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# EXISTENCE AND UNIQUENESS RESULTS FOR IMPLICIT FRACTIONAL DIFFERENTIAL EQUATIONS INVOLVING GENERALIZED KATUGAMPOLA DERIVATIVE

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**Abstract:** In this paper, we investigate the existence and uniqueness results for the solutions to implicit fractional differential equations involving generalized Katugampola derivative with nonlocal initial condition. By means of some classical fixed point theorem techniques such as Krasnosel'skii fixed point theorem and Banach contraction principle we established our main results. A suitable example is given to illustrate the applicability of our main results.

**Keywords and Phrases:** Generalized Katugampola derivative, implicit differential equation, existence, Uniqueness, fixed point.

**2020 Mathematics Subject Classification:** 26A33, 34A08, 34A12.

## 1. Introduction

In recent years, fractional calculus have been extensively used in many fields of applied sciences and engineering. This is the reason, that the fractional calculus and their applications have remains in spotlight in many research works. Certainly, fractional differential equations have grabbed a great attention of several authors and proved to be an important tool in the modeling of many physical phenomena. One can see [1, 2, 3, 6, 7, 8, 9, 10, 12, 14, 15, 18, 17] and references therein.

Recently, in [4], Benchohra et al., discussed the existence and stability results of implicit fractional differential equations with Hadamard fractional derivatives. Kucche et al., [11], obtained the existence, uniqueness results and other important properties of nonlinear implicit fractional differential equations. In [16] Vivek et al., obtained the existence and stability results for implicit differential equations involving Hilfer-Hadamard fractional derivative with nonlocal initial conditions.

In [13] D. S. Oliveira et al., introduced a new fractional differential operator: Hilfer–Katugampola frational derivative (also known as Generalized Katugampola Derivative) and derived various properties. Further, they studied the fractional differential equation with generalized Katugampola derivative to obtain the corresponding existence and uniqueness results. This new definition of fractional order derivative interpolates the Hilfer, Hilfer–Hadamard, Hadamard, Riemann–Liouville, Caputo, Caputo–Hadamard derivative, as well as the Liouville and Weyl fractional derivatives. In [3], Bagwan et al., established the existence and stability results for a class of initial value problems involving generalized Katugampola derivative with the nonlocal initial conditions. In [5], Benchohra et al., studied the existence and uniqueness results for a class of terminal type boundary value problems for fractional differential equations involving Hilfer–Katugampola fractional derivative.

In this paper, we discussed the existence and uniqueness of solution to the implicit fractional differential equations (IFDEs) involving generalized Katugampola derivative of the form:

$$^{\rho}D_{a^+}^{\mu,\nu}u\left(t\right)=f\left(t,u\left(t\right),{}^{\rho}D_{a^+}^{\mu,\nu}u\left(t\right)\right), \tag{1}$$

with the nonlocal initial condition

$$^{\rho}I_{a^+}^{1-\alpha}u\left(a\right)=\sum_{i=1}^{m}\lambda_i u\left(\omega_i\right), \tag{2}$$

where $\mu\in\left(0,1\right), \nu\in\left[0,1\right], t\in(a,b], \quad \mu\leq\alpha=\mu+\nu\left(1-\mu\right)<1, \omega_i\in\left(a,b\right].$ $f$ is a given function such that $f:(a,b]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$, $\rho>0$. The operator $^{\rho}D_{a^+}^{\mu,\nu}$ is the generalized Katugampola fractional derivative of order $\mu$ and type $\nu$ and the operator $^{\rho}I_{a^+}^{1-\alpha}$ is the Katugampola fractional integral of order $1-\alpha$ with $a>0$. $\omega_i, i=1,2,...,m$ are prefixed points satisfying $a<\omega_1\leq\omega_2\leq...\leq\omega_m<b$.

This paper is arranged as follows: In section 2, we introduce some basic definitions, important results and preliminary facts about generalized Katugampola derivative. Further, we derived the equivalent Volterra integral equation of mixed type for the IFDE (1)–(2). In section 3, we established two existence results for the IFDE (1)–(2). First result is derived by means of Krasnosel'skii fixed point theorem and second one is by means of Banach contraction principle which also ensures the uniqueness of solution. Finally, in section 4, an illustrative example is given to show the applicability of our main results.

## 2. Preliminary results

In this section, we state some basic definitions of fractional integrals and derivatives, few important results and preliminaries about generalized Katugampola derivative that are very useful to us in the sequel.

Let $C\,[a,b]$, where $0 < a < b < \infty$ be a finite interval on $\mathbb{R}^+$, be the Banach space of all continuous functions $\varphi : [a,b] \to \mathbb{R}$ with the norm

$$\|\varphi\|_C = \max\{|\varphi(t)| : t \in [a,b]\}.$$

We define the weighted space of continuous functions $\varphi$ on $(a,b]$ by

$$C_{\alpha,\rho}\,[a,b] = \left\{\varphi : (a,b] \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha \varphi(t) \in C\,[a,b]\right\},\ \ 0 \le \alpha < 1,\ \ \rho > 0,$$

with the norm

$$\|\varphi\|_{C_{\alpha,\rho}} = \left\|\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha \varphi(t)\right\|_C = \max_{t\in[a,b]}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha \varphi(t)\right|. \tag{3}$$

Clearly, $C_{0,\rho}\,[a,b] = C\,[a,b]$.

