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SOME SPECIAL FAMILIES OF HOLOMORPHIC AND AL-OBOUDI TYPE BI-UNIVALENT FUNCTIONS ASSOCIATED WITH (m, n)-LUCAS POLYNOMIALS INVOLVING MODIFIED SIGMOID ACTIVATION FUNCTION

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Abstract: The aim of the present paper is to introduce some special families of holomorphic and Al-Oboudi type bi-univalent functions associated with (m, n)-Lucas polynomials involving modified sigmoid activation function $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$ in the open unit disc $\mathfrak{D}$. We investigate the upper bounds on initial coefficients for functions of the form $g_\phi(z) = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j$, in these newly introduced special families and also discuss the Fekete-Szegö problem. Some interesting consequences of the results established here are also indicated.

Keywords and Phrases: Holomorphic function, Bi-univalent function, Fekete - Szegö inequality, (m, n)-Lucas polynomials, Modified sigmoid function.

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1. Introduction and Preliminaries

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers and $\mathbb{C}$ be the set of complex numbers. Let $\mathcal{A}$ be the family of normalized functions that have the form

$$g(z) = z + \sum_{j=2}^{\infty} d_j z^j, \tag{1}$$

which are holomorphic in $\mathfrak{D} = \{z \in \mathbb{C} : |z| < 1\}$ and let $\mathcal{S}$ be the collection of all members of $\mathcal{A}$ that are univalent in $\mathfrak{D}$. It is well- known (see [6]) that every function $g \in \mathcal{S}$ has an inverse g^{-1} satisfying $z = g^{-1}(g(z))$, $z \in \mathfrak{D}$ and $\omega = g(g^{-1}(\omega))$, $|\omega| < r_0(g)$, $r_0(g) \geq 1/4$, where

$$g^{-1} = f(\omega) = \omega - d_2\omega^2 + (2d_2^2 - d_3)\omega^3 - (5d_2^3 - 5d_2d_3 + d_4)\omega^4 + \dots . \tag{2}$$

A member g of $\mathcal{A}$ is said to be bi-univalent in $\mathfrak{D}$ if both g and g^{-1} are univalent in $\mathfrak{D}$. We denote the family of bi-univalent functions that have the form (1), by $\sum$. For detailed study and various subfamilies of the family $\sum$, one can refer the works of [2], [5], [10], [13] and [16].

Let $m(x)$ and $n(x)$ be polynomials with real coefficients. The (m, n)- Lucas polynomials $L_j(m(x), n(x), x)$ or briefly $L_j(x)$ are given by the following recurrence relation (see [12]):

$$L_j(x) = m(x)L_{j-1}(x) + n(x)L_{j-2}(x), \quad L_0(x) = 2,\ L_1(x) = m(x), \tag{3}$$

where $j \in \mathbb{N} - \{1\}$. It is clear from (3) that $L_2(x) = m^2(x) + 2n(x)$, $L_3(x) = m^3(x) + 3m(x)n(x)$. The generating function of the (m, n)-Lucas polynomial sequence $L_j(x)$ is given by

$$\mathcal{G}(x, z) := \sum_{j=0}^{\infty} L_j(x)z^j = \frac{2 - m(x)z}{1 - m(x)z - n(x)z^2}. \tag{4}$$

Note that for particular choices of $m(x)$ and $n(x)$, the (m, n)-Lucas polynomial $L_j(x)$ leads to various polynomials, among those we list following few here (see, for more details [3]):
i) $L_j(x, 1, x) = \mathcal{L}_j(x)$, the Lucas polynomials, *ii)* $L_j(2x, 1, x) = P_j(x)$, the Pell-Lucas polynomials, *iii)* $L_j(1, 2x, x) = J_j(x)$, the Jacobsthal polynomials, *iv)* $L_j(3x, -2, x) = F_j(x)$, the Fermat-Lucas polynomials, *v)* $L_j(2x, -1, x) = T_j(x)$, the first kind Chebyshev polynomials.

In the literature, the coefficient estimates and celebrated Fekete- Szegö inequality are found for bi-univalent functions associated with certain polynomials like the Chebyshev polynomials, the Horadam polynomials, the Fibonacci polynomials and the (m,n)-Lucas polynomials. We also note that the above polynomials and other special polynomials are potentially important in the mathematical, physical, statistical and engineering sciences. More details associated with these polynomials can be found in [1], [9], [11], [14] [18], [19] and [20].

