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ON SOME GROWTH PROPERTIES OF COMPOSITE ENTIRE AND MEROMORPHIC FUNCTIONS FROM THE VIEW POINT OF THEIR GENERALIZED TYPE (α, β) AND GENERALIZED WEAK TYPE (α, β)

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Abstract: The main aim of this paper is to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their generalized type (α, β) and generalized weak type (α, β), where α and β are continuous non-negative functions defined on $(-\infty, +\infty)$.

Keywords and Phrases: Entire function, meromorphic function, growth, generalized order (α, β), generalized type (α, β), generalized weak type (α, β).

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1. Introduction, Definitions and Notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [7, 9, 14]. We also use the standard notations and definitions of the theory of entire functions which are available in [13] and therefore we do not explain those in details. Let f be an entire function and $M_f(r) = \max\{|f(z)| : |z| = r\}$.

When f is meromorphic, the Nevanlinna's characteristic function $T_f(r)$ (see [7, p. 4]) plays the same role as $M_f(r)$, which is defined as

$$T_f(r) = N_f(r) + m_f(r),$$

wherever the function $N_f(r,a)(\overline{N}_f(r,a))$ known as counting function of a-points (distinct a-points) of meromorphic f is defined as follows:

$$N_f(r,a) = \int_0^r \frac{n_f(t,a) - n_f(0,a)}{t} dt + n_f(0,a)\log r$$

$$\left(\overline{N}_f(r,a) = \int_0^r \frac{\overline{n}_f(t,a) - \overline{n}_f(0,a)}{t} dt + \overline{n}_f(0,a)\log r\right),$$

in addition we represent by $n_f(r,a)(\overline{n}_f(r,a))$ the number of a-points (distinct a-points) of f in $|z| \le r$ and an ∞ -point is a pole of f. In many occasions $N_f(r,\infty)$ and $\overline{N}_f(r,\infty)$ are symbolized by $N_f(r)$ and $\overline{N}_f(r)$ respectively.

On the other hand, the function $m_f(r,\infty)$ alternatively indicated by $m_f(r)$ known as the proximity function of f is defined as:

$$m_f(r) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta, \quad \text{where}$$

$$\log^+ x = \max(\log x, 0) \text{ for all } x \geqslant 0 .$$

Also we may employ $m(r, \frac{1}{f-a})$ by $m_f(r,a)$.

For an entire function f, the Nevanlinna's characteristic function $T_f(r)$ of f is defined as

$$T_f(r) = m_f(r).$$

Moreover, if f is non-constant entire then $T_f(r)$ is also strictly increasing and continuous function of r. Therefore its inverse $T_f^{-1} : (T_f(0),\infty) \to (0,\infty)$ exists and is such that $\lim\limits_{s\to\infty} T_f^{-1}(s) = \infty$. For $x \in [0,\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, we define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$, with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$, and $\exp^{[-1]} x = \log x$. Further we assume that p and q always denote positive integers. Now considering this, let us recall that Juneja et al. [8] defined the (p,q)-th order and (p,q)-th lower order of an entire function as follows:

Definition 1. [8] *Let $p \geq q$. The (p,q)-th order $\rho^{(p,q)}(f)$ and (p,q)-th lower order $\lambda^{(p,q)}(f)$ of an entire function f are defined as:*

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r}.$$

If f is a meromorphic function, then

$$\rho^{(p,q)}(f) = \limsup_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \text{ and } \lambda^{(p,q)}(f) = \liminf_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.$$

For any entire function f, using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.$ [7]$\}$, one can easily verify that

$$\begin{aligned}\rho^{(p,q)}(f) &= \limsup_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \limsup_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r} \\ \text{and } \lambda^{(p,q)}(f) &= \liminf_{r\to+\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} = \liminf_{r\to+\infty} \frac{\log^{[p-1]} T_f(r)}{\log^{[q]} r}.\end{aligned}$$

when $p \geq 2$.

