South East Asian J. of Mathematics and Mathematical Sciences
Vol. 17, No. 1 (2021), pp. 95-114

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

A SIMPLE SOLUTION TO DIOPHANTINE EQUATIONS-FOURTH POWER

Wadhawan Narinder Kumar and Wadhawan Priyanka*

Civil Servant, Indian Administrative Service, Now Retired,
House No. 563, Sector 2, Panchkula-134112, Haryana, INDIA

E-mail : narinderkw@gmail.com

*Program Manager- Space Management (TCS),
Walgreen Co.304 Winter Road,
Deerfield, Il. 60015 USA

(**Received:** *Jul. 16, 2019* **Accepted:** *Nov. 18, 2020* **Published:** *Apr. 30, 2021)*

Abstract: In this research paper, a method has been devised to solve some Diophantine equations of fourth power. To begin with, an integer is expressed as an algebraic quantity, then utilising these algebraic quantities, a quartic Diophantine equation is written as an algebraic equation of fourth power with real and rational coefficients. The quartic is, then reduced to a linear equation that gives straightway solution. The process of reduction of the quartic to linear equation entails some conditions which are incorporated in the solution. Last, use of elementary and only elementary functions makes this paper easily comprehensible to scholars and students alike.

Keywords and Phrases: Integers, Rational Quantity, Quartic, Linear, Diophantine Equation.

2020 Mathematics Subject Classification: 11D25.

1. Introduction

A Diophantine equation of fourth power is traditionally written as $4.q.r$ equation where first figure 4 denotes its power, second figure q its number of terms in right hand side RHS or left hand side LHS whichever is less and third figure r its number of terms in right hand side RHS or left hand side LHS which is more than the other. That is $4.q.r$ equation is a Diophantine equation of fourth power with number of terms say in LHS as q and number of terms in RHS as r such that $q \leq r$. If a Diophantine equation of fourth power has equal number of terms in LHS and RHS say n, it is written as $4.n.n$. Symbol $\sum_{i=1}^{n} a_i$ used in this paper will mean sum of terms a_i when i varies from 1 to n, in other words, $\sum_{i=1}^{n} a_i = a_1 + a_2 + a_3 + ... + a_n$. Diophantine equations of fourth power i.e. 4.3.3, 4.4.4, 4.2.3, 4.3.4 and $4.n.n$ as given below will be solved and their parametrisation found.

$$A^4 + B^4 + C^4 = D^4 + E^4 + F^4,$$
$$A^4 + B^4 + C^4 + D^4 = E^4 + F^4 + G^4 + H^4,$$
$$A^4 + B^4 + C^4 = D^4 + E^4,$$
$$A^4 + B^4 + C^4 = D^4 + D^4 + (2D)^4,$$
$$A^4 + B^4 + C^4 = D^4 + E^4 + F^4 + G^4$$

and

$$A_1^4 + A_2^4 + A_3^4 + ... + A_n^4 = A_{n+1}^4 + A_{n+2}^4 + A_{n+3}^4 + ... + A_{2n}^4.$$

Such Diophantine equations have been dealt by Piezas and Weisstein [Piezas and Weisstein 7]. Ramanujan gave the equations

$$2^4 + 4^4 + 7^4 = 3^4 + 6^4 + 6^4,$$
$$3^4 + 7^4 + 8^4 = 1^4 + 2^4 + 9^4,$$

and

$$6^4 + 9^4 + 12^4 = 2^4 + 2^4 + 13^4$$

[Berndt and Bhargava 1; 1994, p. 101]. Similarly, examples were given by [Martin 6; (1896)]. Ramanujan also gave the general expression

$$3^4 + (2x^4 - 1)^4 + (4x^5 + x)^4 = (4x^4 + 1)^4 + (6x^4 - 3)^4 + (4x^5 - 5x)^4$$

[Berndt and Bhargava 1; 1994, p. 106]. Several formulas [Dickson 2; (2005, pp. 653-655)] have been cited giving solutions to this equation and a general formula [Haldeman 3; (1904)] was also given. Another identity given by Ramanujan is

$$(a + b + c)^4 + (b + c + d)^4 + (a - d)^4 = (c + d + a)^4 + (d + a + b)^4 + (b - c)^4,$$

where $a/b = c/d$ and 4 may also be replaced by 2 [Hirschhorn 4; (1998)].
Parametric solutions to the equation

$$A^4 + B^4 + C^4 = D^4 + E^4$$

are known. The smallest solution is

$$3^4 + 5^4 + 8^4 = 7^4 + 7^4$$

[Lander, Parkin, and Selfridge 5; (1967)]. Although solutions to some of the equations out of six, are known, all the six are being considered in the paper owing to the fact that the solutions provided by us, are easy to apply. To start with, a rational and real quantity say n can always be expressed as

$$n = a\cdot x + M, \tag{1.1}$$

where a, x and M are rational and real quantities which can be assigned rational real values as large as infinity to satisfy $n = a\cdot x + M$. For example, n say 5 can be written as

$$5 = 2(2) + 1 = 1(2) + 3 = 10 - 5(1) = \frac{1}{2}(3) + \frac{7}{2} = -\left\{\frac{5}{6}(-4) - \frac{5}{3}\right\} \text{ likewise}$$

Lemma 1.1. *A rational and real quantity n can always be expressed as algebraic equation that is, $n = a\cdot x + M$, where a, x and M are rational quantities. If a is fixed, x and M can have infinite rational real values that satisfy above said equation. If a and x are fixed, then M also gets fixed at one and only one value.*

If real and rational quantities A, B, C, D, E and F satisfy relation $A^4 + B^4 + C^4 = D^4 + E^4 + F^4$, then using Lemma 1.1, the relation can be written as

$$(x + a_1)^4 + (x + a_2)^4 + (x + a_3)^4 = (x + a_4)^4 + (x + a_5)^4 + (x + a_6)^4, \tag{1.2}$$

where a_1, a_2, a_3, a_4, a_5 and a_6 are real rational values. For solving this quartic equation, A, B, C, D, E and F are written using Lemma 1.1 so that terms containing x^4 may cancel other.

