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IMPROVEMENT OF THE RULE OF ARYABHATA IN THE CASE OF DIFFERENCES BETWEEN TWO PROJECTIONS OF CONSECUTIVE ARC DIVISIONS

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Abstract: In the present study we modify the value of differences between two projections of consecutive arc divisions mentioned in the work of A. A. Krishnaswami Ayyangar entitled "The Mathematics of Aryabhata". With the help of circumference-diameter ratio of Aryabhata, we will obtain two more corrected values. Also we will obtain the result with modern circumference-diameter ratio. We can find error percentage in three cases 0.14338558 %, 0.000467054 % and 0.14291853 % respectively.

Keywords and Phrases: Aryabhata, Mathematics, Arc, Chord, Projection.

2020 Mathematics Subject Classification: 01A32, 97A30, 01A11.

1. Introduction

A. A. Krishnaswami Ayyangar (1926) described the mathematics of Aryabhata in which he has given the rule of Aryabhata for the calculation of differences between two projections of consecutive arc divisions.

Let the quadrant AOB be formed by radii OA, OB and the arc AB. Let the arc AB be divided into 24 equal parts such that each part consists 3.75^o. Let us draw perpendiculars from three consecutive points of division arc AB on OB. Let this perpendicular are $A_{n-1}B_{n-1}, A_nB_n$ and $A_{n+1}B_{n+1}$ respectively. Therefore projections of consecutive arc divisions are $B_{n-1}B_n$ and A_nB_{n+1} respectively.

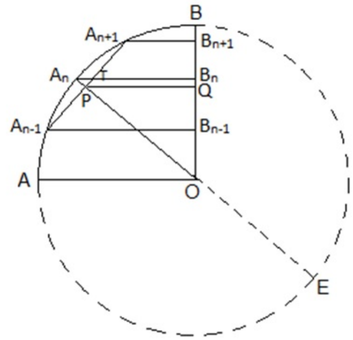


Figure-1.

$A_{n-1}A_{n+1}$ be a chord which intersects the radius OA_n at the point P. Let us draw a perpendicular PQ from the point P on OB.

With the help of Aryabhata's circumference-diameter ratio, Ayyangar (1926) has calculated as follows:

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2QB_n \\ &= \frac{2OB_n}{OA_n}PA_n = \frac{OB_n(A_nA_{n+1})^2}{OA_n} \\ &= \frac{OB_n}{225} \\ For \qquad \frac{OA_n}{A_nA_{n+1}} &= \frac{\text{radius}}{\frac{1}{96}\text{ circumference}} \quad \text{approximately} \\ &= \frac{960000}{62832} \quad \text{(Using Aryabhata's values)} \\ &= 15 \ \text{ to the nearest integer.} \end{aligned}$$

Here, we shall try to make some corrections of the above work.

2. Description for Correction

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= B_{n-1}Q + QB_n - (QB_{n+1} - QB_n) \\ &= B_{n-1}Q + QB_n - QB_{n+1} + QB_n \\ &= B_{n-1}Q + 2QB_n - QB_{n+1} \\ &= B_{n-1}Q + 2QB_n - QB_{n-1} \qquad (\because B_{n-1}Q = QB_{n+1}) \\ &= 2QB_n \qquad (1) \end{aligned}$$

Since, $\triangle OA_nB_n \sim \triangle PA_nT$

$$\therefore \quad \frac{OA_n}{PA_n} = \frac{OB_n}{QB_n}$$
$$\Rightarrow QB_n = \frac{OB_n \times PA_n}{OA_n} \tag{2}$$