Furthermore, We define the following weighted spaces

$$C^{\mu,\nu}_{1-\alpha,\rho}\,[a,b] = \left\{\varphi \in C_{1-\alpha,\rho}\,[a,b]\,,\ {}^\rho D^{\mu,\nu}_{a^+}\varphi \in C_{\alpha,\rho}\,[a,b]\right\}$$

and

$$C^{\alpha}_{1-\alpha,\rho}\,[a,b] = \left\{\varphi \in C_{1-\alpha,\rho}\,[a,b]\,,\ {}^\rho D^{\alpha}_{a^+}\varphi \in C_{1-\alpha,\rho}\,[a,b]\right\},$$

with the norm defined in (3), where the parameters $\alpha, \mu, \nu$ and $\rho$ are such that $\alpha = \mu + \nu(1-\mu)$, for $0 < \mu, \nu < 1$ and $0 \le \alpha < 1$. Note that, $C^{\alpha}_{1-\alpha,\rho}\,[a,b] = C^{\mu,\nu}_{1-\alpha,\rho}\,[a,b]$.

For $c \in \mathbb{R}$ and $1 \le r \le \infty$ consider the space $Z^r_c\,(a,b)$ of those complex valued Lebesgue measurable functions $h$ on $[a,b]$ for which $\|h\|_{Z^r_c} < \infty$, where

$$\|h\|_{Z^r_c} = \left(\int_a^b |x^c h(x)|^r \frac{dx}{x}\right)^{1/r} < \infty, \quad c \in \mathbb{R}, 1 \le r < \infty$$

and for $r = \infty$

$$\|h\|_{Z_c^\infty} = ess \sup_{x\in[a,b]} \left[x^c \left|h(x)\right|\right], \qquad (c \in \mathbb{R}).$$

**Definition 1.** (Katugampola fractional integral [9], [13]) *Let $\mu$, $c \in \mathbb{R}$ with $\mu > 0$, $u \in Z_c^r(a,b)$, where $Z_c^r(a,b)$ be the space of Lebesgue measurable functions with complex values. The left-sided Katugampola fractional integral of order $\mu$ is defined by*

$$\left({}^\rho I_{a^+}^\mu u\right)(t) = \frac{\rho^{1-\mu}}{\Gamma(\mu)} \int_a^t \frac{x^{\rho-1} u(x)}{\left(t^\rho - x^\rho\right)^{1-\mu}} dx \ , \qquad (t > a). \tag{4}$$

**Definition 2.** (Katugampola fractional derivative [9] [13]) *Let $\mu$, $\rho \in \mathbb{R}$ such that $\mu \notin \mathbb{N}$, $0 < \mu, \rho$. The left-sided Katugampola fractional derivative of order $\mu$ is defined by*

$$\begin{aligned}\left({}^\rho D_{a^+}^\mu u\right)(t) &= \delta_\rho^n \left({}^\rho I_{a^+}^{n-\mu} u\right)(t) \\ &= \frac{\rho^{1-n+\mu}}{\Gamma(n-\mu)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t \frac{x^{\rho-1} u(x)}{\left(t^\rho - x^\rho\right)^{1-n+\mu}} dx,\end{aligned} \tag{5}$$

*where $n = [\mu] + 1$ such that $[\mu]$ is the integer part of $\mu$.*

**Definition 3.** (Generalized Katugampola fractional derivative [13]) *Let $0 < \mu \le 1$, and $0 \le \nu \le 1$. The generalized Katugampola fractional derivative (of order $\mu$ and type $\nu$) with respect to $t$ is defined by*

$$\begin{aligned}\left({}^\rho D_{a^+}^{\mu,\nu} u\right)(t) &= \left\{\pm {}^\rho I_{a\pm}^{\nu(1-\mu)} \left(t^{\rho-1}\frac{d}{dt}\right)^1 {}^\rho I_{a\pm}^{(1-\nu)(1-\mu)} u\right\}(t) \\ &= \left\{\pm {}^\rho I_{a\pm}^{\nu(1-\mu)} \delta_\rho {}^\rho I_{a\pm}^{(1-\nu)(1-\mu)} u\right\}(t),\end{aligned} \tag{6}$$

*where $\rho > 0$, $u \in C_{1-\alpha,\rho}[0,1]$ and $I$ is Katugampola fractional integral defined in (4).*

**Remark 1.** ([13]) *For $\alpha = \mu + \nu(1-\mu)$, the generalized Katugampola fractional derivative operator ${}^\rho D_{a^+}^{\mu,\nu}$ can be expressed as*

$${}^\rho D_{a^+}^{\mu,\nu} = {}^\rho I_{a^+}^{\nu(1-\mu)} \delta_\rho {}^\rho I_{a^+}^{1-\alpha} = {}^\rho I_{a^+}^{\nu(1-\mu)} {}^\rho D_{a^+}^\alpha. \tag{7}$$

**Lemma 1.** ([13]) *Let $\mu > 0, 0 \le \alpha < 1$ and $u \in C_{\alpha,\rho}[a,b]$, then*

$$\left({}^\rho D_{a^+}^\mu {}^\rho I_{a^+}^\mu u\right)(t) = u(t), \qquad for\ all \ \ t \in (a,b].$$

**Lemma 2.** (Semigroup property [13]) *Let $\mu > 0, \nu > 0, 1 \le r \le \infty$. $a, b \in (0, \infty)$ such that $a < b$ and $\rho$, $s \in \mathbb{R}$, $s \le \rho$. Then the following property hold true:*

$$\left({}^{\rho}I_{a^+}^{\mu}\,{}^{\rho}I_{a^+}^{\nu}u\right)(t) = \left({}^{\rho}I_{a^+}^{\mu+\nu}u\right)(t),$$