Let $\mathcal{A}_\phi$ denote the family of functions of the form

$$g_\phi(z) = z + \sum_{j=2}^{\infty} \frac{2}{1+e^{-s}} d_j z^j = z + \sum_{j=2}^{\infty} \phi(s) d_j z^j,$$

where $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$, is a modified sigmoid function. Clearly $\phi(0) = 1$ and hence $\mathcal{A}_1 := \mathcal{A}$ (see [7]).

Definition 1.1. *For $g_\phi \in \mathcal{A}_\phi$, $k \in \mathbb{N}_0, \beta \geq 0$, an Al-Oboudi type differential operator $D_\beta^k : \mathcal{A}_\phi \to \mathcal{A}_\phi$, is defined by*

$$\begin{aligned}
D_\beta^0 g_\phi(z) &= g_\phi(z), \\
D_\beta^1 g_\phi(z) &= (1-\beta) g_\phi(z) + \beta z g_\phi'(z), \\
&\vdots \\
D_\beta^k g_\phi(z) &= D_\beta(D_\beta^{k-1} g_\phi(z)),\ z \in \mathfrak{D}.
\end{aligned}$$

Remark 1.1. *If $g_\phi(z) = z + \sum\limits_{j=2}^{\infty} \phi(s) d_j z^j \in \mathcal{A}_\phi$, $z \in \mathfrak{D}$, then*

$$D_\beta^k g_\phi(z) = z + \sum_{j=2}^{\infty} (1+(j-1)\beta)^k \phi(s) d_j z^j,\ z \in \mathfrak{D}.$$

When $\phi(s) = 1$, we have the differential operator defined by Al-Oboudi [4], which reduces to Sălăgean differential operator, when $\beta = 1$ [15].

We recall the principle of subordination between two holomorphic functions $g(z)$ and $f(z)$ in $\mathfrak{D}$. It is known that $g(z)$ is subordinate to $f(z)$, written as $g(z) \prec f(z)$, $z \in \mathfrak{D}$, if there is a $\psi(z)$ holomorphic in $\mathfrak{D}$, with $\psi(0) = 0$ and $|\psi(z)| < 1$, $z \in \mathfrak{D}$, such that $g(z) = f(\psi(z))$. Moreover, $g(z) \prec f(z)$ is equivalent to $g(0) = f(0)$ and $g(\mathfrak{D}) \subset f(\mathfrak{D})$, if f is univalent in $\mathfrak{D}$.

Inspired by recent trends on bi-univalent functions, we define the following special families of $\sum$ by making use of the (m,n)-Lucas polynomials, which are given by the recurrence relation (3) and the generating function (4).

Definition 1.2. *A function $g(z)$ in $\sum$ of the form (1) is said to be in the family*

$\mathfrak{S}_{\sum}(x, \gamma, \mu, \beta, k, \phi(s))$, $0 \leq \gamma \leq 1, \mu \geq 0, \mu \geq \gamma, \beta \geq 0, k \in \mathbb{N} \cup \{0\}$ *and* $\phi(s) = \frac{2}{1+e^{-s}}, s \geq 0,$ *if*

$$\frac{z(D^k_\beta g_\phi(z))' + \mu z^2(D^k_\beta g_\phi(z))''}{(1-\gamma)D^k_\beta g_\phi(z) + \gamma z(D^k_\beta g_\phi(z))'} \prec \mathcal{G}(x, z) - 1, \; z \in \mathfrak{D}$$

and

$$\frac{\omega(D^k_\beta f_\phi(\omega))' + \mu\omega^2(D^k_\beta f_\phi(\omega))''}{(1-\gamma)D^k_\beta f_\phi(\omega) + \gamma\omega(D^k_\beta f_\phi(\omega))'} \prec \mathcal{G}(x, \omega) - 1, \; \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ *is an extension of* g^{-1} *to* $\mathfrak{D}$ *given by (2) and* $\mathcal{G}$ *is as in (4).*

It is interesting to note that the special values of γ and μ lead the family $\mathfrak{S}_{\sum}(x, \gamma, \mu, \beta, k, \phi(s))$ to the following various subfamilies:

1. For $\gamma = \mu = \frac{1}{2}$, we get the family $\mathscr{K}_{\sum}(x, \beta, k, \phi(s)) = \mathfrak{S}_{\sum}(x, \frac{1}{2}, \frac{1}{2}, \beta, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$\frac{(z^2(D^k_\beta g_\phi(z))')'}{(zD^k_\beta g_\phi(z))'} \prec \mathcal{G}(x, z) - 1 \quad \text{and} \quad \frac{(\omega^2(D^k_\beta f_\phi(\omega))')'}{(\omega D^k_\beta f_\phi(\omega))'} \prec \mathcal{G}(x, \omega) - 1, \; z, \, \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

