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SUFFICIENT CONDITIONS FOR CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract: In this paper, sufficient conditions for normalized analytic functions defined on unit disk to be in the subclasses of close-to-convex , close-to-star and quasi-convex functions are obtained.

Keywords and Phrases: Normalized, analytic, close-to-convex, close-to-star, quasi-convex functions.

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the class of all analytic functions $f : \Delta \to \mathbb{C}$, where $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ which are normalized by $f(0) = 0$ and $f'(0) = 1$. Then

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1}$$

Denote by $\mathcal{S}$ the subclass of $\mathcal{A}$ containing univalent functions. The well known Bieberbach conjecture [3] says that for functions $f \in \mathcal{S}$ of the form (1), $|a_n| \leq n,\ \forall n \geq 2$. This was settled positively by Louis de Branges [2] and henceforth is known as "de Branges Theorem." However, in attempting to prove this result, researchers had defined various subclasses of $\mathcal{S}$ and had verified the same. Some of the standard subclasses of $\mathcal{S}$ introduced and studied for this purpose were subclasses of starlike, convex, close-to-convex, close-to-star, quasi convex functions and so on. Details about these subclases can be found in [3, 4]. Sufficient conditions for functions in the class $\mathcal{S}$ to belong to these standard subclasses are well known which are also necessary for those functions with negative coefficients [7].

Let us now recall certain standard subclasses of the class $\mathcal{S}$ from [3] which will be useful in this present work. Functions $g \in \mathcal{S}$ which maps the unit disk Δ onto a starlike domain are called starlike functions and analytically they satisfy $\Re\left(\frac{zg'(z)}{g(z)}\right) > 0, z \in \Delta$. Functions $g \in \mathcal{S}$ which maps the unit disk Δ onto a convex domain are called convex functions and analytically they satisfy $\Re\left(1 + \frac{zg''(z)}{g'(z)}\right) > 0$, $z \in \Delta$. Alexander's Theorem [3] states that $f \in \mathcal{S}$ is convex if and only if zf' is starlike. Thus, there is a one-to-one correspondence between the subclass of convex functions and that of starlike functions. A function $f \in \mathcal{A}$ is said to be close-to-convex if there is a starlike function $g \in \mathcal{S}$ such that $\Re\left(\frac{zf'(z)}{g(z)}\right) > 0,\ z \in \Delta$. Close-to-convex functions are univalent, but the converse need not be true. Thus in order to establish the univalency of a function in the class $\mathcal{A}$, it is enough to show that it is close-to-convex.

Obtaining sufficiency conditions for the membership of a function in the standard subclasses is an interesting problem. In [1] Bharanedhar *et.al.* had developed certain sufficient conditions for univalence and close - to - convexity of normalised analytic functions.

In this paper, we derive certain sufficiency conditions for functions in the class $\mathcal{A}$ to be close-to-convex, close-to-star and quasi-convex in terms of certain differential inequality involving functions in the class $\mathcal{A}$ and functions in subclasses of starlike and convex functions. We also establish the corresponding conditions of sufficiency in terms of coefficient inequalities,

Definition 1.1. [8] *For $\lambda \in [0,1]$, a function $f \in \mathcal{S}$ is said to be in the subclass $\mathcal{K}_{\lambda g}$ if there is a function $g \in \mathcal{S}^*$ such that*

$$\Re\left(\frac{zf'(z) + \lambda z^2 f''(z)}{(1-\lambda)g(z) + \lambda z g'(z)}\right) > 0,\ z \in \Delta.$$

Remark 1.1. [8] *The class $\mathcal{K}_{\lambda g}$ is a subclass of close-to-convex functions.*

Remark 1.2. *When $\lambda = 0$, the class $\mathcal{K}_{\lambda g}$ reduces to the subclass $\mathcal{K}$ of close-to-convex functions.*

Definition 1.2. [6] *A function $f \in \mathcal{A}$ with $f(z) \neq 0$ for $z \in \Delta - \{0\}$ is called a* *close-to-star function if there exists a univalent starlike function $g : \Delta \to \mathbb{C}$ not necessarily normalized, such that*

$$\Re\left(\frac{f(z)}{g(z)}\right) > 0,\ z \in \Delta.$$

Definition 1.3. [5] *Let f be an analytic function in Δ with $f(0) = 0$ and $f'(0) = 1$. Then f is said to be quasi-convex in Δ if there exists a convex function g with $g(0) = 0,\ g'(0) = 1$ such that for $z \in \Delta$,*

$$\Re\frac{(zf'(z))'}{g'(z)} > 0,\ z \in \Delta.$$

The class of quasi-convex functions is denoted by $\mathcal{Q}$.

Remark 1.3. [5] *Every quasi-convex function is close-to-convex and hence quasi-convex functions form a subclass of the class of close-to-convex functions.*

Remark 1.4. [5] *f is quasi-convex if and only if zf' is close-to-convex.*

2. Main Results

Theorem 2.1. *Let $\lambda \in [0, 1],\ f \in \mathcal{A}$ and g be a convex univalent function defined on Δ. If*

$$|f'(z) + \lambda z f''(z) - (g'(z) + \lambda z g''(z))| < m$$

where $m = inf_{z\in\Delta}|g'(z) + \lambda z g''(z)|$, then $f \in \mathcal{K}_{\lambda g}$.