2. Theory and concept

After expansion, Equation (1.2) gets reduced to a cubic equation,

$$\begin{aligned} 4x^3(a_1 + a_2 + a_3 - a_4 - a_5 - a_6) + 6x^2(a_1^2 + a_2^2 + a_3^2 - a_4^2 - a_5^2 - a_6^2) \\ + 4x(a_1^3 + a_2^3 + a_3^3 - a_4^3 - a_5^3 - a_6^3) \\ = -(a_1^4 + a_2^4 + a_3^4 - a_4^4 - a_5^4 - a_6^4). \end{aligned} \tag{2.1}$$

If the above equation is solvable in terms of $a_1, a_2, a_3, ..., a_6$ and on putting that value of x in Equation (1.2) and after normalisation, integers satisfying equation (1.2) can always be found. Since our interest is to find integer solution and if x

so determined, were a fraction, then putting the values of x in equation (1.2) will yield fractions. Such fractions can be got ridden of by multiplying these with the lowest common multiplier abbreviated as LCM. In this paper, wherever the term normalisation appears, that will denote fraction multiplied with LCM to obtain integers.
To avoid solving tedious cubic or quadratic equations, the cubic equation is reduced to a linear equation by equating coefficients of x^3 and x^2 to zero. That is

$$(a_1 + a_2 + a_3 - a_4 - a_5 - a_6) = 0$$

and

$$(a_1^2 + a_2^2 + a_3^2 - a_4^2 - a_5^2 - a_6^2) = 0.$$

On eliminating a_6 from the above equations, a_1 can be determined,

$$a_1 = a_4 + a_5 - \frac{a_3a_2 - a_4a_5}{a_2 + a_3 - a_4 - a_5} \tag{2.2}$$

and from a_1, a_6 can be determined,

$$a_6 = a_2 + a_3 - \frac{a_3a_2 - a_4a_5}{a_2 + a_3 - a_4 - a_5}. \tag{2.3}$$

On substituting the values of a_1, a_6 in equation (2.1), it is reduced to linear equation,

$$4x(a_1^3 + a_2^3 + a_3^3 - a_4^3 - a_5^3 - a_6^3) = -(a_1^4 + a_2^4 + a_3^4 - a_4^4 - a_5^4 - a_6^4). \tag{2.4}$$

or

$$x = -\frac{(a_1^4 + a_2^4 + a_3^4 - a_4^4 - a_5^4 - a_6^4)}{4(a_1^3 + a_2^3 + a_3^3 - a_4^3 - a_5^3 - a_6^3)}. \tag{2.5}$$

Since a_1, a_6 can be determined as already discussed and a_2, a_3, a_4, a_5 are assigned real rational values by us, therefore, x can always be determined and hence the integers $(x + a_1)$ $(x + a_2)$, $(x + a_3)$, $(x + a_4)$, $(x + a_5)$ and $(x + a_6)$ satisfying the Equation (1.2).

Lemma 2.1. *$(x+a_1)^4+(x+a_2)^4+(x+a_3)^4$ always equals to $(x+a_4)^4+(x+a_5)^4+(x+a_6)^4$, when $a_1 = a_4 + a_5 - \frac{a_3a_2 - a_4a_5}{a_2 + a_3 - a_4 - a_5}$, $a_6 = a_2 + a_3 - \frac{a_3a_2 - a_4a_5}{a_2 + a_3 - a_4 - a_5}$ and $x = -\frac{(a_1^4 + a_2^4 + a_3^4 - a_4^4 - a_5^4 - a_6^4)}{4.(a_1^3 + a_2^3 + a_3^3 - a_4^3 - a_5^3 - a_6^3)}$, where a_2, a_3, a_4 and a_5 are real and rational quantities assigned.*
Based on Lemma 2.1, integers satisfying equation (1.2) are given in Table 2.1.

Table 2.1. Showing Integers Satisfying Equation (1.2)

S. No	a_2, a_3, a_4, a_5	Calculated $(a_1), (a_6), (x)$	Normalised $(x+a_1)^4+(x+a_2)^4+(x+a_3)^4 = (x+a_4)^4+(x+a_5)^4+(x+a_6)^4$
1	2,3,4,-5	-16/3,2/3,1/9	$47^4+19^4+28^4=37^4+44^4+7^4$
2	2,3,-4,5	-11/2,-3/2,1/6	$32^4+13^4+19^4=23^4+31^4+8^4$
3	2,-3,-4,5	8,6,-7/3	$17^4+1^4+16^4=19^4+8^4+11^4$
4	1,-1,3,4	36/7,-13/7,-12/7	$24^4+5^4+19^4=16^4+9^4+25^4$
5	1,-1,-3,4	12,11,-4	$8^4+5^4+3^4=7^4+7^4+0^4$
6	1,-1,-3,-6	-62/9,19/9,62/27	$124^4+89^4+35^4=19^4+100^4+119^4$
7	1,-1,-3,6	26/3,17/3,-26/9	$52^4+17^4+35^4=53^4+28^4+25^4$
8	0,-1,3,6	36/5,-14/5,-31/5	$77^4+31^4+46^4=14^4+59^4+73^4$
9	0,1,3,6	27/4,-5/4,-31/12	$50^4+31^4+19^4=5^4+41^4+46^4$
10	0,-1,3,-6	-12,-10,13/3	$23^4+13^4+10^4=22^4+5^4+17^4$
11	1,3,-5,-2	-70/11,51/11,26/33	$184^4+59^4+125^4=139^4+40^4+179^4$
12	1,3,-5,3	-34/7,15/7,2/7	$32^4+9^4+23^4=33^4+16^4+17^4$
13	1,-3,5,2	50/9,-31/9,-32/27	$118^4+5^4+113^4=103^4+22^4+125^4$
14	-1,3,5,2	22/5,-3/5,-32/15	$34^4+47^4+13^4=43^4+2^4+41^4$
15	0,-3,5,-2	14/3,-4/3,-5/3	$37^4+5^4+32^4=40^4+23^4+17^4$
16	0,2,3,4	23/5,-2/5,-11/15	$12^4+11^4+1^4=4^4+9^4+13^4$
17	0,-2,3,4	17/3,-10/3,-11/9	$40^4+11^4+29^4=16^4+25^4+41^4$
18	0,1,-3,-5	-19/3,8/3,16/9	$41^4+16^4+25^4=11^4+29^4+40^4$
19	1,-1,2,2	11/4,-5/4,-11/12	$22^4+1^4+23^4=13^4+13^4+26^4$
20	2,-2,-1,-1	1/2,5/2,-1/6	$2^4+11^4+13^4=7^4+7^4+14^4$

Next is solution to Diophantine equation $A^4+B^4+C^4+D^4=E^4+F^4+G^4+H^4$. Following the same procedure as detailed earlier, this equation can be written as

$$(x+a_1)^4+(x+a_2)^4+(x+a_3)^4+(x+a_4)^4 \\ =(x+a_5)^4+(x+a_6)^4+(x+a_7)^4+(x+a_8)^4. \tag{2.6}$$

On expansion,

$$4x^3(a_1+a_2+a_3+a_4-a_5-a_6-a_7-a_8) \\ +6x^2(a_1^2+a_2^2+a_3^2+a_4^2-a_5^2-a_6^2-a_7^2-a_8^2) \\ +4x(a_1^3+a_2^3+a_3^3+a_4^3-a_5^3-a_6^3-a_7^3-a_8^3) \\ =-(a_1^4+a_2^4+a_3^4+a_4^4-a_5^4-a_6^4-a_7^4-a_8^4). \tag{2.7}$$