The function f is said to be of regular (p,q) growth when (p,q) -th order and (p,q)-th lower order of f are the same. Functions which are not of regular (p,q) growth are said to be of irregular (p,q) growth.

Extending the notion of (p,q)-th order, recently Shen et al. [11] introduced the new concept of $[p,q]$-φ order of entire and meromorphic functions where $p \geq q$. Later on, combining the definition of (p,q)-order and $[p,q]$-φ order, Biswas (see, e.g., [2]) redefined the (p,q)-order of entire and meromorphic functions without restriction $p \geq q$.

However the above definition is very useful for measuring the growth of entire and meromorphic functions. If $p = l$ and $q = 1$ then we write $\rho^{(l,1)}(f) = \rho^{(l)}(f)$ and $\lambda^{(l,1)}(f) = \lambda^{(l)}(f)$ where $\rho^{(l)}(f)$ and $\lambda^{(l)}(f)$ are respectively known as generalized order and generalized lower order of entire or meromorphic function f. For details about generalized order one may see [10]. Also for $p = 2$ and $q = 1$, we respectively denote $\rho^{(2,1)}(f)$ and $\lambda^{(2,1)}(f)$ by $\rho(f)$ and $\lambda(f)$ which are classical growth indicators such as order and lower order of entire or meromorphic function f.

Now let L be a class of continuous non-negative functions α defined on $(-\infty, +\infty)$ such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ with $\alpha(x) \uparrow +\infty$ as $x \to +\infty$. For any $\alpha \in L$, we say that $\alpha \in L_1^0$, if $\alpha((1 + o(1))x) = (1 + o(1))\alpha(x)$ as $x \to +\infty$ and

$\alpha \in L_2^0$, if $\alpha(\exp((1+o(1))x)) = (1+o(1))\alpha(\exp(x))$ as $x \to +\infty$. Finally for any $\alpha \in L$, we also say that $\alpha \in L_1$, if $\alpha(cx) = (1+o(1))\alpha(x)$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$ and $\alpha \in L_2$, if $\alpha(\exp(cx)) = (1+o(1))\alpha(\exp(x))$ as $x_0 \leq x \to +\infty$ for each $c \in (0, +\infty)$. Clearly, $L_1 \subset L_1^0$, $L_2 \subset L_2^0$ and $L_2 \subset L_1$. Further we assume that throughout the present paper $\alpha_2, \beta, \beta_1, \beta_2 \in L_1$ and $\alpha_1 \in L_2$ unless otherwise specifically stated.

Considering the above, Sheremeta [12] introduced the concept of generalized order (α, β) of an entire function. For details about generalized order (α, β) one may see [12].

Now, we shall give the definition of the generalized order (α, β) of a entire function which considerably extend the definition of φ-order introduced by Chyzhykov et al. [6]. In order to keep accordance with Definition 1, have gave a minor modification to the original definition of generalized order (α, β) of an entire function (e.g. see, [12]).

Definition 2. *The generalized order (α, β) denoted by $\rho_{(\alpha,\beta)}[f]$ and generalized lower order (α, β) denoted by $\lambda_{(\alpha,\beta)}[f]$ of an entire function f are defined as:*

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)}, \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \text{ where } \alpha \in L_1.$$

If f is a meromorphic function, then

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ where } \alpha \in L_2.$$

Using the inequality $T_f(r) \leq \log M_f(r) \leq 3T_f(2r)$ $\{cf.[7]\}$, for an entire function f, one may easily verify that

$$\rho_{(\alpha,\beta)}[f] = \limsup_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} = \limsup_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)}, \text{ and}$$
$$\lambda_{(\alpha,\beta)}[f] = \liminf_{r\to+\infty} \frac{\alpha(M_f(r))}{\beta(r)} \liminf_{r\to+\infty} \frac{\alpha(\exp(T_f(r)))}{\beta(r)} \text{ when } \alpha \in L_2.$$

Definition 1 is a special case of Definition 2 for $\alpha(r) = \log^{[p]} r$ and $\beta(r) = \log^{[q]} r$.