2. When $\gamma = 0, \mu = \frac{1}{2}$, we obtain the family $\mathscr{J}_{\sum}(x, \beta, k, \phi(s)) = \mathfrak{S}_{\sum}(x, 0, \frac{1}{2}, \beta, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$\frac{(z^2(D^k_\beta g_\phi(z))')'}{2\, D^k_\beta g_\phi(z)} \prec \mathcal{G}(x, z) - 1 \quad \text{and} \quad \frac{(\omega^2(D^k_\beta f_\phi(\omega))')'}{2\, D^k_\beta f_\phi(\omega)} \prec \mathcal{G}(x, \omega) - 1, \; z, \, \omega \in \mathfrak{D},$$

where $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

3. On putting $\gamma = \frac{1}{2}, \mu = 1$, we have the family $\mathscr{L}_{\sum}(x, \beta, k, \phi(s)) = \mathfrak{S}_{\sum}(x, \frac{1}{2}, 1, \beta, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$\frac{2z(z(D^k_\beta g_\phi(z))')'}{(zD^k_\beta g_\phi(z))'} \prec \mathcal{G}(x, z) - 1 \quad \text{and} \quad \frac{2\omega(\omega(D^k_\beta f_\phi(\omega))')'}{(\omega D^k_\beta f\omega))'} \prec \mathcal{G}(x, \omega) - 1,$$

where $z, \omega \in \mathfrak{D}$, $f_\phi(\omega) = g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

4. On taking $\gamma = 0$, we get the family $\mathscr{P}_{\sum}(x, \mu, \beta, k, \phi(s)) = \mathfrak{S}_{\sum}(x, 0, \mu, \beta, k, \phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$\left(\frac{z(D^k_\beta g_\phi(z))'}{D^k_\beta g_\phi(z)}\right)\left(1 + \mu\frac{z(D^k_\beta g_\phi(z))''}{(D^k_\beta g_\phi(z))'}\right) \prec \mathcal{G}(x, z) - 1, \; z \in \mathfrak{D}$$

and

$$\left(\frac{\omega(D_\beta^k f_\phi(\omega))'}{D_\beta^k f_\phi(\omega)}\right)\left(1+\mu\frac{\omega(D_\beta^k f_\phi(\omega))''}{(D_\beta^k f_\phi(\omega))'}\right) \prec \mathcal{G}(x,\omega)-1,\ \omega\in\mathfrak{D},$$

where $f_\phi(\omega)=g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

Definition 1.3. *A function $g(z)$ in $\sum$ of the form (??) is said to be in the family $\mathfrak{M}_{\sum}(x,\gamma,\mu,\ \beta,\ k)$, $0\le\gamma\le1$, $\mu\ge0$, $\mu\ge\gamma$, $\beta\ge0$, $k\in\mathbb{N}\cup\{0\}$ and $\phi(s)=\frac{2}{1+e^{-s}}$, $s\ge0$, if*

$$\frac{z(D_\beta^k g_\phi(z))'+\mu z^2(D_\beta^k g_\phi(z))''}{(1-\gamma)z+\gamma z(D_\beta^k g_\phi(z))'} \prec \mathcal{G}(x,z)-1,\ z\in\mathfrak{D}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))'+\mu\omega^2(D_\beta^k f_\phi(\omega))''}{(1-\gamma)\omega+\gamma\omega(D_\beta^k f_\phi(\omega))'} \prec \mathcal{G}(x,\omega)-1,\ \omega\ \in\mathfrak{D},$$

where $f_\phi(\omega)=g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

It is easy to observe that the special values of γ lead the family $\mathfrak{M}_{\sum}$ $(x,\ \gamma,\ \mu,\ \beta,\ k,\phi(s))$ to the following various subfamilies:

1. For $\gamma=0$, we get the family $\mathfrak{K}_{\sum}(x,\mu,\ \beta,\ k,\phi(s))=\mathfrak{M}_{\sum}(x,0,\mu,\ \beta,\ k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$(D_\beta^k g_\phi(z))'+\mu z(D_\beta^k g_\phi(z))'' \prec \mathcal{G}(x,z)-1 \text{ and } (D_\beta^k f_\phi(\omega))'+\mu\omega(D_\beta^k f_\phi(\omega))'' \prec \mathcal{G}(x,\omega)-1,$$

where $z,\ \omega\ \in\mathfrak{D}$, $f_\phi(\omega)=g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