For reducing equation (2.6) to a linear equation, we put

$$(a_1+a_2+a_3+a_4-a_5-a_6-a_7-a_8)=0$$

and

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2 - a_5^2 - a_6^2 - a_7^2 - a_8^2) = 0.$$

Elimination of a_8 yields

$$a_1 = a_5 + a_6 + a_7 - \frac{a_2a_3 + a_3a_4 + a_4a_2 - (a_5a_6 + a_6a_7 + a_7a_5)}{a_2 + a_3 + a_4 - a_5 - a_6 - a_7} \tag{2.8}$$

and since

$$a_8 = a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7,$$

therefore, putting value of a_1 in above equation, yields

$$a_8 = a_2 + a_3 + a_4 - \frac{a_2a_3 + a_3a_4 + a_4a_2 - (a_5a_6 + a_6a_7 + a_7a_5)}{a_2 + a_3 + a_4 - a_5 - a_6 - a_7}. \tag{2.9}$$

From Equation (2.7),

$$x = -\frac{a_1^4 + a_2^4 + a_3^4 + a_4^4 - a_5^4 - a_6^4 - a_7^4 - a_8^4}{4(a_1^3 + a_2^3 + a_3^3 + a_4^3 - a_5^3 - a_6^3 - a_7^3 - a_8^3)}. \tag{2.10}$$

Since a_1, a_8 can be determined as already discussed and a_2, a_3, a_4, a_5, a_6, and a_7 are real rational values assigned, therefore, x can always be determined and hence the integers satisfying the equation (2.6).

Lemma 2.2. $(x + a_1)^4 + (x + a_2)^4 + (x + a_3)^4 + (x + a_4)^4$ *always equals to* $(x + a_5)^4 + (x + a_6)^4 + (x + a_7)^4 + (x + a_8)^4$, *when*

$$a_1 = a_5 + a_6 + a_7 - \frac{a_2a_3 + a_3a_4 + a_4a_2 - (a_5a_6 + a_6a_7 + a_7a_5)}{a_2 + a_3 + a_4 - a_5 - a_6 - a_7},$$
$$a_8 = a_2 + a_3 + a_4 - \frac{a_2a_3 + a_3a_4 + a_4a_2 - (a_5a_6 + a_6a_7 + a_7a_5)}{a_2 + a_3 + a_4 - a_5 - a_6 - a_7}$$

and

$$x = -\frac{a_1^4 + a_2^4 + a_3^4 + a_4^4 - a_5^4 - a_6^4 - a_7^4 - a_8^4}{4(a_1^3 + a_2^3 + a_3^3 + a_4^3 - a_5^3 - a_6^3 - a_7^3 - a_8^3)},$$

where a_2, a_3, a_4, a_6 *and* a_7 *are real and rational quantities.*

Based on Lemma 2.2, integers satisfying Equation (2.7) are given in Table 2.2.

Table 2.2. Showing Integers Satisfying Equation (2.6)

$a_2, a_3, a_4,$ a_5, a_6, a_7	Calculated $(a_1), (a_8), (x)$	Normalised $(x+a_1)^4+(x+a_2)^4+(x+a_3)^4$ $+(x+a_4)^4=(x+a_5)^4+(x+a_6)^4$ $+(x+a_7)^4+(x+a_8)^4$
0,1,5,3,-3,-4	(-27/5),(-23/5),(106/45)	$137^4+106^4+151^4+331^4$ $=241^4+29^4+74^4+313^4$
1,3,5,7,9,-11	(-29),(-25),(255/29)	$586^4+284^4+342^4+400^4$ $=458^4+516^4+64^4+470^4$
0,1,-3,3,4,-4	(28/5),(3/5),(-82/85)	$394^4+82^4+3^4+337^4$ $=173^4+258^4+422^4+31^4$
0,-1,-3,4,3,-4	(40/7),(-9/7),(-94/91)	$426^4+94^4+185^4+367^4$ $=179^4+270^4+458^4+211^4$
-4,-1,-3,4,3,0	(112/15),(-113/15),(112/2655)	$19936^4+10508^4+2543^4+7853^4$ $=10732^4+8077^4+112^4+19889^4$
1,5,3,-3,-4,0	(-123/16),(133/16),(-1297/4464)	$35614^4+3167^4+21023^4+12095^4$ $=14689^4+19153^4+1297^4+35810^4$
1,-5,3,-3,-4,0	(-13/6),(23/6),(317/522)	$814^4+839^4+2293^4+1883^4$ $=1249^4+1771^4+317^4+2318^4$
-1,-5,3,-3,-4,0	(-3/4),(13/4),(115/76)	$58^4+39^4+265^4+343^4$ $=113^4+189^4+115^4+362^4$
-1,-5,3,-3,4,1	(8/5),(-17/5),(42/65)	$146^4+23^4+283^4+237^4$ $=153^4+302^4+107^4+179^4$
1,-5,3,-3,-4,-1	(-20/7),(29/7),(72/203)	$508^4+275^4+943^4+681^4$ $=537^4+740^4+131^4+913^4$

Next is solution to $A_1^4+A_2^4+A_3^4+\ldots+A_n^4=A_{n+1}^4+A_{n+2}^4+A_{n+3}^4+\ldots+A_{2n}^4$. Following the same procedure as explained earlier, this equation can be written as

$$(x+a_1)^4+(x+a_2)^4+(x+a_3)^4+\ldots+(x+a_n)^4$$
$$=(x+a_{n+1})^4+(x+a_{n+2})^4+(x+a_{n+3})^4+\ldots+(x+a_{2n})^4. \quad (2.11)$$

On expansion,

$$4x^3\{(a_1+a_2+a_3+\ldots+a_n)-(a_{n+1}+a_{n+2}+a_{n+3}+\ldots+a_{2n})\}$$
$$+6x^2\{(a_1^2+a_2^2+a_3^2+\ldots+a_n^2)-(a_{n+1}^2+a_{n+2}^2+a_{n+3}^2+\ldots+a_{2n}^2)\}$$
$$+4x\{(a_1^3+a_2^3+a_3^3+\ldots+a_n^3)-(a_{n+1}^3+a_{n+2}^3+a_{n+3}^3+\ldots+a_{2n}^3)\}$$
$$=-\{(a_1^4+a_2^4+a_3^4+\ldots+a_n^4)-(a_{n+1}^4+a_{n+2}^4+a_{n+3}^4+\ldots+a_{2n}^4)\}. \quad (2.12)$$

