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A GENERALIZATION OF ORLICZ SEQUENCE SPACES DERIVED BY QUADRUPLE SEQUENTIAL BAND MATRIX

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Abstract: In this article we have introduced a new Orlicz sequence space $l_p^\lambda(M, B)$ derived by a quadruple sequential band matrix associated with an Orlicz function and lambda matrix. Further, we have studied some topological properties and inclusion relations of this space.

Keywords and Phrases: Orlicz function, Four band matrix, Lambda matrix, AK-space, BK-space.

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1. Introduction

Let w represent the space of all real or complex sequences and any subspace of w is called a sequence space. By $c, c_0, l_\infty, l_1, l_p$ and bv_p, we denote the space of all convergent, null, bounded, absolutely summable, p-absolutely summable and p-bounded variation sequences respectively, where $0 < p < \infty$.

As the theory of sequence spaces has been a subject of interest to several mathematicians, Cesàro, Nörlund, Abel, Riesz and others studied the theory of sequence spaces through summability theory while Nakano [24], Simons [28], Maddox [19] and many others have constructed different sequence spaces by using the modern techniques of functional analysis. Later on Kızmaz [16], Et and Çolak [13], Başar and Dutta [6], Dutta and Başar [12] and many others gave a new direction for the development of the structural properties of Orlicz sequence spaces.

For any two sequence spaces X and Y, with $A : X \to Y$ the A-transform of $x = (x_k) \in X$ written as $y = Ax$ and is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \text{ for all } n \in \mathbb{N}$$

where $A = (a_{nk})$ is an infinite matrix of real or complex numbers and each of these series are convergent. The set of all infinite matrices $A = (a_{nk})$ where $Ax \in Y$ for all $x \in X$ is denoted by (X, Y). For an arbitrary sequence space X, the set X_A is also called matrix domain of an infinite matrix $A = (a_{nk})$ defined by

$$X_A = \{x = (x_k) \in w : Ax \in X\}$$

is also a sequence space. The study of matrix transformations has been enriched by several mathematicians considering different sequence spaces. Recent work includes Altay and Başar [1], Aydın and Başar [2], Candan [9] and many others. For detailed knowledge of sequence spaces, matrix transformations and the domain of triangular matrices in the normed sequence spaces, a reader should refer to the monographs Nanda [25], Dash [11], Başar [4], Mursaleen and Başar [21].

The domains $c_0(\Delta^F), c_0(\Delta^F)$ and $l_\infty(\Delta^F)$ of the forward difference matrix Δ^F in the spaces c_0, c and l_∞ are introduced by Kızmaz [16]. Afterwards, the domain bv_p of the backward difference matrix Δ^B in the space l_p have recently been investigated for $0 < p < 1$ by Altay and Başar [1] and for $1 \le p < \infty$ by Başar and Altay [5]. Later Kirişçi and Başar [15] have constructed the difference sequence spaces

$$\hat{X} = \{x = (x_k) \in w : B(r,s)x \in X\}$$

for $X = c, c_0, l_\infty$ and l_p, where $1 \le p < \infty$ and $(B(r,s)x)_k = (sx_{k-1} + rx_k)(r, s \neq 0)$. Candan [9] generalized this space by choosing $\tilde{r} = (r_n)_{n=0}^{\infty}$ and $\tilde{s} = (s_n)_{n=0}^{\infty}$ as convergent sequence of positive real numbers. Sönmez [29] constructed the sequence space with triple band matrix which was further been generalized by Bişgin [8] using a Quadruple band matrix $Q = Q(r,s,t,u) = (q_{nk}(r,s,t,u))$ defined by,

$$q_{nk}(r,s,t,u) = \begin{cases} r & k = n \\ s & k = n-1 \\ t & k = n-2 \\ u & k = n-3 \\ 0 & \text{otherwise} \end{cases}$$

for all $n, k \in \mathbb{N}$ and $r, s, t, u \in \mathbb{R} - \{0\}$.