*for all $u \in Z_s^r(a, b)$.*

**Lemma 3.** ([13]) *Let $t > a$, and for $\mu \ge 0$ and $\nu > 0$, we have*

$$\left[{}^{\rho}D_{a^+}^{\mu}\left(\frac{x^{\rho} - a^{\rho}}{\rho}\right)^{\mu-1}\right](t) = 0, \qquad 0 < \mu < 1,$$

$$\left[{}^{\rho}I_{a^+}^{\mu}\left(\frac{x^{\rho} - a^{\rho}}{\rho}\right)^{\nu-1}\right](t) = \frac{\Gamma(\nu)}{\Gamma(\mu+\nu)}\left(\frac{x^{\rho} - a^{\rho}}{\rho}\right)^{\mu+\nu-1}.$$

**Lemma 4.** ([13]) *Let $\mu > 0$, $0 \le \alpha < 1$ and $a, b \in (0, \infty)$ such that $a < b$ and $u \in C_{\alpha,\rho}[a, b]$. Then*

$$\left({}^{\rho}I_{a^+}^{\mu}u\right)(a) = \lim_{t \to a^+}\left({}^{\rho}I_{a^+}^{\mu}u\right)(t) = 0$$

*and ${}^{\rho}I_{a^+}^{\mu}u$ is continuous on $[a, b]$ if $\alpha < \mu$.*

**Lemma 5.** ([13]) *Let $\mu \in (0, 1)$, $\nu \in [0, 1]$ and $\alpha = \mu + \nu - \mu\nu$. If $u \in C_{1-\alpha}^{\alpha}[a, b]$ then*

$${}^{\rho}I_{a^+}^{\alpha}\,{}^{\rho}D_{a^+}^{\alpha}u = {}^{\rho}I_{a^+}^{\mu}\,{}^{\rho}D_{a^+}^{\mu,\nu}u$$

and

$${}^{\rho}D_{a^+}^{\alpha}\,{}^{\rho}I_{a^+}^{\mu}u = {}^{\rho}D_{a^+}^{\nu(1-\mu)}u.$$

**Lemma 6.** ([13]) *Let $\mu \in (0, 1)$, $0 \le \alpha < 1$. If $u \in C_{\alpha}[a, b]$ and ${}^{\rho}I_{a^+}^{1-\mu}u \in C_{\alpha}^{1}[a, b]$ then for all $t \in (a, b]$*

$$\left({}^{\rho}I_{a^+}^{\mu}\,{}^{\rho}D_{a^+}^{\mu}u\right)(t) = -\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\mu-1}\frac{\left({}^{\rho}I_{a^+}^{1-\alpha}u\right)(a)}{\Gamma(\mu)} + u(t).$$

**Lemma 7.** ([13]) *Let $u \in L^1(a, b)$. If ${}^{\rho}D_{a^+}^{\nu(1-\mu)}u$ exists on $L^1(a, b)$, then*

$${}^{\rho}D_{a^+}^{\mu,\nu}\,{}^{\rho}I_{a^+}^{\mu}u = {}^{\rho}I_{a^+}^{\nu(1-\mu)}\,{}^{\rho}D_{a^+}^{\nu(1-\mu)}u.$$

**Theorem 1.** (Krasnosel'skii fixed point theorem [14]) *Let $E$ be a nonempty closed, bounded and convex subset of a Banach space $(\mathcal{B}, \|\cdot\|)$. Further, Assume that $F$ and $G$ be two operators defined on $E$ which map $E$ into $\mathcal{B}$ such that*

1. $F(x) + G(y) \in E$ *for all* $x, y \in E$,

2. $F$ *is a contraction,*

3. $G$ *is continuous and compact.*

*Then,* $F + G$ *has a fixed point in* $E$.

**Theorem 2.** (Banach's fixed point theorem [14]) *Let* $E$ *be a non-empty closed subset of a Banach space* $\mathcal{B}$, *then any contraction mapping* $\Delta$ *of* $E$ *into itself has a unique fixed point.*

**Lemma 8.** *Let* $f : (a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *be a function such that* $f\left(\cdot, u(\cdot), {}^{\rho}D_{a^+}^{\mu,\nu} u(.)\right) \in C_{1-\alpha,\rho}[a, b]$ *for any* $u \in C_{1-\alpha,\rho}[a, b]$ *where* $\alpha = \mu + \nu(1 - \mu)$ *with* $\mu \in (0, 1)$, $\nu \in [0, 1]$. *A function* $u \in C_{1-\alpha,\rho}^{\alpha}[a, b]$ *is a solution of generalized Katugampola type IFDE:*

$$ {}^{\rho}D_{a^+}^{\mu,\nu} u(t) = f\left(t, u(t), {}^{\rho}D_{a^+}^{\mu,\nu} u(t)\right), $$

*with the initial condition*

$$ {}^{\rho}I_{a^+}^{1-\alpha} u\left(a^+\right) = u_0, $$

*if and only if* $u$ *satisfies the following Volterra type integral equation:*

$$ u(t) = \frac{u_0}{\Gamma(\alpha)} \left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\alpha-1} + \frac{1}{\Gamma(\mu)} \int_a^t \left(\frac{t^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} f\left(x, u(x), {}^{\rho}D_{a^+}^{\mu,\nu} u(x)\right) dx. $$

For the sake of brevity, let $A_u : (a, b] \to \mathbb{R}$ be a function such that

$$ {}^{\rho}D_{a^+}^{\mu,\nu} u(t) = A_u(t) = f(t, u(t), A_u(t)). $$

Clearly, $A_u \in C_{1-\alpha,\rho}[a, b]$.

