alpha^\lambda} \int_{-i\infty}^{i\infty} \Gamma(-x)\Gamma(1+x)\Gamma(\lambda+x) \left(\frac{v_1-av_2}{v_2} \right)^{-x} \alpha^{-x} dx \\ &= \frac{1}{(v_1-av_2)\Gamma(\lambda)\alpha^\lambda} H_{1,2}^{2,1} \left[\left(\frac{v_1-av_2}{v_2} \right) \alpha \middle| \begin{matrix} (1,1) \\ (1,1),(\lambda,1) \end{matrix} \right] \end{aligned} \tag{34}$$

The last step is followed from definition of H-function.
In particular if we take $a = 0$ in first equality, one can get

$$S_{\lambda,\alpha}[1, v_1, v_2] = \frac{1}{v_1\Gamma(\lambda)\alpha^{\lambda}} H_{1,2}^{2,1}\left[\frac{v_1\alpha}{v_2}\middle| \begin{matrix} (1,1) \\ (1,1),(\lambda,1) \end{matrix}\right]$$

Similarly one can find Sandip transform of sine and cosine hyperbolic functions.

Let us prove now Sandip transform of Hyper geometric function
Consider, $(t+\alpha)^{\lambda}{}_2F_1(a, b; c : -t)$ be hyper geometric function then

$$S_{\lambda,\alpha}[(t+\alpha)^{\lambda}{}_2F_1(a, b; c : -t), v_1, v_2]$$

$$= \frac{1}{v_2}\int_0^{\infty} \frac{\Gamma(c)e^{-t\frac{v_1}{v_2}}}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b-1}(1+tu)^{-a} du dt$$

Now using the similar procedure as we used to prove equation (26), we get

$$S_{\lambda,\alpha}[(t+\alpha)^{\lambda}{}_2F_1(a, b; c : -t), v_1, v_2]$$

$$= \frac{1}{v_2}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 u^{b-1}(1-u)^{c-b} H_{1,2}^{2,1}\left[\frac{v_1\alpha}{v_2 u}\middle| \begin{matrix} (1,1) \\ (1,1),(a,1) \end{matrix}\right] du$$

$$= \frac{1}{v_2}\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} H_{3,2}^{2,2}\left[\frac{v_1\alpha}{v_2}\middle| \begin{matrix} (1-b,-1),(1,1) \\ (1,1),(a,1),(1-c,-1) \end{matrix}\right]$$

This last step we get using particular values in Euler transform of H-function (more detail see [6], 2.53, page 59)

Let $U(a; c; x)$ be Confluent Hyper geometric function (see [7] for more detail) of the second kind and

$$y(t) = \frac{t^{a-1}}{(\alpha+t)^{a-1}}$$

then

$$S_{\lambda,\alpha}\left[\frac{t^{a-1}}{(\alpha+t)^{a-1}}, v_1, v_2\right] = \frac{v_2\Gamma(a)}{\alpha^{\lambda+a-1}} U\left(a; -\lambda; \frac{v_1\alpha}{v_2}\right) \tag{35}$$

2.2. Applications to Partial differential Equations

Example 2.8. Let us solve inhomogeneous Wave equation, given in [3] (p. 219)

$$\frac{1}{c^2}u_{tt} - u_{xx} = k\sin\left(\frac{\pi x}{a}\right),\ 0 < x < a, t > 0,\ u = u(x,t) \tag{36}$$

subject to the initial and boundary conditions $u(x,0) = 0 = u_t(x,0), \quad 0 < x < a$ and $u(0,t) = 0 = u_t(a,t), \quad t > 0$
Let $\lambda > 0$ then given equation can be written as,

$$\frac{(t+\alpha)^{\lambda}}{c^2}u_{tt} - (t+\alpha)^{\lambda}u_{xx} = (t+\alpha)^{\lambda}k\sin\left(\frac{\pi x}{a}\right)$$

Taking Sandip transform of both sides and using equation (20) with $U = N\,[u(x,t), v_1, v_2]$, using given conditions, one can get

$$\frac{d^2U}{dx^2} - \frac{v_1^2}{v_2^2c^2}U = -\frac{k}{v_1}\sin\left(\frac{\pi x}{a}\right)$$

It is linear differential equation of second order having solution,

$$U = c_1e^{\frac{v_1x}{cv_2}} + c_2e^{-\frac{v_1x}{cv_2}} + \frac{ka^2c^2v_2^2}{v_1(v_1^2a^2+\pi^2c^2v_2^2)}sin\left(\frac{\pi x}{a}\right)$$

With the help of remaining conditions, one can get $c_1 = 0, c_2 = 0$.

$$U = \frac{ka^2c^2v_2^2}{v_1(v_1^2a^2+\pi^2c^2v_2^2)}\sin\left(\frac{\pi x}{a}\right) = \frac{ka^2}{\pi^2}\sin\left(\frac{\pi x}{a}\right)\left[\frac{1}{v_1} - \frac{v_1a^2}{v_1^2a^2+\pi^2c^2v_2^2}\right]$$

Using inverse Normal transform, one can get

$$u(x,t) = \frac{ka^2}{\pi^2}\sin\left(\frac{\pi x}{a}\right)\left[1-\cos\left(\frac{\pi ct}{a}\right)\right]$$

Example 2.9. Let us solve the equation

$$xu_t + u_x = \frac{x}{(t+\alpha)^{\lambda}},\ x>0,\ t>0,\ 0\le\alpha<\pi \tag{37}$$

Rewriting given equation and taking Sandip transform of both sides with $v_2 = 1$, $\bar{U} = L[u(x,t), v_1]$ Using given conditions, one can get

$$\frac{d\bar{U}}{dx} + xv_1\bar{U} = \frac{x}{v_1}\frac{1}{\Gamma(\lambda)\alpha^{\lambda}}H_{1,2}^{2,1}\left[v_1\alpha\middle|\begin{matrix}(1,1)\\(1,1),(\lambda,1)\end{matrix}\right]$$

It is first order linear differential equation with integrating factor $e^{\frac{x^2v_1}{2}}$ and general solution using given initial condition is,

$$\bar{U} = \left(1-e^{-\frac{x^2v_1}{2}}\right)\frac{1}{v_1^2}\frac{1}{\Gamma(\lambda)\alpha^{\lambda}}H_{1,2}^{2,1}\left[v_1\alpha\middle|\begin{matrix}(1,1)\\(1,1),(\lambda,1)\end{matrix}\right]$$

Taking inverse Laplace transform as given in [6] (p. 2.21, 2.23) and using inversion theorem of Sandip transform (2.2) with little bit simplification one can get

$$u(x,t) = \frac{1}{(1-\lambda)}\left(\frac{1}{(t+\alpha)^{(\lambda-1)}} - \frac{1}{\alpha^{(\lambda-1)}}\right)$$
$$-\frac{1}{(1-\lambda)}\left(\frac{1}{\left(t-\frac{x^2}{2}+\alpha\right)^{(\lambda-1)}} - \frac{1}{\alpha^{(\lambda-1)}}\right) H\left(t-\frac{x^2}{2}\right)$$

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