Suppose $\tilde{r} = (r_k), \tilde{s} = (x_k), \tilde{t} = (t_k)$ and $\tilde{u} = (u_k)$ are convergent sequences of distinct real numbers and $x = (x_k)$ is any sequence in either c_0 or l_1. Baliarsingh and Dutta [3] introduced the difference sequence space using difference operator $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ where

$$(B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})x)_k = r_k x_k + s_{k-1}x_{k-1} + t_{k-2}x_{k-2} + u_{k-3}x_{k-3}. \tag{1.1}$$

The quadruple sequential band matrix $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u}) = (b_{nk})$ is defined as follows

$$b_{nk} = \begin{cases} r_k & k = n \\ s_k & k = n-1 \\ t_k & k = n-2 \\ u_k & k = n-3 \\ 0 & \text{otherwise} \end{cases}$$

and any term of (b_{nk}) having negative subscript is zero.

Lindenstrauss and Tzafriri [18] defined the Orlicz sequence space

$$l_M = \inf \left\{ x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\},$$

which is a Banach space with the norm $\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \right\}$. Later on Parashar and Choudhary [27] introduced and studied the space $l_M(p)$ for $p = (p_k)$ a bounded sequence of positive real numbers.

Motivated by the earlier work we have introduced a new sequence space $l_p^\lambda(M, B)$, as

$$l_p^\lambda(M, B) = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho \lambda_n} \right) \right]^{p_n} < \infty, \text{for some } \rho > 0 \right\},$$

where $X_k = r_k x_k + s_{k-1}x_{k-1} + t_{k-2}x_{k-2} + u_{k-3}x_{k-3}$, $\lambda = (\lambda_k)_{k=0}^{\infty}$ consist of positive reals such that $0 < \lambda_0 < \lambda_1 < \ldots$ and $\lim_{k\to\infty} \lambda_k = \infty$ and the λ matrix $\Lambda = (\lambda_{nk})$ is defined by

$$\lambda_{nk} = \begin{cases} \frac{\lambda_k - \lambda_{k-1}}{\lambda_n}, & 0 \le k \le n \\ 0, & k > n \end{cases}$$

for all $n, k \in \mathbb{N}$ and (p_n) is a bounded sequence of positive real numbers.

For suitable choice of p_n, M, B and Λ the space $l_p^\lambda(M, B)$ generalizes the following cases:

(i) For $M(x) = x$ and $\tilde{r} = re, \tilde{s} = se$ and $\tilde{t} = \tilde{u} = \theta$, it reduces to $l_p^\lambda(B)$ introduced by Başar and Karaisa [7].

(ii) For $M(x) = x, p_n = p$, for all $n \in \mathbb{N}$ and $B = I$, the identity matrix it reduces to the space l_p^λ studied by Mursaleen and Noman ([22], [23]).

(iii) For $M(x) = x, p_n = 1$ for all $n \in \mathbb{N}$ and $B = \Lambda = I$, the identity matrix it reduces to l_M studied by Lindenstrauss and Tzafriri [18].

(iv) For $B = \Lambda = I$ it reduces to $l_M(p)$ studied by Parasar and Choudhary [27].

(v) For $\tilde{r} = re, \tilde{s} = se, \tilde{t} = \tilde{u} = \theta, M(x) = x, \Lambda = I$ and $p_n = p(1 \leq p < \infty)$ it reduces to $\hat{l}_p$ studied by Kirişçi and Başar [15].

(vi) For $\tilde{r} = re, \tilde{s} = se, \tilde{t} = \tilde{u} = \theta, M(x) = x$ and $\Lambda = I$ it reduces to $\hat{l}(p)$ studied by Aydın and Başar [2].

(vii) For $\tilde{t} = \tilde{u} = \theta, M(x) = x, \Lambda = I$ and $p_n = p$ $(1 \leq p < \infty)$ for all $n \in \mathbb{N}$ it reduces to $\tilde{l}(p)$ studied by Candan [9].