## 3. Existence Results

Using the above fundamental results, first we prove the equivalence between the IFDE (1)–(2) and an improved Volterra integral equation of mixed type.

**Theorem 3.** *Let* $f : (a, b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *be a function such that* $f\left(\cdot, u(\cdot), {}^{\rho}D_{a^+}^{\mu,\nu} u(.)\right) \in C_{1-\alpha,\rho}[a, b]$ *for any* $u \in C_{1-\alpha,\rho}[a, b]$ *where* $\alpha = \mu + \nu(1 - \mu)$ *with* $\mu \in (0, 1)$, $\nu \in [0, 1]$. *A function* $u \in C_{1-\alpha,\rho}^{\alpha}[a, b]$ *is a solution of generalized Katugampola type IFDE (1)–(2) if and only if it satisfies the following Volterra integral equation of* *mixed type*

$$u(t) = \frac{H}{\Gamma(\mu)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx$$
$$+\frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx, \tag{8}$$

*where* $H = \left\{\Gamma(\alpha) - \sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right\}^{-1}$, *provided* $\Gamma(\alpha) \neq \sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\alpha-1}$.

**Proof.** Let $u \in C_{1-\alpha}^{\alpha}[a,b]$ be a solution of IFDE (1)–(2), then by the Lemma 8 the solution of IFDE (1)–(2) can be written as

$$u(t) = \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\frac{\left({}^\rho I_{a^+}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\mu)}\int_a^t\left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx. \tag{9}$$

Now, substitute $t = \omega_i$ in the above equation

$$u(\omega_i) = \left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\alpha-1}\frac{\left({}^\rho I_{a^+}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)} + \frac{1}{\Gamma(\mu)}\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx.$$

Multiplying both sides of above equation by $\lambda_i$, we get

$$\lambda_i u(\omega_i) = \lambda_i\left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\alpha-1}\frac{\left({}^\rho I_{a^+}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)} + \frac{\lambda_i}{\Gamma(\mu)}\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx.$$

Thus, we have

$$\begin{aligned}{}^\rho I_{a^+}^{1-\alpha}u(a) &= \sum_{i=1}^{m}\lambda_i u(\omega_i)\\ &= \frac{\left({}^\rho I_{a^+}^{1-\alpha}u\right)(a)}{\Gamma(\alpha)}\sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\alpha-1}\\ &\quad + \frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx,\end{aligned}$$

that implies,

$$\left({}^{\rho}I_{a^+}^{1-\alpha}u\right)(a) = \frac{\Gamma(\alpha)}{\Gamma(\mu)} H \sum_{i=1}^{m} \lambda_i \int\limits_a^{\omega_i} \left(\frac{{\omega_i}^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} A_u(x)\, dx. \tag{10}$$

Substituting (10) in (9) we get (8). It proved that $u$ also satisfies integral equation (8) when it satisfies the IFDE (1)–(2) which proved the necessity.

Now, to prove the sufficiency, apply ${}^{\rho}I_{a^+}^{1-\alpha}$ to both sides of the integral equation (8), we get

$$\begin{aligned} {}^{\rho}I_{a^+}^{1-\alpha}u(t) =& {}^{\rho}I_{a^+}^{1-\alpha}\left(\frac{t^{\rho} - a^{\rho}}{\rho}\right)^{\alpha-1} \frac{H}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int\limits_a^{\omega_i} \left(\frac{{\omega_i}^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} A_u(x)\, dx \\ &+ {}^{\rho}I_{a^+}^{1-\alpha}\,{}^{\rho}I_{a^+}^{\mu} A_u(x) \end{aligned}$$

using Lemma 2, Lemma 1 and Lemma 3, we have

$${}^{\rho}I_{a^+}^{1-\alpha}u(t) = \frac{\Gamma(\alpha)}{\Gamma(\mu)} H \sum_{i=1}^{m} \lambda_i \int\limits_a^{\omega_i} \left(\frac{{\omega_i}^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} A_u(x)\, dx + {}^{\rho}I_{a^+}^{1-\nu(1-\mu)} A_u(x).$$

Since, $1 - \nu(1-\mu) > 1 - \alpha$. By taking the limit as $t \to a$ and using Lemma 4, we have

$$\left({}^{\rho}I_{a^+}^{1-\alpha}u\right)(a) = \frac{\Gamma(\alpha)}{\Gamma(\mu)} H \sum_{i=1}^{m} \lambda_i \int\limits_a^{\omega_i} \left(\frac{{\omega_i}^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} A_u(x)\, dx. \tag{11}$$

Now, substituting $t = \omega_i$ in (8), we have

$$\begin{aligned} u(\omega_i) =& \left(\frac{{\omega_i}^{\rho} - a^{\rho}}{\rho}\right)^{\alpha-1} \frac{H}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int\limits_a^{\omega_i} \left(\frac{{\omega_i}^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} A_u(x)\, dx \\ &+ \frac{1}{\Gamma(\mu)} \int\limits_a^{\omega_i} \left(\frac{{\omega_i}^{\rho} - x^{\rho}}{\rho}\right)^{\mu-1} x^{\rho-1} A_u(x)\, dx. \end{aligned}$$