2. When $\gamma=1$, we have the family $\mathfrak{L}_{\sum}(x,\mu,\ \beta,\ k,\phi(s))=\mathfrak{M}_{\sum}(x,1,\mu,\ \beta,\ k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$1+\mu\left(\frac{z(D_\beta^k g_\phi(z))''}{(D_\beta^k g_\phi(z))'}\right) \prec \mathcal{G}(x,z)-1 \quad\text{and}\quad 1+\mu\left(\frac{\omega(D_\beta^k f_\phi(\omega))''}{(D_\beta^k f_\phi(\omega))'}\right) \prec \mathcal{G}(x,\omega)-1,$$

where $z,\ \omega\in\mathfrak{D}$, $\mu\ge1$, $f_\phi(\omega)=g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

Definition 1.4. *A function $g(z)$ in $\sum$ of the form (1) is said to be in the family $\mathfrak{B}_{\sum}(x,\xi,\tau,\ \beta,\ k,\ \phi(s))$, $\xi\ge1$, $\tau\ge1$, $\beta\ge0$, $k\in\mathbb{N}\cup\{0\}$ and $\phi(s)=\frac{2}{1+e^{-s}}$, $s\ge0$, if*

$$\frac{(1-\xi)+\xi[(z(D_\beta^k g_\phi(z))')']^\tau}{(D_\beta^k g_\phi(z))'} \prec \mathcal{G}(x,z)-1, z\in\mathfrak{D}$$

and

$$\frac{(1-\xi)+\xi[(\omega(D^k_\beta f_\phi(\omega))')']^\tau}{(D^k_\beta f_\phi(\omega))'} \prec \mathcal{G}(x,\omega)-1, \omega\in\mathfrak{D},$$

where $f_\phi(\omega)=g_\phi^{-1}(\omega)$ *is an extension of* g^{-1} *to* $\mathfrak{D}$ *given by (2) and* $\mathcal{G}$ *is as in (4).*

Note that the particular values of ξ and τ lead the family $\mathfrak{B}_{\sum}(x,\xi,\tau,\ \beta,\ k,\phi(s))$ to the following various subfamilies:

1. When $\tau=1$, we have the family $\mathscr{M}_{\sum}(x,\xi,\ \beta,\ k,\phi(s))=\mathfrak{B}_{\sum}(x,\xi,1,\ \beta,\ k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$(1-\xi)\frac{1}{(D^k_\beta g_\phi(z))'}+\xi\left(1+\frac{z(D^k_\beta g_\phi(z))''}{(D^k_\beta g_\phi(z))'}\right)\prec\mathcal{G}(x,z)-1,\ z\in\mathfrak{D}$$

and

$$(1-\xi)\frac{1}{(D^k_\beta f_\phi(\omega))'}+\xi\left(1+\frac{\omega(D^k_\beta f_\phi(\omega))''}{(D^k_\beta f_\phi(\omega))'}\right)\prec\mathcal{G}(x,\omega)-1,\ \omega\in\mathfrak{D},$$

where $f_\phi(\omega)=g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

2. For $\xi=1$, we have the family $\mathfrak{N}_{\sum}(x,\tau,\ \beta,\ k,\phi(s))=\mathfrak{B}_{\sum}(x,1,\tau,\ \beta,\ k,\phi(s))$ of functions $g(z)$ in $\sum$ of the form (1) satisfying

$$\frac{[(z(D^k_\beta g_\phi(z))')']^\tau}{(D^k_\beta g_\phi(z))'}\prec\mathcal{G}(x,z)-1,\quad\text{and}\quad\frac{[(\omega(D^k_\beta f_\phi(\omega))')']^\tau}{(D^k f_\phi(\omega))'}\prec\mathcal{G}(x,\omega)-1,$$

where $z,\,\omega\in\mathfrak{D},\ f_\phi(\omega)=g_\phi^{-1}(\omega)$ is an extension of g^{-1} to $\mathfrak{D}$ given by (2) and $\mathcal{G}$ is as in (4).

The families $\mathfrak{S}_{\sum}(x,\gamma,\mu,\ 1,\ k,\ 1)$, $\mathfrak{M}_{\sum}(x,\gamma,\mu,\ 1,k,\ 1)$ and $\mathfrak{B}_{\sum}(x,\xi,\tau,\ 1,k,\ 1)$ were studied in [17].

For functions belonging to these newly defined families $\mathfrak{S}_{\sum}(x,\gamma,\mu,\ \beta,k,\phi(s))$, $\mathfrak{M}_{\sum}(x,\gamma,\mu,\ \beta,k,\ \phi(s))$ and $\mathfrak{B}_{\sum}(x,\xi,\tau,\ \beta,k,\ \phi(s))$, we derive the estimates for the coefficients $|d_2|$ and $|d_3|$ and also, we consider the celebrated Fekete- Szegö problem [8] in Section 2.