(viii) For $\tilde{t} = \tilde{u} = \theta, M(x) = x$ and $\Lambda = I$ it reduces to $l(\tilde{B}, p)$ introduced by Nergiz and Başar [26].

(ix) For $\tilde{r} = re, \tilde{s} = se, \tilde{t} = te, \tilde{u} = \theta, M(x) = x, \Lambda = I$ and $p_n = p$ $(1 < p < \infty)$ for all $n \in \mathbb{N}$ it reduces to $l_p(B)$ studied by Sönmez [29].

(x) For $M(x) = x, B = I, p_n = p$ for all $n \in \mathbb{N}$ and $\Lambda = I$ it reduces to classical sequence l_p space.

2. Definitions and Preliminaries

Definition 2.1. [14] *A sequence space X is called a K-space if the co-ordinate function $P_k : X \to K$ given by $P_k(x) = x_k$ is continuous for each $k \in \mathbb{N}$.*

Definition 2.2. [30] *An FK-space is a Fréchet sequence space with continuous co-ordinates.*

Definition 2.3. [31] *A linear space X is called BK-space, if it is equipped with a norm under which it is a Banach space with continuous co-ordinates.*

Definition 2.4. [30] *An FK-space X is said to be an AK-space if $X \supset \phi$, the set of all finitely non-zero sequences and $\{\delta^n\}$ is a basis for X, i.e., for each x, $x^{[n]} \to x$, where $x^{[n]}$ denotes the n^{th} section of x is $\sum_{k=1}^{n} x_k \delta^k$, otherwise expressed as $x = \sum x_k \delta^k$ for all $x \in X$. For example, $l(p), c_0(p), w_0(p)$ are AK-spaces.*

Definition 2.5. [10] *A sequence space X is said to be convergence free when, if $x = (x_k)$ in X and if $y_k = \theta$ whenever $x_k = \theta$, then $y = (y_k)$ is in X.*

Definition 2.6. [14] *A sequence space X is said to be normal if $(x_k) \in X$ implies $(\alpha_k x_k) \in X$ for all sequence of scalars (α_k) with $|\alpha_k| \leq 1$ for all $k \in \mathbb{N}$.*

Definition 2.7. [10] *A sequence space X is said to be symmetric if,when $x = (x_k)$ is in X, then $y = (y_k)$ is in X when the co-ordinates of y are those of x, but in a different order.*

Definition 2.8. [14] *An Orlicz function is a function $M : [0, \infty) \longrightarrow [0, \infty)$ which is continuous, non-decreasing and convex with $M(0) = 0, M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.*

Lemma 2.1. [17] *An Orlicz function M is said to satisfy Δ_2-condition for all values of u, if there exists a constant $K > 0$ such that $M(2u) = KM(u), u \geq 0$.*

The Δ_2-condition is equivalent to the inequality $M(lu) \leq K'lM(u)$, for some $K' > 0$ which holds for all values of u and $l > 1$.

Lemma 2.2. [20] *Let $p = (p_n)$ be a bounded sequence of positive real numbers. Then for any complex numbers a_n and b_n, $|a_n + b_n|^{p_n} \leq D\left(|a_n|^{p_n} + |b_n|^{p_n}\right)$, where $0 < p_n \leq \sup p_n = G$ and $D = \max\{1, 2^{G-1}\}$.*

Lemma 2.3. [20] *Let $0 < p \leq 1$. Then for any complex numbers a and b, $|a + b|^p \leq |a|^p + |b|^p$.*

3. Main Result

Theorem 3.1. *$l_p^\lambda(M, B)$ is a linear space over $\mathbb{C}$.*

Proof. Let $x = (x_n), y = (y_n) \in l_p^\lambda(M, B)$ and $\alpha, \beta \in \mathbb{C}$. So there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho_1 \lambda_n} \right) \right]^{p_n} < \infty$$

and

$$\sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) Y_k|}{\rho_2 \lambda_n} \right) \right]^{p_n} < \infty$$