Then, we have

$$\begin{aligned}
\sum_{i=1}^{m}\lambda_i u(\omega_i) &= \frac{H}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx \\
&\quad+\frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx \\
&=\frac{1}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx \\
&\quad\cdot\left\{H\sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}+1\right\} \\
&=\frac{\Gamma(\alpha)}{\Gamma(\mu)}H\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}A_u(x)\,dx. \qquad (12)
\end{aligned}$$

It follows from (11) and (12), that

$$^{\rho}I_{a^+}^{1-\alpha}u(a)=\sum_{i=1}^{m}\lambda_i u(\omega_i).$$

It follows from Lemma 3 and Lemma 5 and by applying $^{\rho}D_{a^+}^{\alpha}$ to both sides of (8) that

$$^{\rho}D_{a^+}^{\alpha}u(t)={}^{\rho}D_{a^+}^{\nu(1-\mu)}f(t,u(t),A_u(t)). \qquad (13)$$

Since, $u\in C_{1-\alpha,\rho}^{\alpha}[a,b]$ and by the definition of $C_{1-\alpha,\rho}^{\alpha}[a,b]$, we have $^{\rho}D_{a^+}^{\alpha}u\in C_{1-\alpha,\rho}^{\alpha}[a,b]$, then $^{\rho}D_{a^+}^{\nu(1-\mu)}f={}^{\rho}D^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\alpha,\rho}[a,b]$. It is obvious that for any $f\in C_{1-\alpha,\rho}[a,b]$, $^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\alpha,\rho}[a,b]$, then $^{\rho}I_{a^+}^{1-\nu(1-\mu)}f\in C_{1-\alpha,\rho}^{1}[a,b]$. Thus, both $f$ and $^{\rho}I_{a^+}^{1-\nu(1-\mu)}f$ satisfies the conditions of Lemma 6.

Now, it follows from Lemma 6, by applying $^{\rho}I_{a^+}^{\nu(1-\mu)}$ on both sides of (13), that

$$\left(^{\rho}D_{a^+}^{\mu,\nu}u\right)(t)=-\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\nu(1-\mu)-1}\frac{^{\rho}I_{a^+}^{1-\nu(1-\mu)}f(a)}{\Gamma(\nu(1-\mu))}+A_u(x). \qquad (14)$$

By Lemma 4, it implies that $^{\rho}I_{a^+}^{1-\nu(1-\mu)}f(a)=0$. Hence, equation (14) reduces to

$$\left(^{\rho}D_{a^+}^{\mu,\nu}u\right)(t)=A_u(t)=f(t,u(t),A_u(t)).$$

This completes the proof.

In the sequel, let us introduce the following hypotheses:

**Q1:** Let $f : (a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any $u \in C_{1-\alpha,\rho}[a,b]$, $f(\cdot, u(\cdot), v(.)) \in C_{1-\alpha,\rho}^{\nu(1-\mu)}[a,b]$.

**Q2:** For all $u, v, \overline{u}, \overline{v} \in \mathbb{R}$, there exists positive constants $J > 0$ and $0 < L < 1$ such that

$$|f(t,u,v) - f(t,\overline{u},\overline{v})| \leq J|u - \overline{u}| + L|v - \overline{v}|,$$

for $t \in (a,b]$.

**Q3:** The constant

$$\xi := \frac{J\mathbf{B}(\mu,\alpha)}{(1-L)\Gamma(\mu)} \left\{ |H| \sum_{i=1}^{m} \lambda_i \left( \frac{{\omega_i}^\rho - a^\rho}{\rho} \right)^{\mu+\alpha-1} + \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\mu} \right\} < 1 \quad (15)$$

where $H$ is defined in the Theorem 3.

Now, we will establish our first existence result for the IFDE (1)–(2) by means of Krasnosel'skii fixed point theorem.

**Theorem 4.** *Assume that the hypotheses* $[Q1]$, $[Q2]$ *and* $[Q3]$ *are satisfied. Then IFDE (1)–(2) has at least one solution in* $C_{1-\alpha,\rho}^{\alpha}[a,b]$.

**Proof.** According to Theorem 3, it is sufficient to prove the existence result for the Volterra integral equation of mixed type (8).

Now, define the operator $\Delta$ by

$$\begin{aligned}(\Delta u)(t) =& \frac{H}{\Gamma(\mu)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left( \frac{{\omega_i}^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} A_u(x)\, dx \\ &+ \frac{1}{\Gamma(\mu)} \int_a^t \left( \frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} A_u(x)\, dx. \end{aligned} \quad (16)$$

It is obvious that the operator $\Delta$ is well defined and maps $C_{1-\alpha,\rho}[a,b]$ into $C_{1-\alpha,\rho}[a,b]$. Let $\hat{f} = \max_{t\in[a,b]} |f(t,0,0)|$ and

$$\delta := \frac{\mathbf{B}(\mu,\alpha)}{\Gamma(\mu)} \left\{ |H| \sum_{i=1}^{m} \lambda_i \left( \frac{{\omega_i}^\rho - a^\rho}{\rho} \right)^{\mu+\alpha-1} + \left( \frac{b^\rho - a^\rho}{\rho} \right)^{\mu} \right\} \left\| \hat{f} \right\|_{C_{1-\alpha,\rho}}. \quad (17)$$

Consider a ball $B_\tau := \left\{ u \in C_{1-\alpha,\rho}[a,b] : \|u\|_{C_{1-\alpha,\rho}} \leq \tau \right\}$ with $\frac{\delta}{1-\xi} \leq \tau$, $(\xi < 1)$.