Proof. By hypothesis,

$$|f'(z) + \lambda z f''(z) - (g'(z) + \lambda z g''(z))| < m.$$

Then

$$|f'(z) + \lambda z f''(z) - (g'(z) + \lambda z g''(z))| < |g'(z) + \lambda z g''(z)|.$$

Equivalently,

$$\left|\frac{f'(z) + \lambda z f''(z)}{g'(z) + \lambda z g''(z)} - 1\right| < 1$$

which implies $f \in \mathcal{K}_{\lambda g}$.

When $\lambda = 0$, we obtain the following result of Bharanedhar *et.al.*

Corollary 2.1. [1] *Let $f \in \mathcal{A}$ and g be a convex univalent function in Δ such that $m = inf_{z\in\Delta}|g'(z)|$. If*

$$|f'(z) - g'(z)| < m,\ z \in \Delta$$

then f is close-to-convex with respect to g in Δ.

Theorem 2.2. *Let $\lambda \in [0,1]$, $f \in \mathcal{A}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be a convex univalent function defined on the unit disk Δ. If*

$$\sum_{n=2}^{\infty} [\lambda n^2 + (1-\lambda)n]|a_n - b_n| < m - |1 - b_1|$$

where $m = inf_{z\in\Delta}|g'(z) + \lambda z g''(z)|$, then $f \in \mathcal{K}_{\lambda g}$.

Proof. It is enough to show that $|f'(z) + \lambda z f''(z) - (g'(z) + \lambda z g''(z))|$ is bounded above by m. Substituting the Taylor's series for f and g we have

$$\begin{aligned} &|f'(z) + \lambda z f''(z) - (g'(z) + \lambda z g''(z))| \\ &= \left|1 - b_1 + \sum_{n=2}^{\infty}[n + \lambda n(n-1)](a_n - b_n)z^{n-1}\right| \\ &\leq |1 - b_1| + \sum_{n=2}^{\infty}[\lambda n^2 + (1-\lambda)n]|a_n - b_n| < m. \end{aligned}$$

By Theorem 2.1, $f \in \mathcal{K}_{\lambda g}$.

When $\lambda = 0$ we get the following result obtained by Bharanedhar *et.al.*

Corollary 2.2. [1] *Let $f \in \mathcal{A}$ and $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be a convex univalent function in Δ with $m = inf_{z\in\Delta}|g'(z)|$. If*

$$\sum_{n=2}^{\infty} n|a_n - b_n| < m - |1 - b_1|$$

then f is close-to-convex in Δ.

Theorem 2.3. *Let $f \in \mathcal{A}$ and g be a convex univalent function in Δ. If*

$$|f(z) - zg'(z)| < m, \ z \in \Delta$$

where $m = inf_{z\in\Delta}|zg'(z)|$ then f is close-to-star with respect to g in Δ.

Proof. The inequality

$$|f(z) - zg'(z)| < m, \ z \in \Delta$$

together with m satisfying the condition in the hypothesis gives

$$|f(z) - zg'(z)| < |zg'(z)|, \ z \in \Delta.$$

This implies

$$\left|\frac{f(z)}{zg'(z)} - 1\right| < 1, \ z \in \Delta.$$

Now, g being convex, by Alexander's Theorem, zg' is starlike and hence f is close-to-star in Δ.

Theorem 2.4. *Let $f \in \mathcal{A}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be an analytic, convex and univalent function on Δ. If*

$$\sum_{n=2}^{\infty} |a_n - nb_n| < m - |1 - b_1|$$

where $m = inf_{z\in\Delta}|zg(z)|$ then f is close-to-star in Δ.
Proof. Using the Taylor's series expansions of f and g, we have

$$\begin{aligned} &|f(z) - zg'(z)| \\ &= |1 - b_1 + \sum_{n=1}^{\infty}(a_n - nb_n)z^n| \\ &\leq |1 - b_1| + \sum_{n=2}^{\infty} |a_n - nb_n| \\ &< m. \end{aligned}$$

By Theorem 2.3, it follows that f is close-to-star in Δ.

Theorem 2.5. *Let $f \in \mathcal{A}$ and $g \in S^*$ in Δ such that $m = inf_{z\in\Delta}|g(z)|$. If*

$$|z^2 f''(z) + zf'(z) - g(z)| < m, \text{ for } z \in \Delta$$

then f is quasi-convex with respect to g.
Proof. By hypothesis,

$$|z^2 f''(z) + zf'(z) - g(z)| < |g(z)|, \ z \in \Delta,$$

from which it follows that

$$\left| \frac{z^2 f''(z) + zf'(z)}{g(z)} - 1 \right| < 1, \ z \in \Delta,$$

which implies $\Re\left(\frac{z(zf'(z))'}{g(z)}\right) > 0$. Hence f is quasi-convex with respect to g.

Theorem 2.6. *Let $f \in \mathcal{A}$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be a starlike univalent function in Δ such that $m = inf_{z\in\Delta}|g(z)|$. If*

$$\sum_{n=2}^{\infty} |n^2 a_n - b_n| < m - |b_1|$$

then f is quasi-convex in Δ.

Proof. Substituting the Taylor's series expansion of f and g, we have

$$\begin{aligned} &|z^2 f''(z) + zf'(z) - g(z)| \\ &= |b_1 + \sum_{n=2}^{\infty}(n(n-1)+n)a_n - b_n| \\ &\leq |b_1| + \sum_{n=2}^{\infty}|n^2 a_n - b_n| \\ &< m. \end{aligned}$$

The result follows by Theorem 2.5.

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