Using the value of equation (2) in (1), we get

$$B_{n-1}B_n - B_nB_{n+1} = \frac{2OB_n}{OA_n}PA_n \tag{3}$$

From the intersection of two chords

$$PE \times PA_n = PA_{n-1} \times PA_{n+1}$$
$$\therefore \quad (OP + OE) \times PA_n = PA_{n-1} \times PA_{n+1}$$
$$\therefore \quad (OP + OA_n) \times PA_n = PA_{n-1} \times PA_{n+1}; \text{ where, } OE = OA_n = \text{radius}$$
$$\therefore \quad PA_n = \frac{(PA_{n+1})^2}{OP + OA_n} \qquad (\because PA_{n-1} = PA_{n+1})$$
$$\therefore \quad PA_n = \frac{(PA_{n+1})^2}{(OA_n - PA_n) + OA_n}$$
$$\therefore \quad \frac{PA_n}{1} = \frac{(PA_{n+1})^2}{2OA_n - PA_n}$$
$$\therefore \quad 2OA_n \times PA_n - (PA_n)^2 = (PA_{n+1})^2$$
$$\therefore \quad (PA_n)^2 - 2OA_nPA_n + (PA_{n+1})^2 = 0$$
$$\therefore \quad PA_n = \frac{-(-2OA_n) \pm \sqrt{(-2OA_n)^2 - 4 \times 1 \times (PA_{n+1})^2}}{2 \times 1}$$
$$= \frac{2OA_n \pm \sqrt{4(OA_n)^2 - 4(PA_{n+1})^2}}{2}$$
$$= \frac{2OA_n \pm \sqrt{4\{(OA_n)^2 - (PA_{n+1})^2\}}}{2}$$
$$= \frac{2OA_n \pm 2\sqrt{(OA_n)^2 - (PA_{n+1})^2}}{2}$$
$$= \frac{2\{OA_n \pm \sqrt{(OA_n)^2 - (PA_{n+1})^2}\}}{2}$$
$$= OA_n \pm \sqrt{(OA_n)^2 - (PA_{n+1})^2} \tag{4}$$

Now, using the value of equation (4) in (3), we get

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} =& \frac{2OB_n}{OA_n} \times \left[OA_n \pm \sqrt{(OA_n)^2 - (PA_{n+1})^2}\right] \\ =& \frac{2OB_n}{R} \times \left[R \pm \sqrt{R^2 - (PA_{n+1})^2}\right] ;\ \text{where}, R = OA_n = \text{radius}. \\ =& \frac{2OB_n}{R} \times \left[R \pm \sqrt{R^2 \left\{1 - \frac{(PA_{n+1})^2}{R^2}\right\}}\right] \\ =& \frac{2OB_n}{R} \times \left[R \pm R\sqrt{1 - \frac{(PA_{n+1})^2}{R^2}}\right] \\ =& \frac{2OB_n}{R} \times R\left[1 \pm \sqrt{1 - \frac{(PA_{n+1})^2}{R^2}}\right] \\ =& 2OB_n \times \left[1 \pm \sqrt{1 - \frac{(PA_{n+1})^2}{R^2}}\right] \end{aligned} \tag{5}$$

Case I. Taking $PA_{n+1} \approx A_nA_{n+1}$, and $Arc(A_nA_{n+1}) = \frac{1}{24}Arc(AB) = \frac{1}{4 \times 24}$ circumference, [$\because$ The quadrant has been divided by 24 equal parts.]
Equation (5) reduces to

$$B_{n-1}B_n - B_nB_{n+1} = 2OB_n \times \left[1 \pm \sqrt{1 - \frac{C^2}{96^2 \times R^2}}\right] \tag{6}$$

Aryabhata assumed a circle of diameter 20000 unit in which he declared that the circumference will be about 62832 unit. Using this result in (6), we get

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} =& 2OB_n \times \left[1 \pm \sqrt{1 - \frac{(62832)^2}{96^2 \times (10000)^2}}\right] \\ =& 2OB_n \times \left[1 \pm \sqrt{1 - \frac{(62832)^2}{(960000)^2}}\right] \\ =& 2OB_n \times \left[1 \pm \sqrt{1 - \left(\frac{62832}{960000}\right)^2}\right] \\ =& 2OB_n \times \left[1 \pm \sqrt{1 - (0.06545)^2}\right] \end{aligned}$$

$$
\begin{aligned}
&= 2OB_n \times \left[1 \pm \sqrt{1 - 0.004283702}\right] \\
&= 2OB_n \times \left[1 \pm \sqrt{0.9957163}\right] \\
&= 2OB_n \times [1 \pm 0.99785585] \qquad (7)
\end{aligned}
$$