Now, let us subdivide the operator $\Delta$ into two operators $\Phi$ and $\Psi$ on $B_\tau$ as follows:

$$(\Phi u)(t) = \frac{H}{\Gamma(\mu)} \left( \frac{t^\rho - a^\rho}{\rho} \right)^{\alpha-1} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left( \frac{{\omega_i}^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} A_u(x)\, dx$$

and

$$(\Psi u)(t) = \frac{1}{\Gamma(\mu)} \int_a^t \left( \frac{t^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} A_u(x)\, dx.$$

The proof is divided into following steps.
**Step I:** For every $u, v \in B_\tau$, $\Phi u + \Psi v \in B_\tau$.

For any $t \in (a,b]$ and the operator $\Phi$, we have

$$(\Phi u)(t) \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\alpha} = \frac{H}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left( \frac{{\omega_i}^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} A_u(x)\, dx,$$

then,

$$\left| (\Phi u)(t) \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\alpha} \right| \leq \frac{|H|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left( \frac{{\omega_i}^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} |A_u(x)|\, dx. \tag{18}$$

From the hypothesis $[Q2]$, we have

$$\begin{aligned} |A_u(t)| &= |f(t, u(t), A_u(t)) - f(t,0,0) + f(t,0,0)| \\ &\leq |f(t, u(t), A_u(t)) - f(t,0,0)| + |f(t,0,0)| \\ &\leq J|u(t)| + L|A_u(t)| + \hat{f} \\ &\leq \frac{J|u(t)| + \hat{f}}{(1-L)}. \end{aligned} \tag{19}$$

Using (18) in (19), we get

$$\left| (\Phi u)(t) \left( \frac{t^\rho - a^\rho}{\rho} \right)^{1-\alpha} \right| \leq \frac{|H|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left( \frac{{\omega_i}^\rho - x^\rho}{\rho} \right)^{\mu-1} x^{\rho-1} \frac{J|u(x)| + \hat{f}}{(1-L)} dx.$$

Now, using the fact

$$\int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} |u(x)|\, dx \le \left\{ \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} \left(\frac{x^\rho - a^\rho}{\rho}\right)^{\alpha-1} x^{\rho-1} dx \right\} \|u\|_{C_{1-\alpha,\rho}}$$
$$= \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1} \mathbf{B}(\mu,\alpha) \|u\|_{C_{1-\alpha,\rho}}, \tag{20}$$

we have

$$\left| (\Phi u)(t) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \right| \le \frac{|H|}{\Gamma(\mu)} \sum_{i=1}^m \lambda_i \left\{ \left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1} \frac{\mathbf{B}(\mu,\alpha)}{(1-L)} \cdot \left( J\|u\|_{C_{1-\alpha,\rho}} + \left\|\hat{f}\right\|_{C_{1-\alpha,\rho}} \right) \right\}$$

which gives,

$$\|\Phi u\|_{C_{1-\alpha,\rho}} \le \frac{|H|\,\mathbf{B}(\mu,\alpha)}{(1-L)\Gamma(\mu)} \sum_{i=1}^m \lambda_i \left\{ \left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1} \left( J\|u\|_{C_{1-\alpha,\rho}} + \left\|\hat{f}\right\|_{C_{1-\alpha,\rho}} \right) \right\}. \tag{21}$$

Now, for $t \in (a,b]$ and the operator $\Psi$, we have

$$(\Psi v)(t) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} = \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} A_v(x)\, dx.$$

Then,

$$\left| (\Psi v)(t) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \right| \le \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \int_a^t \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} |A_v(x)|\, dx.$$

Again using (19) and (20), we have

$$\left| (\Psi v)(t) \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \right| \le \frac{1}{\Gamma(\mu)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \left\{ \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1} \cdot \frac{\mathbf{B}(\mu,\alpha)}{(1-L)} \left( J\|v\|_{C_{1-\alpha,\rho}} + \left\|\hat{f}\right\|_{C_{1-\alpha,\rho}} \right) \right\}$$
$$\le \frac{\mathbf{B}(\mu,\alpha)}{(1-L)\,\Gamma(\mu)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu} \left( J\|v\|_{C_{1-\alpha,\rho}} + \left\|\hat{f}\right\|_{C_{1-\alpha,\rho}} \right),$$

which gives

$$\|(\Psi v)\|_{C_{1-\alpha,\rho}} \leq \frac{\mathbf{B}(\mu,\alpha)}{(1-L)\,\Gamma(\mu)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu}\left(J\|v\|_{C_{1-\alpha,\rho}} + \left\|\hat{f}\right\|_{C_{1-\alpha,\rho}}\right). \tag{22}$$

Combining equations (21) and (22), for every $u, v \in B_\tau$, we have

$$\begin{aligned}\|\Phi u + \Psi v\|_{C_{1-\alpha,\rho}} &\leq \|\Phi u\|_{C_{1-\alpha,\rho}} + \|(\Psi v)\|_{C_{1-\alpha,\rho}} \\ &\leq \xi\tau + \delta \leq \tau,\end{aligned}$$

which implies that $\Phi u + \Psi v \in B_\tau$.