Now in order to refine the growth scale namely the generalized order (α, β), we introduce the definitions of another growth indicators, called generalized type (α, β) and generalized lower type (α, β) respectively of *an entire* function which are as follows:

Definition 3. *The generalized type* (α, β) *denoted by* $\sigma_{(\alpha,\beta)}[f]$ *and generalized lower type* (α, β) *denoted by* $\overline{\sigma}_{(\alpha,\beta)}[f]$ *of an entire function* f *having finite positive generalized order* (α, β) $(0 < \rho_{(\alpha,\beta)}[f] < \infty)$ *are defined as :*

$$\begin{aligned} \sigma_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}} \text{ and} \\ \overline{\sigma}_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}}, \ (\alpha \in L_1). \end{aligned}$$

If f *is a meromorphic function, then*

$$\begin{aligned} \sigma_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}} \text{ and} \\ \overline{\sigma}_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\rho_{(\alpha,\beta)}[f]}}, \ (\alpha \in L_2). \end{aligned}$$

It is obvious that $0 \le \overline{\sigma}_{(\alpha,\beta)}[f] \le \sigma_{(\alpha,\beta)}[f] \le \infty$.

Analogously, to determine the relative growth of two entire functions having same non-zero finite generalized lower order (α, β), one can introduced the definition of generalized weak type (α, β) and generalized upper weak type (α, β) of a entire function f of finite positive generalized lower order (α, β), $\lambda_{(\alpha,\beta)}[f]$ in the following way:

Definition 4. *The generalized upper weak type* (α, β) *denoted by* $\overline{\tau}_{(\alpha,\beta)}[f]$ *and generalized weak type* (α, β) *denoted by* $\tau_{(\alpha,\beta)}[f]$ *of an entire function* f *having finite positive generalized lower order* (α, β) $(0 < \lambda_{(\alpha,\beta)}[f] < \infty)$ *are defined as:*

$$\begin{aligned} \overline{\tau}_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and} \\ \tau_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\exp(\alpha(M_f(r)))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}, (\alpha \in L_1). \end{aligned}$$

If f *is a meromorphic function, then*

$$\begin{aligned} \overline{\tau}_{(\alpha,\beta)}[f] &= \limsup_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}} \text{ and} \\ \tau_{(\alpha,\beta)}[f] &= \liminf_{r\to+\infty} \frac{\exp(\alpha(\exp(T_f(r))))}{(\exp(\beta(r)))^{\lambda_{(\alpha,\beta)}[f]}}, (\alpha \in L_2). \end{aligned}$$

It is obvious that $0 \leq \tau_{(\alpha,\beta)}[f] \leq \overline{\tau}_{(\alpha,\beta)}[f] \leq \infty$.

In this paper we wish to prove some results related to the growth rates of composite entire and meromorphic functions on the basis of their generalized order (α, β), generalized type (α, β) and generalized weak type (α, β). In fact some works in this direction have already been explored in [3, 4, 5].

2. Main Results

First we present a lemma which will be needed in the sequel.

Lemma 1. [1] *Let f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f(g)}(r) \leqslant \{1 + o(1)\} \frac{T_g(r)}{\log M_g(r)} T_f(M_g(r)).$$

Now we present the main results of the paper.

Theorem 1. *Let f be meromorphic and g be an entire function such that $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$ and $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ where $\beta_1(r) \leq \exp(\alpha_2(r))$. Then*

$$\limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Proof. We get from Lemma 1 and the inequality $T_g(r) \leq \log M_g(r)$ $\{cf.[7]\}$ for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f(g)}(r))) \leqslant (1 + o(1))\alpha_1(\exp(T_f(M_g(r))))$$

$$i.e.,\ \alpha_1(\exp(T_{f(g)}(r))) \leqslant (1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)\beta_1(M_g(r))$$

$$i.e.,\ \alpha_1(\exp(T_{f(g)}(r))) \leqslant (1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon) \exp(\alpha_2(M_g(r)))$$

$$i.e.,\ \alpha_1(\exp(T_{f(g)}(r))) \leqslant$$

$$(1 + o(1))(\rho_{(\alpha_1,\beta_1)}[f] + \varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g] + \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \tag{2.1}$$