2. Coefficient estimates and Fekete-Szegö inequality

Theorem 2.1. *Let* $0\le\gamma\le1$, $\mu\ge0$, $\mu\ge\gamma$, $\beta\ge0$, $k\in\mathbb{N}\cup\{0\}$ *and* $\phi(s)=\frac{2}{1+e^{-s}}, s\ge0$. *If* $g(z)$ *of the form (1) is in* $\mathfrak{S}_{\sum}(x,\gamma,\mu,\ \beta,k,\ \phi(s))$, *then*

$$|d_2|\le\frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{2|\mu(2\mu-\gamma)\ m^2(x)+\lambda^2\,n(x)|}},\tag{5}$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{m^2(x)}{\lambda^2}+\frac{|m(x)|}{2(\lambda+\mu)}\right] \tag{6}$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{2(1+2\beta)^k\phi(s)(\lambda+\mu)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J \\ \frac{|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{2(1+2\beta)^k\phi(s)|\mu(2\mu-\gamma)m^2(x)+\lambda^2 n(x)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J, \end{cases} \tag{7}$$

where

$$J = \frac{1}{(\lambda+\mu)}\left|\mu(2\mu-\gamma)+\lambda^2\left(\frac{n(x)}{m^2(x)}\right)\right|, \tag{8}$$

$$\lambda = (1-\gamma+2\mu). \tag{9}$$

Proof. Let $g(z) \in \mathfrak{S}_{\sum}(x,\gamma,\mu,\ \beta,k,\ \phi(s))$. Then, for two holomorphic functions r and s such that $r(0)=s(0)=0, |r(z)|<1$ and $|s(\omega)|<1,\ z,\omega \in \mathfrak{D}$, and using Definition 1.2, we can write

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{(1-\gamma)D_\beta^k g_\phi(z) + \gamma z(D_\beta^k g_\phi(z))'} = \mathcal{G}(x, r(z)) - 1$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{(1-\gamma)D_\beta^k f_\phi(\omega) + \gamma\omega(D_\beta^k f_\phi(\omega))'} = \mathcal{G}(x, s(\omega)) - 1.$$

Or, equivalently

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{(1-\gamma)D_\beta^k g_\phi(z) + \gamma z(D_\beta^k g_\phi(z))'} = -1 + L_0(x) + L_1(x)r(z) + L_2(x)r^2(z) + ... \tag{10}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))' + \mu\omega^2(D_\beta^k f_\phi(\omega))''}{(1-\gamma)D_\beta^k f_\phi(\omega) + \gamma\omega(D_\beta^k f_\phi(\omega))'} = -1+L_0(x)+L_1(x)s(\omega)+L_2(x)s^2(\omega)+.... \tag{11}$$

From (10) and (11), in view of (3), we obtain

$$\frac{z(D_\beta^k g_\phi(z))' + \mu z^2(D_\beta^k g_\phi(z))''}{(1-\gamma)D_\beta^k g_\phi(z) + \gamma z(D_\beta^k(z))'} = 1 + L_1(x)r_1 z + [L_1(x)r_2 + L_2(x)r_1^2]z^2 + ... \tag{12}$$

and

$$\frac{\omega(D^k_\beta f_\phi(\omega))' + \mu\omega^2(D^k_\beta f_\phi(\omega))''}{(1-\gamma)D^k_\beta f_\phi(\omega) + \gamma\omega(D^k_\beta f_\phi(\omega))'} = 1+L_1(x)s_1\omega+[L_1(x)s_2+L_2(x)s_1^2]\omega^2+\dots. \quad (13)$$

It is well known that if $|r(z)| = |r_1 z + r_2 z^2 + r_3 z^3 + ...| < 1, \quad z \in \mathfrak{D}$ and $|s(\omega)| = |s_1\omega + s_2\omega^2 + s_3\omega^3 + ...| < 1, \quad \omega \in \mathfrak{D}$, then

$$|r_i| \le 1 \,\text{and}\, |s_i| \le 1 \,(i \in \mathbb{N}). \quad (14)$$

Comparing the corresponding coefficients in (12) and (13), we have

$$(1+\beta)^k\phi(s)\lambda\, d_2 = L_1(x)r_1 \quad (15)$$

$$\begin{aligned} 2(1+2\beta)^k\phi(s)(\lambda+\mu)d_3 - &(1+\beta)^{2k}\phi^2(s)(1+\gamma)\lambda\, d_2^2 \\ &= L_1(x)r_2 + L_2(x)r_1^2 \end{aligned} \quad (16)$$