**Step II:** The operator $\Phi$ is contraction mapping.

For any $u, v \in B_\tau$ and the operator $\Phi$,

$$\begin{aligned}&\left\{(\Phi u)(t) - (\Phi v)(t)\right\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \\ &\qquad = \frac{H}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1}\left[A_u(x) - A_v(x)\right]dx.\end{aligned}$$

Then,

$$\begin{aligned}\left|\left\{(\Phi u)(t) - (\Phi v)(t)\right\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha}\right| \leq \frac{|H|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1} \\ \cdot\left|A_u(x) - A_v(x)\right|dx.\end{aligned} \tag{23}$$

From the hypothesis [$Q2$], we have

$$\begin{aligned}\left|A_u(t) - A_v(t)\right| &= \left|f(t, u(t), A_u(t)) - f(t, v(t), A_v(t))\right| \\ &\leq J\left|u(t) - v(t)\right| + L\left|A_u(t) - A_v(t)\right| \\ &\leq \frac{J}{(1-L)}\left|u(t) - v(t)\right|.\end{aligned} \tag{24}$$

Using (24) in (23) and then using (20), we get

$$\begin{aligned}\left|\left\{(\Phi u)(t) - (\Phi v)(t)\right\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha}\right| &\leq \frac{|H|}{\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\int_a^{\omega_i}\left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1}x^{\rho-1} \\ &\quad\cdot\frac{J}{(1-L)}\left|u(x) - v(x)\right|dx \\ &\leq \frac{J\,|H|\,\mathbf{B}(\mu,\alpha)}{(1-L)\Gamma(\mu)}\sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1}\cdot\|u - v\|_{C_{1-\alpha,\rho}},\end{aligned}$$

which gives

$$\begin{aligned}\|\Phi u-\Psi v\|_{C_{1-\alpha,\rho}} &\leq \frac{J\left|H\right|\mathbf{B}\left(\mu,\alpha\right)}{(1-L)\Gamma\left(\mu\right)}\sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\alpha-1}\|u-v\|_{C_{1-\alpha,\rho}}\\ &\leq \xi\|u-v\|_{C_{1-\alpha,\rho}}.\end{aligned} \tag{25}$$

Hence, from the hypothesis [$Q3$] and the equation (25) it implies that the operator $\Phi$ is a contraction mapping.

**Step III:** The operator $\Psi$ is compact and continuous.

Since, the function $f\in C_{1-\alpha}\left[a,b\right]$ is continuous, it is obvious that the operator $\Psi$ is continuous.

Next, we prove the compactness.

For any $a<t_1<t_2\leq b$, we have

$$\begin{aligned}\left|\left(\Psi u\right)\left(t_1\right)-\left(\Psi u\right)\left(t_2\right)\right| &= \left|\frac{1}{\Gamma\left(\mu\right)}\int_a^{t_1}\left(\frac{{t_1}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}A_u\left(x\right)dx\right.\\ &\qquad\left.-\frac{1}{\Gamma\left(\mu\right)}\int_a^{t_2}\left(\frac{{t_2}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}x^{\rho-1}A_u\left(x\right)dx\right|\\ &\leq \frac{\|A_u\|_{C_{1-\alpha}}}{\Gamma\left(\mu\right)}\left|\int_a^{t_1}\left(\frac{{t_1}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}x^{\rho-1}dx\right.\\ &\qquad\left.-\int_a^{t_2}\left(\frac{{t_2}^{\rho}-x^{\rho}}{\rho}\right)^{\mu-1}\left(\frac{x^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}x^{\rho-1}dx\right|\\ &\leq \frac{\|A_u\|_{C_{1-\alpha}}\mathbf{B}\left(\mu,\alpha\right)}{\Gamma\left(\mu\right)}\left|\left(\frac{{t_1}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\alpha-1}-\left(\frac{{t_2}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\alpha-1}\right|\end{aligned}$$

tending to zero as $t_2\to t_1$ whether $\mu+\alpha-1\geq 0$ or $\mu+\alpha-1<0$. Thus, $\Psi(B_\tau)$ is equicontinuous. From the equation (22) clearly, $\Psi$ is uniformly bounded on $B_\tau$. Hence, by Arzela–Ascoli theorem, the operator $\Psi$ is compact on $B_\tau$.

It follows from Krasnosel'skii fixed point theorem that the IFDE (1)–(2) has at least one solution $u\in C_{1-\alpha,\rho}\left[a,b\right]$. Using the Lemma 7 and repeating the process of proof in Theorem 3, one can show that this solution is actually in $C^{\alpha}_{1-\alpha,\rho}\left[a,b\right]$. This completes the proof.

Now, in the next theorem we will prove our second existence and uniqueness result for IFDE (1)–(2) by means of Banach contraction principle.

**Theorem 5.** *Assume that the hypothesis [Q1], [Q2] and [Q3] are satisfied. Then by Banach contraction principle IFDE (1)–(2) has a unique solution.*

**Proof.** By Theorem 3, it is clear that the fixed points of the operator $\Delta$ defined in (16) are the solutions of IFDE (1)–(2). Now, we prove that the operator $\Delta$ has a unique fixed point in $C_{1-\alpha,\rho}[a,b]$.