Sub Case I (i). Considering positive value inside the bracket

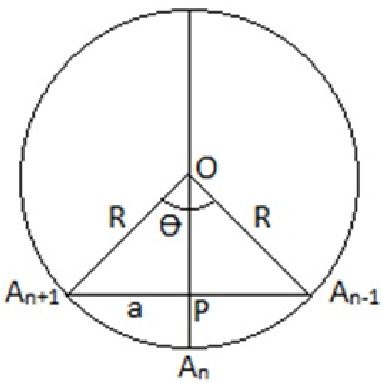
$$
\begin{aligned}
B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 + 0.99785585] \\
&= OB_n \times 2 \times 1.99785585 \\
&= OB_n \times 3.9957117; \quad \text{which is not possible} \qquad (8)
\end{aligned}
$$

Sub Case I (ii). Considering negative value inside the bracket

$$
\begin{aligned}
B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 - 0.99785585] \\
&= OB_n \times 2 \times 0.00214415 \\
&= OB_n \times 0.00428830 \qquad (9)
\end{aligned}
$$

Case II. Steps for more accuracy

Let, $\angle A_{n-1}OA_{n+1} = \theta$, therefore $\angle POA_{n+1} = \frac{\theta}{2}$



In $\triangle OPA_{n+1}$

$$\sin\frac{\theta}{2} = \frac{PA_{n+1}}{OA_{n+1}} = \frac{PA_{n+1}}{R} \qquad (10)$$

Since,

$$\theta = \frac{arc(A_{n+1}A_{n-1})}{R}$$

$$\therefore \quad \theta = \frac{2arc(A_nA_{n+1})}{R}$$

$$\therefore \quad \frac{\theta}{2} = \frac{2arc(A_nA_{n+1})}{2R}$$

$$\therefore \quad \frac{\theta}{2} = \frac{arc(A_nA_{n+1})}{R} \tag{11}$$

Using this value in the equation (10), we get

$$\therefore \ \sin\left(\frac{arc(A_nA_{n+1})}{R}\right) = \frac{PA_{n+1}}{R}$$

$$\therefore \ PA_{n+1} = R\sin\left(\frac{arc(A_nA_{n+1})}{R}\right)$$

$$= R\sin\left(\frac{C}{96\times R}\right); \text{ where C=circumference and R=Radius}$$

[∵ The quadrant has been divided by 24 equal parts.]
Using this value in the equation (5), we get

$$B_{n-1}B_n - B_nB_{n+1} = 2OB_n \times \left[1 \pm \sqrt{1 - \frac{\left(R\sin\left(\frac{C}{96\times R}\right)\right)^2}{R^2}}\right] \tag{12}$$

$$= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{R^2\left(\sin\left(\frac{62832}{96\times 10000}\right)\right)^2}{R^2}}\right], \text{using Aryabhata}$$

$$= 2OB_n \times \left[1 \pm \sqrt{1 - \left(\sin\left(\frac{62832}{960000}\right)\right)^2}\right]$$

$$= 2OB_n \times \left[1 \pm \sqrt{1 - (\sin(0.06545))^2}\right]$$

$$= 2OB_n \times \left[1 \pm \sqrt{1 - (0.06540328)^2}\right]$$

$$= 2OB_n \times \left[1 \pm \sqrt{1 - 0.004277589}\right]$$

$$= 2OB_n \times \left[1 \pm \sqrt{0.99572241}\right]$$

$$= 2OB_n \times [1 \pm 0.99785891] \tag{13}$$

Sub Case II (i). Considering positive value inside the bracket

$$B_{n-1}B_n - B_nB_{n+1} = 2OB_n \times [1 + 0.99785891]$$

$$= OB_n \times 2 \times 1.99785891$$
$$= OB_n \times 3.99571782; \quad \text{which is not possible} \tag{14}$$

Sub Case II (ii). Considering negative value inside the bracket

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 - 0.99785891] \\ &= OB_n \times 2 \times 0.00214109 \\ &= OB_n \times 0.00428218 \end{aligned} \tag{15}$$

Case III. To find the correct value by using modern circumference-diameter ratio

Since, $\dfrac{circumference}{diameter} = \pi$, therefore $\dfrac{C}{2R} = \pi$, where C = circumference and R = radius

$$\begin{aligned} \therefore \quad \frac{C}{R} &= 2\pi \\ \therefore \quad \frac{C}{R} &= 2 \times 3.14159265... \\ \therefore \quad \frac{C}{R} &= 6.2831853... \end{aligned} \tag{16}$$