For reducing equation (2.12) to a linear equation,

$$\{(a_1 + a_2 + a_3 + ... + a_n) - (a_{n+1} + a_{n+2} + a_{n+3} + ... + a_{2n})\} = 0 \tag{2.13}$$

and

$$\{(a_1^2 + a_2^2 + a_3^2 + ... + a_n^2) - (a_{n+1}^2 + a_{n+2}^2 + a_{n+3}^2 + ... + a_{2n}^2)\} = 0. \tag{2.14}$$

Eliminating a_{2n} from equations (2.13) and (2.14) yield

$$a_1 = S' - \frac{(s_2 + s_3 + s_4 + ...s_{n-1}) - (s'_2 + s'_3 + s'_4 + ...s'_{n-1})}{S - S'} \tag{2.15}$$

or in mathematical notation,

$$a_1 = \sum_{i=n+1}^{2n-1} a_i - \frac{\sum_{i=2}^{n-1} s_i - \sum_{i=2}^{n-1} s'_i}{\sum_{i=2}^{n} a_i - \sum_{i=n+1}^{2n-1} a_i}$$

and

$$a_{2n} = S - \frac{(s_2 + s_3 + s_4 + ...s_{n-1}) - (s'_2 + s'_3 + s'_4 + ...s'_{n-1})}{S - S'} \tag{2.16}$$

or in mathematical notation,

$$a_{2n} = \sum_{i=2}^{n} a_i - \frac{\sum_{i=2}^{n-1} s_i - \sum_{i=2}^{n-1} s'_i}{\sum_{i=2}^{n} a_i - \sum_{i=n+1}^{2n-1} a_i}$$

where $S, S', s_2, s_3...s_{n-1}, s'_2 s'_3..s'_{n-1}$ are given by relations described below.

$$S = a_2 + a_3 + ... + a_n = \sum_{i=2}^{n} a_i, \quad S' = a_{n+1} + a_{n+2} + ... + a_{2n-1} = \sum_{i=n+1}^{2n-1} a_i,$$

$$s_2 = a_2(a_3 + a_4 + ...a_n) = a_2 \sum_{i=3}^{n} a_i, \;\; s'_2 = a_{n+1}(a_{n+2} + a_{n+3} + ...a_{2n-1}) = a_{n+1} \sum_{i=n+2}^{2n-1} a_i,$$

$$s_3 = a_3(a_4 + a_5 + ...a_n) = a_3 \sum_{i=4}^{n} a_i, \;\; s'_3 = a_{n+2}(a_{n+3} + a_{n+4} + ...a_{2n-1}) = a_{n+2} \sum_{i=n+3}^{2n-1} a_i,$$

$$s_4 = a_4(a_5 + a_6 + ...a_n) = a_4 \sum_{i=5}^{n} a_i, \;\; s'_4 = a_{n+3}(a_{n+4} + a_{n+5} + ...a_{2n-1}) = a_{n+3} \sum_{i=n+4}^{2n-1} a_i,$$

....

$$s_{n-2} = a_{n-2}(a_{n-1} + a_n) = a_{n-2} \sum_{i=n-1}^{n} a_i, s'_{n-2} = a_{n+3}(a_{n+4} + a_{2n-1}) = a_{2n-3} \sum_{i=2n-2}^{2n-1} a_i,$$

$$s_{n-1} = a_{n-1}.a_n, \qquad s'_{n-1} = a_{2n-2}.a_{2n-1}.$$

Coming to equation (2.12),

$$x = -\frac{(a_1^4 + a_2^4 + a_3^4 + ... + a_n^4) - (a_{n+1}^4 + a_{n+2}^4 + a_{n+3}^4 + ... + a_{2n}^4)}{4\{(a_1^3 + a_2^3 + a_3^3 + ... + a_n^3) - (a_{n+1}^3 + a_{n+2}^3 + a_{n+3}^3 + ... + a_{2n}^3)\}}$$
$$= -\frac{1}{4}.\frac{\sum_{i=1}^{n} a_i^4 - \sum_{i=n+1}^{2n} a_i^4}{\sum_{i=1}^{n} a_i^3 - \sum_{i=n+1}^{2n} a_i^3}, \tag{2.17}$$

since a_1, a_{2n} can be determined from above equations, therefore, x can always be determined from equation (2.17) and hence the integers satisfying the equation

$$A_1^4 + A_2^4 + A_3^4 + ... + A_n^4 = A_{n+1}^4 + A_{n+2}^4 + A_{n+3}^4 + ... + A_{2n}^4.$$

Lemma 2.3. $(x+a_1)^4+(x+a_2)^4+(x+a_3)^4+...+(x+a_n)^4$ *always equals to* $(x+a_{n+1})^4+(x+a_{n+2})^4+(x+a_{n+3})^4+...+(x+a_{2n})^4$, *when* a_1 *and* a_{2n} *are given by equations (2.15) and (2.16), x by equation* $x = -\frac{(a_1^4+a_2^4+a_3^4+...+a_n^4)-(a_{n+1}^4+a_{n+2}^4+a_{n+3}^4+...+a_{2n}^4)}{4\{(a_1^3+a_2^3+a_3^3+...+a_n^3)-(a_{n+1}^3+a_{n+2}^3+a_{n+3}+...+a_{2n}^3)\}}$ *, where* $a_2, a_3, a_4, ...a_{2n-1}$ *are real and rational quantities.*

Based on Lemma 2.3, integers satisfying equation (2.11) are given in Table 2.3.

Table 2.3. Showing Integers Satisfying Equation (2.11)

$2n$	$a_2, a_3, a_4, ..., a_{2n-1}$	a_1, a_{2n}, x	Normalised $A_1^4 + A_2^4 + A_3^4 + ... + A_n^4 = A_{n+1}^4 + A_{n+2}^4 + A_{n+3}^4 + ... + A_{2n}^4$
10	0,1,2,3,5 -3,-2,-1	-37/7,12/7,5/28	$143^4 + 5^4 + 33^4 + 61^4 + 89^4 = 145^4 + 79^4 + 51^4 + 53^4 + 23^4$
12	0,1,2,3,4,5, -4,-3,-2,-1	-25/3,20/3,25/36	$275^4 + 25^4 + 61^4 + 97^4 +133^4 + 169^4 = 205^4 + 119^4 + 83^4 + 47^4 +11^4 + 265^4$
14	0,1,2,3,4,5,6, -5,-4,-3,-2,-1	-51/4,45/4,81/116	$1398^4 + 81^4 + 197^4 + 313^4 +429^4 + 545^4 + 661^4 = 777^4 + 35^4 + 151^4 + 267^4 +383^4 + 499^4 + 1386^4$
16	0,1,2,3,4,5,6,7, -6,-5,-4,-3,-2,-1	-91/5,84/5,98/145	$2541^4 + 98^4 + 243^4 + 388^4 +533^4 + 678^4 + 823^4 + 968^4 = 1113^4 + 772^4 + 627^4 + 482^4 +337^4 + 192^4 + 47^4 + 2534^4$
18	0,1,2,3,4,5,6,7,8, -7,-6,-5,-4,-3,-2,-1	-74/3,70/3,100/153	$3674^4 + 100^4 + 253^4 + 406^4 + 559^4 +712^4 + 865^4 + 1018^4 + 1171^4 = 1324^4 + 971^4 + 818^4 + 665^4 +512^4 + 359^4 + 206^4 + 53^4 + 3670^4$