$$-(1+\beta)^k\phi(s)\lambda\, d_2 = L_1(x)s_1 \quad (17)$$

$$\begin{aligned} -2(1+2\beta)^k\phi(s)(\lambda+\mu)d_3 + &(1+\beta)^{2k}\phi^2(s)(\gamma^2-(4+2\mu)\gamma+3+10\mu)d_2^2 \\ &= L_1(x)s_2 + L_2(x)s_1^2. \end{aligned} \quad (18)$$

where λ is as in (9). From (15) and (17), we can easily see that

$$r_1 = -s_1 \quad (19)$$

and also

$$2(1+\beta)^{2k}\phi^2(s)\lambda^2 d_2^2 = (r_1^2+s_1^2)(L_1(x))^2. \quad (20)$$

If we add (16) and (18), then we obtain

$$2(1+\beta)^{2k}\phi^2(s)((1-\gamma)\lambda+2\mu)d_2^2 = L_1(x)(r_2+s_2) + L_2(x)(r_1^2+s_1^2). \quad (21)$$

Substituting the value of $(r_1^2+s_1^2)$ from (20) in (21), we get

$$d_2^2 = \frac{(L_1(x))^3(r_2+s_2)}{2(1+\beta)^{2k}\phi^2(s)\left[((1-\gamma)\lambda+2\mu)(L_1(x))^2 - \lambda^2 L_2(x)\right]}, \quad (22)$$

which yields (5), on using (14).

Using (19) in the subtraction of (18) from (16), we obtain

$$d_3 = \frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k} d_2^2 + \frac{L_1(x)(r_2 - s_2)}{4(1+2\beta)^k\phi(s)(\lambda+\mu)}. \tag{23}$$

Then in view of (20), (23) becomes

$$d_3 = \frac{(L_1(x))^2(r_1^2 + s_1^2)}{2(1+2\beta)^k\phi(s)\lambda^2} + \frac{L_1(x)(r_2 - s_2)}{4(1+2\beta)^k\phi(s)(\lambda+\mu)},$$

which yields (6), on using (14).

From (22) and (23), for $\delta \in \mathbb{R}$, we get

$$|d_3 - \delta d_2^2| =$$
$$|m(x)| \left| \left(T(\delta, x) + \frac{1}{4(1+2\beta)^k\phi(s)(\lambda+\mu)} \right) r_2 + \left(T(\delta, x) - \frac{1}{4(1+2\beta)^k\phi(s)(\lambda+\mu)} \right) s_2 \right|,$$

where

$$T(\delta, x) = \frac{\left(\frac{(1+\beta)^{2k}\phi(s)}{(1+2\beta)^k} - \delta \right) m^2(x)}{4(1+\beta)^{2k}\phi^2(s)\left[\mu(\gamma - 2\mu)m^2(x) + \lambda^2 n(x)\right]}.$$

In view of (3), we conclude that

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{2(1+2\beta)^k\phi(s)(\lambda+\mu)} & ; 0 \leq |T(\delta, x)| \leq \frac{1}{4(1+2\beta)^k\phi(s)(\lambda+\mu)} \\ 2|m(x)||T(\delta, x)| & ; |T(\delta, x)| \geq \frac{1}{4(1+2\beta)^k\phi(s)(\lambda+\mu)}, \end{cases}$$

which yields (7) with J as in (8). This evidently completes the proof of Theorem 2.1.

Remark 2.1. *The results obtained in Theorem 2.1 coincide with Theorem 2 and Theorem 3 of [3], for $k = 0$ and $\mu = \gamma$, $(0 \leq \gamma \leq 1)$, $\phi(s) = 1$.*

Remark 2.2. *The results of Theorem 2.1 reduce to Corollary 1 and Corollary 3 of [3], when $k = \mu = \gamma = 0$, $\phi(s) = 1$.*

Remark 2.3. *Corollary 2 and Corollary 4 of [3] can be obtained from Theorem 2.1, by putting $k = 0$, $\phi(s) = 1$ and $\mu = \gamma = 1$.*