Now from the definition of $\lambda_{(\alpha_1,\beta_1)}[f]$, we obtain for all sufficiently large values of r that

$$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}))) \geq (\lambda_{(\alpha_1,\beta_1)}[f] - \varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}. \tag{2.2}$$

Therefore from (2.1) and (2.2), it follows for all sufficiently large values of r that

$$\frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq$$

$$\frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}{(\lambda_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]}}$$

$$i.e., \limsup_{r\to+\infty}\frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))}\leq\frac{\sigma_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Thus the theorem is established.

Remark 1. *In Theorem 1, if we replace the condition "$\sigma_{(\alpha_2,\beta_2)}[g]<\infty$" by "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$ " and other conditions remain same, then Theorem 1 remains valid with "$\lambda_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" instead of "$\rho_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_2,\beta_2)}[g]$ " respectively.*

Remark 2. *In Theorem 1, if we replace the conditions "$0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<\infty$ and $\sigma_{(\alpha_2,\beta_2)}[g]<\infty$ " by "$\rho_{(\alpha_1,\beta_1)}[f]<\infty$, $\lambda_{(\alpha_2,\beta_2)}[g]>0$ and $\sigma_{(\alpha_2,\beta_2)}[g]<\infty$ where $\alpha_2\in L_2$ " and other condition remains same, then Theorem 1 remains valid with "$\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$ " and "$\lambda_{(\alpha_2,\beta_2)}[g]$" instead of "$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" and "$\lambda_{(\alpha_1,\beta_1)}[f]$ " respectively.*

Remark 3. *In Theorem 1, if we replace the conditions "$0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<\infty$ and $\sigma_{(\alpha_2,\beta_2)}[g]<\infty$ " by "$\rho_{(\alpha_1,\beta_1)}[f]<\infty$, $\lambda_{(\alpha_2,\beta_2)}[g]>0$ and $\overline{\tau}_{(\alpha_2,\beta_2)}[g]<\infty$ where $\alpha_2\in L_2$" and other condition remains same, then Theorem 1 remains valid with "$\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" instead of "$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_1,\beta_1)}[f]$ " and "$\sigma_{(\alpha_2,\beta_2)}[g]$" respectively.*

Using the notion of generalized lower type (α,β) we may state the following theorem without its proof because it can be carried out in the line of Theorem 1.

Theorem 2. *Let f be meromorphic and g be an entire function such that $0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<\infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$ where $\beta_1(r)\leq\exp(\alpha_2(r))$. Then*

$$\liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))}\leq\frac{\overline{\sigma}_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_1,\beta_1)}[f]}.$$

Remark 4. *In Theorem 2, if we replace the conditions "$0<\lambda_{(\alpha_1,\beta_1)}[f]\leq\rho_{(\alpha_1,\beta_1)}[f]<\infty$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$" by "$\rho_{(\alpha_1,\beta_1)}[f]<\infty$, $\lambda_{(\alpha_2,\beta_2)}[g]>0$ and $\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$ where $\alpha_2\in L_2$" and other condition remains same, then Theorem 2 remains valid with "$\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" and "$\lambda_{(\alpha_2,\beta_2)}[g]$" instead of "$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" and "$\lambda_{(\alpha_1,\beta_1)}[f]$ " respectively.*

Remark 5. *In Theorem 2, if we replace the condition "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]<\infty$" by "$\tau_{(\alpha_2,\beta_2)}[g]<\infty$ " and other conditions remain same, then Theorem 2 remains*

valid with
"$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$" *and* "$\tau_{(\alpha_2,\beta_2)}[g]$" *instead of*
"$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" *and* "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$ " *respectively.*