2.1. Parametrisation

Parameterisations has already been given by Lemmas 2.1, 2.2 and 2.3, however, parametrisation will be further discussed and more details will be given hereinafter. Reiterating, when $A^4 + B^4 + C^4 = D^4 + E^4 + F^4$, it is represented by Equation (2.1), where a_1, a_6 and x are given by equations (2.2), (2.3) and (2.5). Let $a_3 = a_2\alpha, a_4 = a_2\beta, a_5 = a_2\gamma$, where α, β, γ are real and rational quantities. On putting these values in equation (2.2) and (2.3) and after simplifying,

$$a_1 = a_2\left(\beta + \gamma - \frac{\alpha - \beta\gamma}{1 + \alpha - \beta - \gamma}\right)$$

and

$$a_6 = a_2\left(1 + \alpha - \frac{\alpha - \beta\gamma}{1 + \alpha - \beta - \gamma}\right).$$

Putting above said values in equation (2.5),

$$x = -a_2\frac{\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}.$$

Since a_1, a_6 and x can be determined from above said equations, therefore, on putting these in Equation (1.2) and cancelling a_2 that appears in LHS and RHS, we get

$$\Bigg[\left(\beta + \gamma - \frac{\alpha - \beta\gamma}{1 + \alpha - \beta - \gamma}\right)$$
$$- \frac{\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}\Bigg]^4$$
$$+ \left[1 - \frac{\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}\right]^4$$
$$+ \left[\alpha - \frac{\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta + \gamma - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1 + \alpha - \frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}\right]^4$$

$$= \left[\beta - \frac{\left\{\left(\beta+\gamma-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta+\gamma-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}\right]^4$$
$$+ \left[\gamma - \frac{\left\{\left(\beta+\gamma-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta+\gamma-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}\right]^4$$
$$+ \left[\left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right) - \frac{\left\{\left(\beta+\gamma-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4 + 1 + \alpha^4 - \beta^4 - \gamma^4 - \left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^4\right\}}{4\left\{\left(\beta+\gamma-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3 + 1 + \alpha^3 - \beta^3 - \gamma^3 - \left(1+\alpha-\frac{\alpha-\beta\gamma}{1+\alpha-\beta-\gamma}\right)^3\right\}}\right]^4 . \quad (2.18)$$

On the basis of this parametrisation, integers satisfying Equation (2.18) are given in Table 2.4.

Table 2.4. Integers Satisfying Equation (2.18)

α	β	γ	Normalised $(x+a_1)^4+(x+a_2)^4+(x+a_3)^4 = (x+a_4)^4+(x+a_5)^4+(x+a_6)^4$
-2	3	4	$46^4+41^4+5^4 = 19^4+31^4+50^4$
-2	-3	4	$13^4+11^4+2^4 = 14^4+7^4+7^4$
-2	-3	-4	$25^4+1^4+26^4 = 10^4+19^4+29^4$
3	-2	4	$22^4+17^4+5^4 = 13^4+23^4+10^4$
-3	-2	-4	$6^4+5^4+11^4 = 1^4+9^4+10^4$
5	-3	2	$26^4+27^4+1^4 = 29^4+6^4+23^4$
4	-3	-2	$73^4+59^4+14^4 = 46^4+31^4+77^4$
-5	3	2	$104^4+33^4+29^4 = 83^4+56^4+139^4$
5	3	-2	$14^4+17^4+3^4 = 7^4+18^4+11^4$
3	5	2	$16^4+1^4+17^4 = 19^4+8^4+11^4$

Parametric solution given by equation (2.18) is difficult to remember, therefore, it needs simplification. Let $\alpha = \beta + \gamma$, then equation (2.18) is simplified to

$$\left[\beta\gamma - \frac{(\beta+1)(\gamma+1)}{3}\right]^4 + \left[1 - \frac{(\beta+1)(\gamma+1)}{3}\right]^4 + \left[\beta+\gamma - \frac{(\beta+1)(\gamma+1)}{3}\right]^4$$
$$= \left[\beta - \frac{(\beta+1)(\gamma+1)}{3}\right]^4 + \left[\gamma - \frac{(\beta+1)(\gamma+1)}{3}\right]^4 + \left[1+\beta\gamma - \frac{(\beta+1)(\gamma+1)}{3}\right]^4. \tag{2.19}$$

$$[3\beta\gamma - (\beta+1)(\gamma+1)]^4 + [3 - (\beta+1)(\gamma+1)]^4 + [3(\beta+\gamma) - (\beta+1)(\gamma+1)]^4$$
$$= [3\beta - (\beta+1)(\gamma+1)]^4 + [3\gamma - (\beta+1)(\gamma+1)]^4$$
$$+ [3(1+\beta\gamma) - (\beta+1)(\gamma+1)]^4. \tag{2.20}$$

$$[2\beta\gamma - \beta - \gamma - 1]^4 + [2 - \beta\gamma - \beta - \gamma]^4 + [2\beta + 2\gamma - \beta\gamma - 1]^4$$
$$= [2\beta - \gamma - \beta\gamma - 1]^4 + [2\gamma - \beta\gamma - \beta - 1]^4 + [2 + 2\beta\gamma - \beta - \gamma]^4. \tag{2.21}$$

Based on this parametrisation, integers satisfying equation (2.21) are given in Table 2.5.