Theorem 2.2. *Let $0 \leq \gamma \leq 1$, $\mu \geq 0$, $\mu \geq \gamma$, $\beta \geq 0$, $k \in \mathbb{N} \cup \{0\}$ and $\phi(s) = \frac{2}{1+e^{-s}}$, $s \geq 0$. If $g(z)$ of the form (1) is in $\mathfrak{M}_{\sum}(x, \gamma, \mu, \beta, k, \phi(s))$, then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{|(4\mu(\mu-\gamma) + (1-\gamma+2\mu))m^2(x) + 8\nu^2 n(x)|}}, \tag{24}$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{m^2(x)}{4\nu^2}+\frac{|m(x)|}{3(\nu+\mu)}\right] \tag{25}$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{3(1+2\beta)^k\phi(s)(\nu+\mu)} & ;\ |1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}| \leq M \\ \frac{|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|(4\mu(\mu-\gamma)+(\nu+\mu))m^2(x)+8\nu^2n(x)|} & ;\ |1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}| \geq M, \end{cases} \tag{26}$$

where

$$M=\frac{1}{3(\nu+\mu)}\left|(4\mu(\mu-\gamma)+(1-\gamma+2\mu))+8\nu^2\left(\frac{n(x)}{m^2(x)}\right)\right|,$$

$$\nu=(1-\gamma+\mu). \tag{27}$$

Proof. Let $g(z) \in \mathfrak{M}_{\sum}(x,\gamma,\mu,\ \beta,k,\ \phi(s))$. Then, for two holomorphic functions r and s such that $r(0)=s(0)=0, |r(z)|=|r_1z+r_2z^2+r_3z^3+...|<1$ and $|s(\omega)|=|s_1\omega+s_2\omega^2+s_3\omega^3+...|<1$, $z,\omega \in \mathfrak{D}$, and using Definition 1.3, we can write

$$\frac{z(D_\beta^k g_\phi(z))'+\mu z^2(D_\beta^k g_\phi(z))''}{(1-\gamma)z+\gamma z(D_\beta^k g_\phi(z))'}=\mathcal{G}(x,r(z))-1 \tag{28}$$

and

$$\frac{\omega(D_\beta^k f_\phi(\omega))'+\mu\omega^2(D_\beta^k f_\phi(\omega))''}{(1-\gamma)\omega+\gamma\omega(D_\beta^k f_\phi(\omega))'}=\mathcal{G}(x,s(\omega))-1. \tag{29}$$

Following (10), (11), (12), and (13) in the proof of Theorem 2.1, one gets in view of (28) and (29)

$$2(1+\beta)^k\phi(s)\,\nu\,d_2=L_1(x)r_1 \tag{30}$$

$$3(1+2\beta)^k\phi(s)(\nu+\mu)d_3-4(1+\beta)^{2k}\phi^2(s)\nu\gamma d_2^2=L_1(x)r_2+L_2(x)r_1^2 \tag{31}$$

$$-2(1+\beta)^k\phi(s)\,\nu\,d_2=L_1(x)s_1 \tag{32}$$

$$\begin{aligned} -(1+2\beta)^k\phi(s)(\nu+\mu)d_3+2(1+\beta)^{2k}\phi^2(s)[2\gamma^2-(5+2\mu)\gamma+3(1+2\mu)]d_2^2 \\ =L_1(x)s_2+L_2(x)s_1^2. \end{aligned} \tag{33}$$

where ν is as in (27).

The results (24)-(26) of this theorem now follow from (30)-(33) by applying the procedure as in Theorem 2.1 with respect to (15)-(18).

Theorem 2.3. *Let* $\xi \geq 1$, $\tau \geq 1$, $\beta \geq 0$, $k \in \mathbb{N} \cup \{0\}$ *and* $\phi(s) = \frac{2}{1+e^{-s}}, s \geq 0$. *If* $g(z)$ *of the form (1) is in* $\mathfrak{B}_{\sum}(x, \xi, \tau, \beta, k, \phi(s))$, *then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k \phi(s)\sqrt{|((8\xi\tau^2 - 7\xi\tau + 1) - 4(2\xi\tau - 1)^2)m^2(x) - 8(2\xi\tau - 1)^2 n(x)|}}, \tag{34}$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k \phi(s)} \left[\frac{m^2(x)}{4(2\xi\tau - 1)^2} + \frac{|m(x)|}{3(3\xi\tau - 1)}\right] \tag{35}$$

and for $\delta \in \mathbb{R}$

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{3(1+2\beta)^k \phi(s)(3\xi\tau - 1)} & ; \quad |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}| \leq \Omega \\ \frac{\left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| |m(x)|^3}{(1+2\beta)^k \phi(s)|((8\xi\tau^2 - 7\xi\tau + 1) - 4(2\xi\tau - 1)^2)m^2(x) - 8(2\xi\tau - 1)^2 n(x)|} & ; \quad |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}| \geq \Omega, \end{cases} \tag{36}$$

where

$$\Omega = \frac{1}{3(3\xi\tau - 1)}\left|((8\xi\tau^2 - 7\xi\tau + 1) - 4(2\xi\tau - 1)^2) - 8(2\xi\tau - 1)^2\left(\frac{n(x)}{m^2(x)}\right)\right|.$$