Using this value in (6), we get

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{C^2}{96^2 \times R^2}}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{1}{96^2} \times (6.2831853...)^2}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{1}{9216} \times 39.4784175...}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{39.4784175...}{9216}}\right] \\ &= 2OB_n \times [1 \pm \sqrt{1 - 0.004283682...}] \\ &= 2OB_n \times \left[1 \pm \sqrt{0.99571632...}\right] \\ &= 2OB_n \times [1 \pm 0.99785586...] \end{aligned} \tag{17}$$

Sub Case III (i). Considering positive value inside the bracket

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 + 0.99785586...] \\ &= OB_n \times 2 \times 1.99785586... \\ &= OB_n \times 3.99571172... \quad \text{; which is not possible} \end{aligned} \tag{18}$$

Sub Case III (ii). Considering negative value inside the bracket

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 - 0.99785586...] \\ &= OB_n \times 2 \times 0.00214414... \\ &= OB_n \times 0.00428828... \end{aligned} \tag{19}$$

Case IV. Using the value of equation (16) in (12), we get-

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{\left(R\sin\left(\frac{1}{96} \times 6.2831853...\right)\right)^2}{R^2}}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{1 - \frac{R^2\left(\sin\left(0.06544985\right)\right)^2}{R^2}}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{1 - (0.06540313)^2}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{1 - 0.004277569}\right] \\ &= 2OB_n \times \left[1 \pm \sqrt{0.99572243}\right] \\ &= 2OB_n \times [1 \pm 0.99785892] \end{aligned} \tag{20}$$

Sub Case IV (i). Considering positive value inside the bracket

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 + 0.99785892...] \\ &= OB_n \times 2 \times 1.99785892... \\ &= OB_n \times 3.99571784... \quad \text{; which is not possible} \end{aligned} \tag{21}$$

Sub Case IV (ii). Considering negative value inside the bracket

$$\begin{aligned} B_{n-1}B_n - B_nB_{n+1} &= 2OB_n \times [1 - 0.99785892...] \\ &= OB_n \times 2 \times 0.00214108... \\ &= OB_n \times 0.00428216... \end{aligned} \tag{22}$$

3. But Ayyangar (1926) found

$$B_{n-1}B_n - B_nB_{n+1} = \frac{OB_n}{225} = OB_n \times \frac{1}{225} = OB_n \times 0.004444444 \quad (23)$$

4. We may show the value of the co-efficient of OB_n in different cases in Tables. Also table for the error percentage may be formed as follow.

Table I. Values of co-efficient of OB_n in different cases

Accurate Value in Case-I (Aryabhata)	Accurate Value in Case-II (Aryabhata)	Less accurate Value in Case-III (modern concept)	More accurate Value in Case-IV (modern concept)
0.00428830	0.00428218	0.00428828	0.00428216

Table- II. Error percentage in different cases

Case-I	Case-II	Case-III
0.14338558 %	0.000467054 %	0.14291853 %

5. Conclusion

As the literature of A. A. Krishnaswami Ayyangar the value of difference between two projections of consecutive arc divisions is $\frac{OB_n}{225} = OB_n \times 0.004444444$ but we have obtained the co-efficient of OB_n with the help of Aryabhata's circumference-diameter ratio. We have found two types of result which are 0.00428830 and 0.00428218 in which the later value is geometrically more accurate. With the help of modern circumference-diameter ratio the co-efficient OB_n are 0.00428828 and 0.00428216. We have found error percentage in three cases 0.14338558 %, 0.000467054 % and 0.14291853 % respectively by assuming 0.00428216 as modern correct value.

References

[1] Burgess, Ebenener., Translation of the Surya-Siddhanta : A Text Book of Hindu Astronomy, Originally published by Burgess and W. D. Whitney, Journal of the American Oriental Society, 6 (1960), pp. 141-498.

[2] Krishnaswami Ayyangar A. A., The Mathematics of Aryabhata, Quarterly Journal of the mythic society, vol. 16 (1926), pp. 165-166.

[3] Naraharayya M., Suryasiddhanta rule for the calculation of successive sine-differences, J. I. M. S., Vol. XV, No 7 (1924), pp. 107-108.

[4] Pandey Ram Chandra, Suryasiddhanta, Chaukhamba Surbharati Prakashan, Varanasi, (2006), 56-68.