Let $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$ where $\alpha, \beta \in \mathbb{C}$. Since M is non-decreasing and convex, we have

$$\sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1})(\alpha X_k + \beta Y_k)|}{\rho_3 \lambda_n} \right) \right]^{p_n}$$

$$\begin{aligned}
&\leq \sum_{n=0}^{\infty}\left[M\left(\frac{|\alpha||\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|+|\beta||\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_3\lambda_n}\right)\right]^{p_n}\\
&\leq \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{p_n}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)+M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_2\lambda_n}\right)\right]^{p_n}\\
&\leq \sum_{n=0}^{\infty}\left[M\left(\frac{|\alpha||\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)+M\left(\frac{|\beta||\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_2\lambda_n}\right)\right]^{p_n}\\
&\leq \sum_{n=0}^{\infty}D\left[M\left(\frac{|\alpha||\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)+M\left(\frac{|\beta||\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_2\lambda_n}\right)\right]^{p_n}.
\end{aligned}$$

Thus from the above inequality with Lemma 2.2 we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(\alpha X_k+\beta Y_k)|}{\rho_3\lambda_n}\right)\right]^{p_n}\\
\leq\ & D\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)\right]^{p_n}+D\sum_{n=1}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_2\lambda_n}\right)\right]^{p_n}\\
<\ & \infty.
\end{aligned}$$

i.e., $\alpha x+\beta y\in l_p^{\lambda}(M,B)$.
Hence $l_p^{\lambda}(M,B)$ is a linear space over $\mathbb{C}$.

Theorem 3.2. *(i) $l_p^{\lambda}(M,B)$ is a normed linear space under the norm defined by*

$$\|x\|=\inf\left\{\rho^{\frac{p_n}{H}}:\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1\right\} \tag{3.1}$$

where $x\in l_p^{\lambda}(M,B)$ and $H=\max(1,\sup_k p_k)$.
(ii) $l_p^{\lambda}(M,B)$ is a Banach space under the norm defined by (3.1).
(iii) $l_p^{\lambda}(M,B)$ is a $BK-$space under the norm defined by (3.1).

Proof. (i) Obviously $\|x\|\geq 0$ and $\|x\|=0$ if $x=0$. Now suppose $\|x\|=0$ i.e.,

$$\|x\|=\inf\left\{\rho^{\frac{p_n}{H}}:\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1\right\}=0.$$

This yields that for a given $\varepsilon>0$, there exists some $\rho_\varepsilon\in(0,\varepsilon)$ such that

$$\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_\varepsilon\lambda_n}\right)\right]\leq 1.$$

Which implies $M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_\varepsilon\lambda_n}\right)\leq 1$ for all $n\in\mathbb{N}$. Thus,

$$M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\varepsilon\lambda_n}\right)\leq M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_\varepsilon\lambda_n}\right)\leq 1 \text{ for all n} \in\mathbb{N}. \tag{3.2}$$

Suppose $\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_\varepsilon\lambda_n}\neq 0$ for some $k\in\mathbb{N}$. Then $\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\varepsilon\lambda_n}\to\infty$ as $\varepsilon\to 0$ which implies that $M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\varepsilon\lambda_n}\right)\to\infty$ as $\varepsilon\to 0$ for some $k\in\mathbb{N}$ (as M is an Orlicz function) which leads to a contradiction. Therefore, $\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_\varepsilon\lambda_n}=0$ for all $n\in\mathbb{N}$. This follows that $X_k=0$ for all $k=1,2,3,\ldots,n$ and $n\in\mathbb{N}$. Since (λ_n) is a sequence of positive integers, this implies $x_n=0$ for all $n\in\mathbb{N}$ i.e., $x=0$. Now let $x,y\in l_p^\lambda(M,B)$. For $\rho_1>0$ and $\rho_2>0$,

$$\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1$$

and

$$\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_2\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1.$$