Table 2.5. Integers Satisfying Equation (2.21)

β	γ	$[2\beta\gamma - \beta - \gamma - 1]^4 + [2 - \beta\gamma - \beta - \gamma]^4 + [2\beta + 2\gamma - \beta\gamma - 1]^4 = [2\beta - \gamma - \beta\gamma - 1]^4 + [2\gamma - \beta\gamma - \beta - 1]^4 + [2 + 2\beta\gamma - \beta - \gamma]^4$
-2	3	$14^4 + 7^4 + 7^4 = 2^4 + 13^4 + 11^4$
-2	-3	$16^4 + 1^4 + 17^4 = 8^4 + 11^4 + 19^4$
-3	-4	$30^4 + 3^4 + 27^4 = 15^4 + 18^4 + 33^4$
-3	4	$26^4 + 13^4 + 13^4 = 1^4 + 22^4 + 23^4$
-5	4	$40^4 + 23 + 17^4 = 5^4 + 32^4 + 37^4$
-5	-4	$48^4 + 9^4 + 39^4 = 24^4 + 27^4 + 51^4$
5	-4	$42^4 + 21^4 + 21^4 = 33^4 + 6^4 + 39^4$
-2	-5	$26^4 + 1^4 + 25^4 = 10^4 + 19^4 + 29^4$
-2	-6	$31^4 + 2^4 + 29^4 = 11^4 + 23^4 + 34^4$
-3	-6	$44^4 + 7^4 + 37^4 = 19^4 + 28^4 + 47^4$

Next is solution to $A^4 + B^4 + C^4 = D^4 + E^4$. In this equation, sixth term F^4 is zero, therefore, equating term say $2\beta - \beta\gamma - \gamma - 1$ to zero gives

$$\beta = \frac{1+\gamma}{2-\gamma}.$$

Putting this value of β in equation (2.21), it gets reduced to

$$(\gamma^2-1)^4+(1-2\gamma)^4+(2\gamma-\gamma^2)^4=(\gamma-\gamma^2-1)^4+(1-\gamma+\gamma^2)^4.$$

Or

$$(\gamma^2-1)^4+(1-2\gamma)^4+(2\gamma-\gamma^2)^4=(1-\gamma+\gamma^2)^4+(1-\gamma+\gamma^2)^4. \qquad (2.22)$$

since even power of negative quantity is positive, $(\gamma-\gamma^2-1)^4$ can be written as $(1-\gamma+\gamma)^4$. Integers satisfying equation (2.22) are listed in Table 2.6.

Table 2.6. Showing Integers Satisfying Equation (2.22)

γ	$(\gamma^2-1)^4+(1-2\gamma)^4+(2\gamma-\gamma^2)^4$ $=(1-\gamma-\gamma^2)^4+(1-\gamma+\gamma^2)^4$	γ	$(\gamma^2-1)^4+(1-2\gamma)^4+(2\gamma-\gamma^2)^4$ $=(1-\gamma-\gamma^2)^4+(1-\gamma+\gamma^2)^4$
3	$8^4+5^4+3^4=7^4+7^4$	13	$168^4+25^4+143^4=157^4+157^4$
4	$15^4+7^4+8^4=13^4+13^4$	14	$195^4+27^4+168^4=183^4+183^4$
5	$24^4+9^4+15^4=21^4+21^4$	15	$224^4+29^4+195^4=211^4+211^4$
6	$35^4+11^4+24^4=31^4+31^4$	16	$255^4+31^4+224^4=241^4+241^4$
7	$48^4+13^4+35^4=43^4+43^4$	17	$288^4+33^4+255^4=273^4+273^4$
8	$63^4+15^4+48^4=57^4+57^4$	18	$323^4+35^4+288^4=307^4+307^4$
9	$80^4+17^4+63^4=73^4+73^4$	19	$360^4+37^4+323^4=343^4+343^4$
10	$99^4+19^4+80^4=91^4+91^4$	20	$399^4+39^4+360^4=381^4+381^4$
11	$120^4+21^4+99^4=111^4+111^4$	21	$440^4+41^4+399^4=421^4+421^4$
12	$143^4+23^4+120^4=133^4+133^4$	22	$483^4+43^4+440^4=463^4+463^4$

Next is parametric solution to equation $A^4+B^4+C^4=D^4+D^4+(2D)^4$. When $\gamma=\beta$, Equation (2.21) transforms to

$$(2\gamma^2-2\gamma-1)^4+(2-\gamma^2-2\gamma)^4+(4\gamma-\gamma^2-1)^4$$
$$=(\gamma-\gamma^2-1)^4+(\gamma-\gamma^2-1)^4+(2+2\gamma^2-2\gamma)^4. \qquad (2.23)$$

Based on this equation, integers satisfying it are given in Table 2.7.

Table 2.7. Showing Integers Satisfying Equation (2.23)

γ	$(2\gamma^2-2\gamma-1)^4+(2-\gamma^2-2\gamma)^4$ $+(4\gamma-\gamma^2-1)^4$ $=(\gamma-\gamma^2-1)^4+(\gamma-\gamma^2-1)^4$ $+(2+2\gamma^2-2\gamma)^4$ or $A^4+B^4+C^4=D^4+D^4+(2D)^4$	γ	$(2\gamma^2-2\gamma-1)^4+(2-\gamma^2-2\gamma)^4$ $+(4\gamma-\gamma^2-1)^4$ $=(\gamma-\gamma^2-1)^4+(\gamma-\gamma^2-1)^4$ $+(2+2\gamma^2-2\gamma)^4$ or $A^4+B^4+C^4=D^4+D^4+(2D)^4$
3	$11^4+13^4+2^4=7^4+7^4+14^4$	11	$219^4+141^4+78^4=111^4+111^4+222^4$
4	$23^4+22^4+1^4=13^4+13^4+26^4$	12	$263^4+166^4+97^4=133^4+133^4+266^4$
5	$39^4+33^4+6^4=21^4+21^4+42^4$	13	$311^4+193^4+118^4=157^4+157^4+314^4$
6	$59^4+46^4+13^4=31^4+31^4+62^4$	14	$363^4+222^4+141^4=183^4+183^4+366^4$
7	$83^4+61^4+22^4=43^4+43^4+86^4$	15	$419^4+253^4+166^4=211^4+211^4+422^4$

8	$111^4+78^4+33^4=57^4+57^4+114^4$	16	$479^4+286^4+193^4=241^4+241^4+482^4$
9	$143^4+97^4+46^4=73^4+73^4+146^4$	17	$543^4+321^4+222^4=273^4+273^4+546^4$
10	$179^4+118^4+61^4=91^4+91^4+182^4$	18	$611^4+358^4+253^4=307^4+307^4+614^4$

Next is $A^4+B^4+C^4=D^4+E^4+F^4+G^4$. Equations (2.22) and (2.23) are true for all rational values of γ and are, in fact, universal identities, therefore, on subtracting identity (2.22) from identity (2.23), resultant identity is parametric solution for the desired equation. Given below is the parametrisation.

$$(2\gamma^2-2\gamma-1)^4+(2-\gamma^2-2\gamma)^4+(4\gamma-\gamma^2-1)^4 = (2+2\gamma^2-2\gamma)^4+(\gamma^2-1)^4+(1-2\gamma)^4+(2\gamma-\gamma^2)^4. \quad (2.24)$$

Based on this equation, integers satisfying it are given in Table 2.8.