Remark 6. *In Theorem 2, if we replace the conditions* "$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$ *and* $\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" *by* "$\rho_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ *and* $\tau_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\alpha_2 \in L_2$" *and other condition remains same, then Theorem 2 remains valid with* "$\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$ ", "$\lambda_{(\alpha_2,\beta_2)}[g]$" *and* "$\tau_{(\alpha_2,\beta_2)}[g]$" *instead of* "$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$", "$\lambda_{(\alpha_1,\beta_1)}[f]$" *and* "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$ " *respectively.*

Now we state the following theorem without its proof as it can easily be carried out in the line in the line of Theorem 1.

Theorem 3. *Let f be meromorphic and g be an entire function such that* $0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ *or* $0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\beta_1(r) \leq \exp(\alpha_2(r))$. *Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \sigma_{(\alpha_2,\beta_2)}[g].$$

Remark 7. *In Theorem 3, if we replace the condition* "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" *by* "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ " *and other conditions remain same, then Theorem 3 remains valid with*
"$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))$" *and* "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$ " *instead of*
"$\alpha_1(\exp(T_f(\beta_1^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))$" *and* "$\sigma_{(\alpha_2,\beta_2)}[g]$" *respectively.*

Remark 8. *In Theorem 3, if we replace the conditions* "$0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ *or* $0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" *by* "$\lambda_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\alpha_2 \in L_2$" *and other condition remains same, then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g] \cdot \lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Remark 9. *In Theorem 3, if we replace the conditions* "$0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ *or* $0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" *by* "$\rho_{(\alpha_1,\beta_1)}[f] < \infty$, $\rho_{(\alpha_2,\beta_2)}[g] > 0$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\alpha_2 \in L_2$ " *and other condition remains same, then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\rho_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\sigma_{(\alpha_2,\beta_2)}[g] \cdot \rho_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_2,\beta_2)}[g]}.$$

Remark 10. *In Theorem 3, if we replace the conditions* "$0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ *or* $0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$ *and* $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" *by* "$\lambda_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$ *and*

$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *where* $\alpha_2 \in L_2$*" and other condition remains same, then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g]\cdot\lambda_{(\alpha_1,\beta_1)}[f]}{\lambda_{(\alpha_2,\beta_2)}[g]}.$$

Remark 11. *In Remark 10, if we replace the conditions "*$\lambda_{(\alpha_1,\beta_1)}[f] < \infty$, $\lambda_{(\alpha_2,\beta_2)}[g] > 0$*" by "*$\rho_{(\alpha_1,\beta_1)}[f] < \infty$, $\rho_{(\alpha_2,\beta_2)}[g] > 0$ *" and other conditions remain same, then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\alpha_2(\exp(T_g(\beta_2^{-1}(\exp(\beta_2(r)))^{\lambda_{(\alpha_2,\beta_2)}[g]})))} \leq \frac{\overline{\tau}_{(\alpha_2,\beta_2)}[g]\cdot\rho_{(\alpha_1,\beta_1)}[f]}{\rho_{(\alpha_2,\beta_2)}[g]}.$$

Theorem 4. *Let* f *be meromorphic and* g *be an entire function such that* (i) $0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$, (ii) $\rho_{(\alpha_1,\beta_1)}[f] = \rho_{(\alpha_2,\beta_2)}[g]$, (iii) $\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ *and* (iv) $0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ *where* $\beta_1(r) \leq \exp(\alpha_2(r))$ *. Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}.$$

Proof. In view of condition (ii),we obtain from (2.1) for all sufficiently large values of r that

$$\alpha_1(\exp(T_{f(g)}(r))) \leqslant (1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{2.3}$$

Again in view of Definition 3 we get for a sequence of values of r tending to infinity that

$$\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r)))))) \geq (\sigma_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]}. \tag{2.4}$$