Let $\rho=\rho_1+\rho_2$. Then as M is convex, we have

$$M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(X_k+Y_k)|}{\rho\lambda_n}\right)$$
$$\leq\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sum_{n=0}^{\infty}M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sum_{n=0}^{\infty}M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho_2\lambda_n}\right).$$

Therefore, by Lemma 2.3 we have

$$\begin{aligned}\|x+y\| &= \inf\left\{\rho^{\frac{p_n}{H}}>0:\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})(X_k+Y_k)|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1\right\}\\
&\leq \inf\left\{\rho_1^{\frac{p_n}{H}}>0:\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1\right\}\\
&\quad+\inf\left\{\rho_2^{\frac{p_n}{H}}>0:\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})Y_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}}\leq 1\right\}\\
&= \|x\|+\|y\|.\end{aligned}$$

Let α be any scalar and define $r = \frac{\rho}{|\alpha|}$. Then, we have

$$\begin{aligned}
\|\alpha x\| &= \inf\left\{\rho^{\frac{p_n}{H}} > 0 : \left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})\alpha X_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1\right\} \\
&= \inf\left\{\rho^{\frac{p_n}{H}} > 0 : \left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\alpha||\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1\right\} \\
&= \inf\left\{r^{\frac{p_n}{H}}|\alpha| > 0 : \left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\alpha||\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1\right\} \\
&= |\alpha|\inf\left\{r^{\frac{p_n}{H}} > 0 : \left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\alpha||\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1\right\} \\
&= |\alpha|\|x\|.
\end{aligned}$$

(ii) Let (x^i) be a Cauchy sequence in $l_p^{\lambda}(M, B)$. Let $\delta > 0$ be fixed and $r > 0$ be given such that $0 < \varepsilon < 1$ and $r\delta \geq 1$. Then there exists a positive integer i_0 such that $\|x^i - x^j\| < \frac{\varepsilon}{r\delta}$ for all $i, j \geq i_0$, by applying the norm in (3.1) we have

$$\inf\left\{\rho^{\frac{p_n}{H}} > 0 : \left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1\right\} < \frac{\varepsilon}{r\delta} \text{ for all } i, j \geq i_0.$$

This implies that

$$\left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\|x^i - x^j\|\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1, \text{ for all } i, j \geq i_0.$$

i.e.,

$$\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\|x^i - x^j\|\lambda_n}\right)\right]^{p_n} \leq 1, \text{ for all } i, j \geq i_0 \text{ and for all } n \in \mathbb{N}.$$

i.e.,

$$\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\|x^i - x^j\|\lambda_n}\right)\right]^{p_n} \leq 1, \text{ for all } i, j \geq i_0 \text{ and for all } n \in \mathbb{N}.$$

For $r > 0$ choosing $M\left(\frac{r\delta}{2}\right) \geq 1$ we have

$$M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\|x^i - x^j\|\lambda_n}\right) \leq M\left(\frac{r\delta}{2}\right), \text{ for all } i, j \geq i_0 \text{ and for all } n \in \mathbb{N}.$$

Since M is non-decreasing, we have

$$\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\|x^i - x^j\|\lambda_n} \leq \frac{r\delta}{2}, \text{ for all } i, j \geq i_0 \text{ and for all } n \in \mathbb{N}.$$

i.e.,

$$\begin{aligned}\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j)|}{\lambda_n} &\leq \frac{r\delta}{2}\|x^i - x^j\|, \text{ for all } i, j \geq i_0 \text{ and for all } n \in \mathbb{N} \\ &\leq \frac{r\delta}{2}.\frac{\varepsilon}{r\delta} \\ &= \frac{\varepsilon}{2}.\end{aligned}$$

So,

$$|(\lambda_k - \lambda_{k-1})(X_k^i - X_k^j) \leq \frac{\varepsilon}{2} \text{ for all } i, j \geq i_0 \text{ and for all } n \in \mathbb{N}.$$

This implies, $\{(\lambda_k - \lambda_{k-1})X_k^i\}$ is a Cauchy sequence of scalars for all $k = 1, 2, 3, \ldots, n$ and for all $n \in \mathbb{N}$ and hence is convergent by the completeness of scalar field.