Table 2.8. Showing Integers Satisfying Equation (2.24)

γ	$(2\gamma^2-2\gamma-1)^4+(2-\gamma^2-2\gamma)^4$ $+(4\gamma-\gamma^2-1)^4$ $=(2+2\gamma^2-2\gamma)^4+(\gamma^2-1)^4$ $+(1-2\gamma)^4+(2\gamma-\gamma^2)^4$ or $A^4+B^4+C^4=D^4+E^4+F^4+G^4$	γ	$(2\gamma^2-2\gamma-1)^4+(2-\gamma^2-2\gamma)^4$ $+(4\gamma-\gamma^2-1)^4$ $=(2+2\gamma^2-2\gamma)^4+(\gamma^2-1)^4$ $+(1-2\gamma)^4+(2\gamma-\gamma^2)^4$ or $A^4+B^4+C^4=D^4+E^4+F^4+G^4$
3	$11^4+13^4+2^4$ $=14^4+8^4+5^4+3^4$	11	$219^4+141^4+78^4$ $=222^4+120^4+21^4+99^4$
4	$23^4+22^4+1^4$ $=26^4+15^4+8^4+7^4$	12	$263^4+166^4+97^4$ $=266^4+143^4+23^4+120^4$
5	$39^4+33^4+6^4$ $=42^4+24^4+9^4+15^4$	13	$311^4+193^4+118^4$ $=314^4+168^4+25^4+143^4$
6	$59^4+46^4+13^4$ $=62^4+35^4+11^4+24^4$	14	$363^4+222^4+141^4$ $=366^4+195^4+27^4+168^4$
7	$83^4+61^4+22^4$ $=86^4+48^4+13^4+35^4$	15	$419^4+253^4+166^4$ $=422^4+224^4+29^4+195^4$
8	$111^4+78^4+33^4$ $=114^4+63^4+15^4+48^4$	16	$479^4+286^4+193^4$ $=482^4+255^4+31^4+224^4$
9	$143^4+97^4+46^4$ $=146^4+80^4+17^4+63^4$	17	$543^4+321^4+222^4$ $=546^4+288^4+33^4+255^4$
10	$179^4+118^4+61^4$ $=182^4+99^4+19^4+80^4$	18	$611^4+358^4+253^4$ $=614^4+323^4+35^4+288^4$

Parametrisation for the above said equation is highlighted in Lemma 2.2, it is taken up to elaborate it further. Equations (2.8) and (2.9) give values of a_1 and a_8 in terms of $a_1, a_2, a_3, a_4, a_5, a_6$. Proceeding further, let $a_3=\alpha a_2$, $a_4=a_2\beta$,

$a_5 = a_2\gamma$, $a_6 = a_2\lambda$, $a_7 = a_2\mu$. On putting these values in equations (2.8) and after simplification,

$$a_1 = \mu + \lambda + \gamma + \frac{\lambda\mu + \gamma\lambda + \gamma\mu - \beta - \alpha\beta - \alpha}{1 + \alpha + \beta - \gamma - \lambda - \mu}.$$

Following the same procedure,

$$a_8 = 1 + \alpha + \beta + \frac{\lambda\mu + \gamma\lambda + \gamma\mu - \beta - \alpha\beta - \alpha}{1 + \alpha + \beta - \gamma - \lambda - \mu}.$$

On putting these values in equation (2.10),

$$x = -\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}} \tag{2.25}$$

As explained in Lemma 2.2, putting values of $x, a_1 \ldots a_8$ in equation (2.6), we obtain parametric

$$\left[\gamma+\lambda+\mu+\frac{\gamma\lambda+\lambda\mu+\mu\gamma-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$+\left[1-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$+\left[\alpha-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$+\left[\beta-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$=\left[\gamma-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$+\left[\lambda-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$+\left[\mu-\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]^4$$

$$\left[1+\alpha+\beta+\frac{\mu\lambda+\lambda\mu+\mu\gamma+\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}+\right.$$

$$\left.\frac{\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4+1+\alpha^4+\beta^4-\gamma^4-\lambda^4-\mu^4-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^4\right\}}{4\left\{\left(\mu+\lambda+\gamma+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3+1+\alpha^3+\beta^3-\gamma^3-\lambda^3-\mu^3-\left(1+\alpha+\beta+\frac{\lambda\mu+\gamma\lambda+\gamma\mu-\beta-\alpha\beta-\alpha}{1+\alpha+\beta-\gamma-\lambda-\mu}\right)^3\right\}}\right]. \tag{2.26}$$

On the basis of this parametric, integers satisfying this equation have already been listed in Table 2.2 and are not again computed here.

3. Results and Conclusions

A real and rational quantity say n is always expressible as $n = a\cdot x + M$, where a, x and M are rational and real quantities Therefore, integers A_1, A_2, A_3, ... A_{2n} that satisfy the equation $A_1^4 + A_2^4 + A_3^4 + ... + A_n^4 = A_{n+1}^4 + A_{n+2}^4 + A_{n+3}^4 + ... + A_{2n}^4$ are also expressible as $(x + a_1)$, $(x + a_2)$, $(x + a_3)$, ...,$(x + a_{2n})$ and accordingly, equation $A_1^4 + A_2^4 + A_3^4 + ... + A_n^4 = A_{n+1}^4 + A_{n+2}^4 + A_{n+3}^4 + ... + A_{2n}^4$ can be written as $(x+a_1)^4+(x+a_2)^4+(x+a_3)^4+...+(x+a_n)^4 = (x+a_{n+1})^4+(x+a_{n+2})^4+(x+a_{n+3})^4+ ... + (x + a_{2n}^4$. Since it has equal terms in left hand side as well as in right hand side, therefore, terms containing x^4 cancel reducing it to a cubic equation. Roots of this cubic equation will give values of x, values of $a_1, a_2, a_3, ...a_{2n}$ are known since these are assigned or calculated by us, therefore, $(x + a_1), (x + a_2), (x + a_3), ...(x + a_{2n})$ will be known and hence solution of Diophantine quartic equation.