Now from (2.3) and (2.4), it follows for a sequence of values of r tending to infinity that

$$\frac{\alpha_1(\exp(T_{f(g)}(r)))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{(1+o(1))(\rho_{(\alpha_1,\beta_1)}[f]+\varepsilon)(\sigma_{(\alpha_2,\beta_2)}[g]+\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]}}{(\sigma_{(\alpha_1,\beta_1)}[f]-\varepsilon)(\exp(\beta_2(r)))^{\rho_{(\alpha_1,\beta_1)}[f]}}.$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\sigma_{(\alpha_1,\beta_1)}[f]}.$$

Remark 12. *In Theorem 4, if we replace the conditions "*$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$*" by "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$*" respectively and other conditions remain same, then Theorem 4 remains valid with "*$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$*" instead of "*$\sigma_{(\alpha_2,\beta_2)}[g]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]$ *" respectively.*

Remark 13. *In Theorem 4, if we replace the conditions "*$0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$*" and "*$0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$ *" by "*$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$ *" and "*$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$*" respectively and other conditions remain same, then Theorem 4 remains valid with "*$\lambda_{(\alpha_1,\beta_1)}[f]$ *" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$*" instead of "*$\rho_{(\alpha_1,\beta_1)}[f]$*" and "*$\sigma_{(\alpha_1,\beta_1)}[f]$*" respectively.*

Remark 14. *In Theorem 4, if we replace the condition "*$0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$*" by "*$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$*" and other conditions remain same, then Theorem 4 remains valid with "limit superior" and "*$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$*" instead of "limit inferior" and "*$\sigma_{(\alpha_1,\beta_1)}[f]$*" respectively.*

Now using the concept of generalized upper weak type (α, β), we may state the following theorem without its proof since it can be carried out in the line of Theorem 4.

Theorem 5. *Let f be meromorphic and g be an entire function such that* (*i*) $0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$, (*ii*) $\lambda_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$, (*iii*) $\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$ *and* (*iv*) $0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ *where* $\beta_1(r) \leq \exp(\alpha_2(r))$. *Then*

$$\liminf_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f] \cdot \overline{\tau}_{(\alpha_2,\beta_2)}[g]}{\overline{\tau}_{(\alpha_1,\beta_1)}[f]}.$$

Remark 15. *In Theorem 5, if we replace the conditions "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$*" by "*$\tau_{(\alpha_2,\beta_2)}[g] < \infty$*" and "*$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$*" respectively and other conditions remain same, then Theorem 5 remains valid with "*$\tau_{(\alpha_2,\beta_2)}[g]$ *" and "*$\tau_{(\alpha_1,\beta_1)}[f]$*" instead of "*$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$*" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$*" respectively.*

Remark 16. *In Theorem 5, if we replace the conditions "*$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$*" and "*$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$*" by "*$0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$ *" and "*$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$*" respectively and other conditions remain same, then Theorem 5 remains valid with "*$\lambda_{(\alpha_1,\beta_1)}[f]$*" and "*$\tau_{(\alpha_1,\beta_1)}[f]$*" instead of "*$\rho_{(\alpha_1,\beta_1)}[f]$ *" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$*" respectively.*

Remark 17. *In Theorem 5, if we replace the condition "*$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$*" by "*$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$ *" and other conditions remain same, then Theorem 5 remains valid with "limit superior" and "*$\tau_{(\alpha_1,\beta_1)}[f]$*" instead of "limit inferior" and "*$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$*" respectively.*

Remark 18. *In Theorem 5, if we replace the conditions "*$\lambda_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$*"*

and "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" by "$\lambda_{(\alpha_1,\beta_1)}[f] = \rho_{(\alpha_2,\beta_2)}[g]$" and "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$" respectively and other conditions remain same, then Theorem 5 remains valid with "$\sigma_{(\alpha_2,\beta_2)}[g]$" instead of "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$".