Let $u, v \in C_{1-\alpha,\rho}[a,b]$, then for any $t \in (a,b]$, we have

$$\begin{aligned}
&\left|\left\{(\Delta u)(t) - (\Delta v)(t)\right\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha}\right| \\
&\qquad \leq \frac{|H|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \left|A_u(x) - A_v(x)\right| dx \\
&\qquad\quad + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \frac{1}{\Gamma(\mu)} \int_a^{t} \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \left|A_u(x) - A_v(x)\right| dx.
\end{aligned}$$

Using (24), we get

$$\begin{aligned}
&\left|\left\{(\Delta u)(t) - (\Delta v)(t)\right\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha}\right| \\
&\qquad \leq \frac{|H|}{\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \int_a^{\omega_i} \left(\frac{{\omega_i}^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \frac{J}{(1-L)} \left|u(t) - v(t)\right| dx \\
&\qquad\quad + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\alpha} \frac{1}{\Gamma(\mu)} \int_a^{t} \left(\frac{t^\rho - x^\rho}{\rho}\right)^{\mu-1} x^{\rho-1} \frac{J}{(1-L)} \left|u(t) - v(t)\right| dx.
\end{aligned}$$

Again by using (20), we have

$$\begin{aligned}
&\left|\left\{(\Delta u)(t) - (\Delta v)(t)\right\}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right| \\
&\leq \frac{J|H|}{(1-L)\Gamma(\mu)} \sum_{i=1}^{m} \lambda_i \left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1} \mathbf{B}(\mu,\alpha) \|u - v\|_{C_{1-\alpha,\rho}} \\
&\quad + \frac{J}{(1-L)\Gamma(\mu)} \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu} \mathbf{B}(\mu,\alpha) \|u - v\|_{C_{1-\alpha,\rho}} \\
&\leq \frac{J}{(1-L)\Gamma(\mu)} \mathbf{B}(\mu,\alpha) \left\{ |H| \sum_{i=1}^{m} \lambda_i \left(\frac{{\omega_i}^\rho - a^\rho}{\rho}\right)^{\mu+\alpha-1} + \left(\frac{b^\rho - a^\rho}{\rho}\right)^{\mu} \right\} \|u - v\|_{C_{1-\alpha,\rho}}.
\end{aligned}$$

Hence,

$$\begin{aligned}\|(\Delta u)-(\Delta v)\|_{C_{1-\alpha,\rho}} &\leq \frac{J}{(1-L)\,\Gamma(\mu)}\mathbf{B}(\mu,\alpha)\left\{|H|\sum_{i=1}^{m}\lambda_i\left(\frac{{\omega_i}^{\rho}-a^{\rho}}{\rho}\right)^{\mu+\alpha-1}\right.\\ &\quad\left.+\left(\frac{b^{\rho}-a^{\rho}}{\rho}\right)^{\mu}\right\}\|u-v\|_{C_{1-\alpha,\rho}}\\ &=\xi\|u-v\|_{C_{1-\alpha,\rho}} \end{aligned}\tag{26}$$

Thus, from $[Q3]$ it follows that the operator $\Delta$ is contraction. Hence, by Banach's fixed point theorem, $\Delta$ has a unique fixed point in $C_{1-\alpha,\rho}$ which is a solution of IFDE (1)–(2). This completes the proof.

## 4. Examples

**Example 1.** Consider the following implicit fractional differential equation:

$$ {}^{\rho}D_{0^+}^{\mu,\nu}u(t)=\frac{|u(t)|\sin t}{100}+\frac{\left|{}^{\rho}D_{0^+}^{\mu,\nu}u(t)\right|}{100\left(2+\left|{}^{\rho}D_{0^+}^{\mu,\nu}u(t)\right|\right)},\tag{27}$$

with the nonlocal initial condition

$$ {}^{\rho}I_{a^+}^{1-\alpha}u(0)=9u\left(\frac{2}{3}\right)+5u\left(\frac{5}{6}\right),\tag{28}$$

where $t\in(0,1]\,,\ \rho>0,\ \mu=\frac{1}{2},\ \nu=\frac{3}{5}$ and $\alpha=\mu+\nu(1-\mu)=\frac{4}{5}$. Set

$$f(t,u,v)=\frac{|u|\sin t}{100}+\frac{|v|}{100\,(2+|v|)}.$$

It is obvious that the function $f$ is continuous. Thus, the condition $[Q1]$ is satisfied.

For any $u,v,\overline{u},\overline{v}\in\mathbb{R}$ and $t\in(0,1]$, we have

$$|f(t,u,v)-f(t,\overline{u},\overline{v})|\leq\frac{1}{100}|u-\overline{u}|+\frac{1}{50}|v-\overline{v}|\,.$$

Thus, the condition $[Q2]$ is satisfied with $J=\frac{1}{100}$ and $L=\frac{1}{50}$. Moreover, with some elementary computation, for $\rho>0$ we have

$$|H|=\left|\left\{\Gamma\left(\frac{4}{5}\right)-\left[9\left(\frac{(2/3)^{\rho}-0^{\rho}}{\rho}\right)^{-1/5}+5\left(\frac{(5/6)^{\rho}-0^{\rho}}{\rho}\right)^{-1/5}\right]^{-1}\right\}\right|<1$$

and

$$\xi = \frac{(1/100)\,\mathbf{B}\,(1/2, 4/5)}{(1-(1/50))\,\Gamma\,(1/2)} \left\{ |H| \left[ 9\left(\frac{(2/3)^{\rho} - 0^{\rho}}{\rho}\right)^{3/10} + 5\left(\frac{(5/6)^{\rho} - 0^{\rho}}{\rho}\right)^{3/10} \right] + \left(\frac{1^{\rho} - 0^{\rho}}{\rho}\right)^{1/2} \right\} < 1.$$

Hence, the condition $[Q3]$ is satisfied. It follows from the Theorem 5 that the IFDE (27)–(28) has a unique solution.

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