However, solving a cubic equation is tedious and cumbersome. Even solving a quadratic equation is cumbersome particularly, when our requirement is to have only and only real and rational roots whereas a linear equation with real and rational coefficients, yields straightway the result. Thus need is to reduce the cubic equation (2.12) into a linear equation. That is only possible, when coefficients of terms containing x^3 and x^2 are zero. If such coefficients are equated to zero that culminates into putting conditions on $a_1, a_2, a_3, ..., a_{2n}$. Equating coefficients of terms containing x^3 and x^2 to zero, two conditions are required to be imposed amongst $a_1, a_2, a_3, ..., a_{2n-1}$. In this research paper, conditions are put on a_1, a_{2n} which are made dependent upon remaining assigned $a_2, a_3, ..., a_{2n-1}$ by relations (2.15) and (2.16). That means, $a_2, a_3, ..., a_{2n-1}$ can be assigned real and rational values, however, values of a_1 and a_{2n} will depend upon the remaining by relations (2.15) and (2.16). After imposing conditions on a_1 and a_{2n}, the cubic is reduced to a linear equation. That is

$$x = -\frac{(a_1^4 + a_2^4 + a_3^4 + ... + a_n^4) - (a_{n+1}^4 + a_{n+2}^4 + a_{n+3}^4 + ... + a_{2n}^4)}{4\{(a_1^3 + a_2^3 + a_3^3 + ... + a_n^3) - (a_{n+1}^3 + a_{n+2}^3 + a_{n+3} + ... + a_{2n}^3)\}}.$$

Now $x, a_2, a_3...a_{2n-1}$ are known, a_1 and a_{2n} are calculated from relations (2.15) and (2.16), therefore, $(x + a_1)^4 + (x + a_2)^4 + (x + a_3)^4 + ... + (x + a_n)^4 = (x + a_{n+1})^4 + (x + a_{n+2})^4 + (x + a_{n+3})^4 + ... + (x + a_{2n})^4$ is known.
Moot question that arises is whether this concept and theory as detailed in this paper are applicable to all Diophantine equations of power four, we are restricting to power four as the paper pertains to equation with power four. Broadly speaking it is applicable but at the same time it needs further examination for its conclusiveness. Prerequisite for application of this concept is, $A_1^4 + A_2^4 + A_3^4 + ... + A_n^4 = B_1^4 + B_2^4 + B_3^4 + ...B_m^4$ i.e. $4.m.n$ equations considering $m < n$, should be reducible to a linear equation. That requires representation of $A_1, A_2...A_n$, and $B_1, B_2...B_n$ in algebraic form as $(l_1x + a_1), (l_2x + a_2), ..., (l_nx + a_n)$ and $(l_{n+1}x + a_{n+1})$, $(l_{n+2}x + a_{n+2})$, ..., $(l_{n+m}x + a_{n+m})$ respectively where $l_1, l_2, l_3, ..., l_(n + m)$ are real rational quantities. In that case,

$$l_1^4 + l_2^4 + l_3^4 + ... + l_n^4 = l_{n+1}^4 + l_{n+2}^4 + l_{n+3}^4 + ... + l_{n+m}^4.$$

We call above equation a seed equation. This concept we have applied to Diophantine equation of third power, we have first generated such seed equations and, then used these for solving Diophantine equations of third power [Wadhawan and Wadhawan 8; (2019)]. Coming to Diophantine equation of fourth power, another requirement is that coefficients required to be equated to zero, must yield linear equations. That is relations of a_1 and a_{2n} with remaining must be a linear. If these were found to be quadratic for a_1 or a_{2n} or both, then that requires solutions of these quadratic equations. Gain in reduction of cubic equation in variable x to a linear equation will be annulled by loss in solving resultant quadratic equation in variables for a_1 or a_{2n}. That requires ones innovative mind to find ways to avoid such situation. If in stead of integers, the terms of this equation on calculation, are found fractions, these are converted to integers by multiplication with Lowest Common Multiplier. General solution of quartic equation can be further simplified by imposing some condition on its variables. For example, in the paper, Equation (2.18) was imposed such condition that $\alpha = \beta + \gamma$ with the purpose of simplifying it to Equation (2.19) which is easy to handle and remember. Imposing condition that vanishes a term of Diophantine equation will yield parametric for a new equation. In this paper, imposition of the condition $\beta = \frac{1+\gamma}{2-\gamma}$ yielded parametric Equation

(2.23) for $A^4 + B^4 + C^4 = D^4 + E^4$ vanishing F that was equalled to zero. Similarly imposition of condition $\gamma = \beta$, parametric Equation (2.21) transformed to Equation (2.23) which is applicable to $A^4 + B^4 + C^4 = D^4 + D^4 + (2D)^4$. Subtracting identity (2.22) from (2.23) resulted in another identity which is parametric for $A^4 + B^4 + C^4 = D^4 + E^4 + F^4 + G^4$. But it is again a research work that requires ones innovative mind for reduction of $4.n.n$ to $4.m.n$ equations. It may also happen that such reduction may not be possible, in that case one will have to resort to method of seed equation already discussed above. For higher degree Diophantine equations, say fifth power, procedure as given above, can be applied but in that case, there will be three conditions to be satisfied. That is besides

$$(a_1 + a_2 + a_3 + a_4 - a_5 - a_6 - a_7 - a_8) = 0,$$

$$(a_1^2 + a_2^2 + a_3^2 + a_4^2 - a_5^2 - a_6^2 - a_7^2 - a_8^2) = 0,$$

third condition

$$(a_1^3 + a_2^3 + a_3^3 + a_4^3 - a_5^3 - a_6^3 - a_7^3 - a_8^3) = 0$$

will have to be satisfied. Thus there will be a situation like that of solution of Tarry Escott problem resulting in solution of Diophantine equations harder.

4. Acknowledgements

We acknowledge the help provided by the wonderful website https://www. desmos. com in calculating the values of tedious exponential terms and successive multiplications of terms.

References

[1] Berndt, B. C. and Bhargava, S., Ramanujan–For Lowbrows, Am. Math. Monthly, 100 (1993), 645-656.

[2] Dickson, L. E., History of the Theory of Numbers, Vol 2: Diophantine Analysis, New York: Dover, 2005.

[3] Haldeman, C. B., On Biquadrate Numbers, Math. Mag. 2 (1904), 285-296.

[4] Hirschhorn, M. D., Two or Three Identities of Ramanujan, Amer. Math. Monthly, 105 (1998), 52-55.

[5] Lander, L. J., Parkin, T. R. and Selfridge, J. L., A Survey of Equal Sums of Like Powers, Math. Comput. 21 (1967), 446-459.

[6] Martin, A., About Biquadrate Numbers whose Sum is a Biquadrate, Math. Mag., 2 (1896), 173-184.

[7] Piezas, Tito III and Weisstein, Eric W., Diophantine Equation-4th Powers, From Math World—A Wolfram Web Resource http://mathworld.wolfram.com/DiophantineEquation4thPowers.html

[8] Wadhawan, Narinder Kumar, Wadhawan, Priyanka, Sum Of Cubes Which Equal Another Cube, Diophantine Equations–Third Power, Annals of the Faculty of Engineering Hunedoara - International Journal of Engineering, Vol. 17 Issue 4 (2019), p105-116. http://annals.fih.upt.ro/pdf-full/2019/ANNALS-2019-4-16.pdf