Remark 19. *In Theorem 5, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$", "$\lambda_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$ " and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ " by "$0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$", "$\rho_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$" and "$0 < \sigma_{(\alpha_1,\beta_1)}[f] < \infty$" respectively and other conditions remain same, then Theorem 5 remains valid with "$\sigma_{(\alpha_1,\beta_1)}[f]$" instead of "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$".*

Remark 20. *In Theorem 5, if we replace the conditions "$\lambda_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ " by "$\lambda_{(\alpha_1,\beta_1)}[f] = \rho_{(\alpha_2,\beta_2)}[g]$", "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$ " respectively and other conditions remain same, then Theorem 5 remains valid with "$\overline{\sigma}_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_1,\beta_1)}[f]$" instead of "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$ " and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$".*

Remark 21. *In Theorem 5, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$", "$\lambda_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$ ", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ " by "$0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$ ", "$\rho_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\tau_{(\alpha_2,\beta_2)}[g] < \infty$ " and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$" respectively and other condition remains same, then Theorem 5 remains valid with "$\tau_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$ " instead of "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\tau}_{(\alpha_1,\beta_1)}[f]$".*

Remark 22. *In Theorem 5, if we replace the conditions "$\lambda_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\tau}_{(\alpha_1,\beta_1)}[f] < \infty$ " by "$\lambda_{(\alpha_1,\beta_1)}[f] = \rho_{(\alpha_2,\beta_2)}[g]$", "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ " and "$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$" respectively and other conditions remain same, then*

$$\liminf_{r\to+\infty}\frac{\alpha_1(\exp(T_{f(g)}(r)))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\lambda_{(\alpha_1,\beta_1)}[f]\cdot\sigma_{(\alpha_2,\beta_2)}[g]}{\tau_{(\alpha_1,\beta_1)}[f]}.$$

Remark 23. *Remark 22 remains also valid with "limit superior" instead of "limit inferior".*

Remark 24. *In Remark 22, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$", "$\lambda_{(\alpha_1,\beta_1)}[f] = \rho_{(\alpha_2,\beta_2)}[g]$", "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ " and "$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$" by "$0 < \lambda_{(\alpha_1,\beta_1)}[f] < \infty$", "$\rho_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and "$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$" respectively and other condition remains same, then conclusion of Remark 22 remains valid with "$\overline{\tau}_{(\alpha_2,\beta_2)}[g]$" and "$\overline{\sigma}_{(\alpha_1,\beta_1)}[f]$" instead of "$\sigma_{(\alpha_2,\beta_2)}[g]$" and "$\tau_{(\alpha_1,\beta_1)}[f]$".*

Remark 25. *In Remark 22, if we replace the conditions "$0 < \lambda_{(\alpha_1,\beta_1)}[f] \leq \rho_{(\alpha_1,\beta_1)}[f] < \infty$", "$\lambda_{(\alpha_1,\beta_1)}[f] = \rho_{(\alpha_2,\beta_2)}[g]$", "$\sigma_{(\alpha_2,\beta_2)}[g] < \infty$ " and "$0 < \tau_{(\alpha_1,\beta_1)}[f] < \infty$" by "$0 < \rho_{(\alpha_1,\beta_1)}[f] < \infty$", "$\rho_{(\alpha_1,\beta_1)}[f] = \lambda_{(\alpha_2,\beta_2)}[g]$ ", "$\overline{\tau}_{(\alpha_2,\beta_2)}[g] < \infty$" and*

"$0 < \overline{\sigma}_{(\alpha_1,\beta_1)}[f] < \infty$" respectively and other condition remains same, then

$$\limsup_{r\to+\infty} \frac{\alpha_1(\exp(T_{f(g)}(r)))}{\exp(\alpha_1(\exp(T_f(\beta_1^{-1}(\beta_2(r))))))} \leq \frac{\rho_{(\alpha_1,\beta_1)}[f]\cdot\overline{\tau}_{(\alpha_2,\beta_2)}[g]}{\overline{\sigma}_{(\alpha_1,\beta_1)}[f]}.$$

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