Now let $\lim_{i\to\infty}(\lambda_k - \lambda_{k-1})X_k^i = (\lambda_k - \lambda_{k-1})X_k$, for $k = 1, 2, 3, \ldots, n$ and for all $n \in \mathbb{N}$.

Let $j \to \infty$ and with the continuity of Orlicz function, we have $(x^i - x) \in l_p^\lambda(M, B)$ and $\inf\left\{\rho^{\frac{p_n}{H}} > 0 : \left[\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})(X_k^i - X_k)|}{\rho\lambda_n}\right)\right]^{p_n}\right]^{\frac{1}{H}} \leq 1\right\}$ for all $i \geq i_0$.

i.e.,$\|x^i - x\| \to 0$ as $i \to \infty$. Since $x^i \in l_p^\lambda(M, B)$, which is a linear space, this implies $x \in l_p^\lambda(M, B)$ and hence $l_p^\lambda(M, B)$ is a Banach space with respect to the norm defined by (3.1).

(iii) From the above proof we can easily conclude that $\|x^i\| \to 0$ as $i \to \infty$ implies that $x_n^i \to 0$ as $n \in \infty$ for each $i \in \mathbb{N}$.

Theorem 3.3. *For different Orlicz functions M_1 and M_2, the following statements hold:*

(i) $l_p^\lambda(M_1, B) \cap l_p^\lambda(M_2, B) \subseteq l_p^\lambda(M_1 + M_2, B)$ and

(ii) $l_p^\lambda(M_2, B) \subseteq l_p^\lambda(M_1, B)$ if $\sup_t \left[\frac{M_1(t)}{M_2(t)}\right] < \infty$.

Proof. (i) Let $x \in \ell_p^\lambda(M_1, B) \cap \ell_p^\lambda(M_2, B)$
Then there exists $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sum_{n=0}^{\infty}\left[M_1\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)\right]^{p_n} < \infty$$

and

$$\sum_{n=0}^{\infty}\left[M_2\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_2\lambda_n}\right)\right]^{p_n} < \infty.$$

Let $\rho = \max(\rho_1, \rho_2)$. Then, by Lemma 2.2 we have

$$\begin{aligned}
&\sum_{n=0}^{\infty}\left[(M_1+M_2)\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\\
&\le \sum_{n=0}^{\infty}\left[M_1\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)+M_2\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_2\lambda_n}\right)\right]^{p_n}\\
&\le D\left[\sum_{n=0}^{\infty}\left[M_1\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_1\lambda_n}\right)\right]^{p_n}+\sum_{n=0}^{\infty}\left[M_2\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho_2\lambda_n}\right)\right]^{p_n}\right]\\
&< \infty.
\end{aligned}$$

Which implies, $x \in \ell_p^\lambda(M_1 + M_2, B)$.
(ii)Let $x \in \ell_p^\lambda(M_2, B)$. Then there exists $\rho > 0$ such that

$$\sum_{n=0}^{\infty}\left[M_2\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n} < \infty.$$

Since $\sup_t\left[\frac{M_1(t)}{M_2(t)}\right] < \infty$, therefore there exists $\eta > 0$ such that,

$$\frac{M_1(t)}{M_2(t)} \le \eta \text{ for all } t \ge 0. \tag{3.3}$$

Replacing t by $\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}$ in (3.3), we get

$$M_1\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right) \le \eta M_2\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)$$

Thus for each $k \in \mathbb{N}$, we have

$$\begin{aligned}
\sum_{n=0}^{\infty}\left[M_1\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n} &\le \max(1,\eta^G)\sum_{n=0}^{\infty}\left[M_2\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}\\
&< \infty,
\end{aligned}$$

which implies that $x \in \ell_p^\lambda(M, B)$, where $G = \sup_k p_k$.

Theorem 3.4. *Let M and M_1 be two Orlicz functions. If M satisfies Δ_2-condition then $\ell_p^\lambda(M, B) \subseteq \ell_p^\lambda(M \circ M_1, B)$.*
Proof. Let $x \in \ell_p^\lambda(M_1, B)$. Then there exists $\rho > 0$ such that

$$\sum_{n=0}^{\infty} \left[M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n} < \infty$$

Case (i). Let $M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \leq 1$.
Then, using convexity of Orlicz function M

$$\sum_{n=0}^{\infty} \left[M \left(M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right) \right]^{p_n} \leq \sum_{n=0}^{\infty} \left[M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) M(1) \right]^{p_n}$$

$$\leq \max \left(1, [M(1)]^H\right) \sum_{n=0}^{\infty} \left[M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n} < \infty.$$

Case (ii). Let $M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) > 1$.
Then, by using Δ_2-condition of Orlicz function M,

$$\sum_{n=0}^{\infty} \left[M \left(M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right) \right]^{p_n} \leq \sum_{n=0}^{\infty} \left[K M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) M(1) \right]^{p_n}$$

$$\leq \max \left(1, [K M(1)]^H\right) \sum_{n=0}^{\infty} \left[M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n} < \infty.$$

From case(i) and case(ii), $\sum_{n=0}^{\infty} \left[M \left(M_1 \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right) \right]^{p_n} < \infty$.
Hence $x \in l_p^\lambda(M \circ M_1, B)$.

Theorem 3.5. *The space $l_p^\lambda(M, B)$ is not convergence free.*
Proof. The result follows from the following example.

Example 3.1. For $M(x) = x, \Lambda = I$, the identity matrix, $\tilde{r} = e, \tilde{s} = 0, \tilde{t} = 0, \tilde{u} = 0, p_n = 2$ for all $n \in \mathbb{N}$ and choose

$$x_k = \begin{cases} \frac{1}{k} & \text{when } k \neq 2^n \\ 0 & \text{when } k = 2^n \end{cases}$$

Then, $(x_k) \in l_2$. Now consider

$$y_k = \begin{cases} k & \text{when } k \neq 2^n \\ 0 & \text{when } k = 2^n \end{cases}$$

Then, $(y_k) \notin l_2$.

This implies the fact that the space $l_p^\lambda(M,B)$ is not convergence free.

Theorem 3.6. *The space $l_p^\lambda(M,B)$ is not symmetric.*
Proof. The result follows from the example given below.

Example 3.2. If we choose $\Lambda = I$, the identity matrix, $M(x) = x, \tilde{r} = e, \tilde{s} = -e, \tilde{t} = 0, \tilde{u} = 0, p_n = 2$ for all $n \in \mathbb{N}$ and $(x_n) = (\frac{1}{n})$, then $(x_n) \in l_2$. But If we consider the sequence $(y_n) = (x_1, x_5, x_8, x_{15}, x_{21}, \dots)$, then $(y_n) \notin l_2$.

Hence the space is not symmetric.

Theorem 3.7. *The space $l_p^\lambda(M,B)$ is normal.*
Proof. Let $x \in l_p^\lambda(M,B)$, i.e.,

$$\sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n} < \infty.$$

For a sequence of scalars $\alpha = (\alpha_k)$ such that $|\alpha_k| \le 1$ for all $k \in \mathbb{N}$, we have

$$\sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) \alpha_k X_k|}{\rho \lambda_n} \right) \right]^{p_n} \le \sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n}.$$

So, $l_p^\lambda(M,B)$ is a normal space.

Now for any Orlicz function M, we define

$$[l_p^\lambda(M,B)] = \left\{ x = (x_k) \in \omega : \sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n} < \infty, \text{ for every } \rho > 0 \right\}$$

Obviously, $[l_p^\lambda(M,B)]$ is a subspace of $l_p^\lambda(M,B)$.

Theorem 3.8. $[l_p^\lambda(M,B)]$ *is a complete normed linear space under the norm defined by (3.1).*
Proof. By using the step (ii) of Theorem 3.2, one can easily prove that $[l_p^\lambda(M,B)]$ is a complete normed linear space.

Theorem 3.9. *Let M be an Orlicz function. Then $[l_p^\lambda(M,B)]$ is an $AK-$space.*
Proof. Let $x = (x_n) \in l_p^\lambda(M,B)$. Therefore, for every $\rho > 0$,

$$\sum_{n=0}^{\infty} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\rho \lambda_n} \right) \right]^{p_n} \le 1.$$

Then for each $\varepsilon \in (0,1)$, we can find n_0 such that

$$\sum_{n \ge n_0} \left[M \left(\frac{|\sum_{k=0}^{n} (\lambda_k - \lambda_{k-1}) X_k|}{\varepsilon \lambda_n} \right) \right]^{p_n} \le 1. \tag{3.4}$$

Define the jth section $x^{[j]}$ of the sequence $x = (x_n)$ by $x^{[j]} = \sum_{n=0}^{j} x_n e^n$, where (e_n) is a Schauder basis for $[l_p^\lambda(M, B)]$. Hence, for $j \geq j_0$,

$$\begin{aligned}\|x - x^{[j]}\| &= \inf\left\{\rho > 0 : \sum_{n \geq j_0}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n} \leq 1\right\} \\ &\leq \inf\left\{\rho > 0 : \sum_{n \geq j}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n} \leq 1\right\}. \qquad (3.5)\end{aligned}$$

From (3.3) and (3.4), we get $\|x - x^{[j]}\| < \varepsilon$ for all $j \geq j_0$. Therefore, $[l_p^\lambda(M, B)]$ is an $AK-$space.

Theorem 3.10. *If an Orlicz function M satisfies the Δ_2- condition, then $l_p^\lambda(M, B) = [l_p^\lambda(M, B)]$.*

Proof. It is obvious that

$$[l_p^\lambda(M, B)] \subseteq l_p^\lambda(M, B). \qquad (3.6)$$

Now let $x = (x_n) \in l_p^\lambda(M, B)$ be any arbitrary element. Then there exists some $\rho > 0$ such that

$$\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n} < \infty.$$

Again let σ be any arbitrary number. Then two cases arise.

Case (i). If $\rho \leq \sigma$, then for each $n \in \mathbb{N}$,

$$\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\sigma\lambda_n}\right)\right]^{p_n} \leq \sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}.$$

i.e.,$x \in [l_p^\lambda(M, B)]$.

Case (ii). If $\rho > \sigma$, then $\frac{\rho}{\sigma} > 1$. From Δ_2-condition of Orlicz function, there exists a constant $k > 0$ such that

$$M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\sigma\lambda_n}\right) \leq \left(\frac{k\rho}{\sigma}\right) M\left(\frac{|\sum_{k=0}^{n}(\lambda_k - \lambda_{k-1})X_k|}{\rho\lambda_n}\right).$$

Consequently, for each $n \in \mathbb{N}$

$$\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\sigma\lambda_n}\right)\right]^{p_n} \leq \sum_{n=0}^{\infty}\left(\frac{k\rho}{\sigma}\right)^{p_n}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n}$$
$$\leq \sup_n\left\{\left(\frac{k\rho}{\sigma}\right)^{p_n}\right\}\sum_{n=0}^{\infty}\left[M\left(\frac{|\sum_{k=0}^{n}(\lambda_k-\lambda_{k-1})X_k|}{\rho\lambda_n}\right)\right]^{p_n} < \infty.$$

i.e.,$x \in [l_p^{\lambda}(M,B)]$.
Hence in both cases we have $x \in [l_p^{\lambda}(M,B)]$.

$$i.e., l_p^{\lambda}(M,B) \subseteq [l_p^{\lambda}(M,B)] \tag{3.7}$$

Combining (3.6) and (3.7), we get $l_p^{\lambda}(M,B) = [l_p^{\lambda}(M,B)]$.

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