Proof. Let $g(z) \in \mathfrak{B}_{\sum}(x, \xi, \tau, \beta, k, \phi(s))$. Then, for two holomorphic functions r and s such that $r(0) = s(0) = 0, |r(z) = r_1 z + r_2 z^2 + r_3 z^3 + ...| < 1$ and $|s(\omega) = s_1\omega + s_2\omega^2 + s_3\omega^3 + ...| < 1$, $z, \omega \in \mathfrak{D}$, and using Definition 1.4, we can write

$$\frac{(1-\xi) + \xi[(z(D_\beta^k g_\phi(z))')']^\tau}{(D_\beta^k g_\phi(z))'} = \mathcal{G}(x, r(z)) - 1,\ z \in \mathfrak{D} \tag{37}$$

and

$$\frac{(1-\xi) + \xi[(\omega(D_\beta^k f_\phi(\omega))')']^\tau}{(D_\beta^k f_\phi(\omega))'} = \mathcal{G}(x, s(\omega)) - 1,\ \omega\ \in \mathfrak{D}. \tag{38}$$

Following (10), (11), (12), and (13) in the proof of Theorem 2.1, one gets in view of (37) and (38)

$$2(1+\beta)^k \phi(s)(2\xi\tau - 1)d_2 = L_1(x)r_1 \tag{39}$$

$$4(1+\beta)^{2k}\phi^2(s)(2\xi\tau^2-4\xi\tau+1)d_2^2+3(1+2\beta)^k\phi(s)(3\xi\tau-1)d_3=L_1(x)r_2+L_2(x)r_1^2 \tag{40}$$

$$-2(1+\beta)^k\phi(s)(2\xi\tau-1)d_2=L_1(x)s_1 \tag{41}$$

$$2(1+\beta)^{2k}\phi^2(s)(4\xi\tau^2+\xi\tau-1)d_2^2-3(1+2\beta)^k\phi(s)(3\xi\tau-1)d_3=L_1(x)s_2+L_2(x)s_1^2. \tag{42}$$

The results (34)-(36) of this theorem now follow from (39)-(42) by applying the procedure as in Theorem 2.1 with respect to (15)-(18).

In next section, we present some interesting consequences of our main result.

3. Corollaries and Consequences

Corollary 3.1. *Let $g(z)$ be in the family $\mathscr{K}_{\sum}(x,\ \beta,k,\ \phi(s))$. Then*

$$|d_2|\leq\frac{\sqrt{2}|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{|m^2(x)+9n(x)|}},\quad |d_3|\leq\frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{4m^2(x)}{9}+\frac{|m(x)|}{4}\right]$$

and for some $\delta\in\mathbb{R}$,

$$|d_3-\delta d_2^2|\leq\begin{cases}\frac{|m(x)|}{4(1+2\beta)^k\phi(s)} & ;\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|\leq\frac{1}{8}\left|1+9\left(\frac{n(x)}{m^2(x)}\right)\right|\\ \frac{2|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|m^2(x)+9\,n(x)|} & ;\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|\geq\frac{1}{8}\left|1+9\left(\frac{n(x)}{m^2(x)}\right)\right|.\end{cases}$$

Corollary 3.2. *Let $g(z)$ be in the family $\mathscr{J}_{\sum}(x,\ \beta,k,\ \phi(s))$. Then*

$$|d_2|\leq\frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{|m^2(x)+8n(x)|}},\quad |d_3|\leq\frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{m^2(x)}{4}+\frac{|m(x)|}{5}\right]$$

and for $\delta\in\mathbb{R}$,

$$|d_3-\delta d_2^2|\leq\begin{cases}\frac{|m(x)|}{5(1+2\beta)^k\phi(s)} & ;\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|\leq\frac{1}{5}\left|1+8\left(\frac{n(x)}{m^2(x)}\right)\right|\\ \frac{|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|m^2(x)+8n(x)|} & ;\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|\geq\frac{1}{5}\left|1+8\left(\frac{n(x)}{m^2(x)}\right)\right|.\end{cases}$$

Corollary 3.3. *Let $g(z)$ be in the family $\mathscr{L}_{\sum}(x,\ \beta,k,\ \phi(s))$. Then*

$$|d_2|\leq\frac{\sqrt{2}|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{|6m^2(x)+25n(x)|}},\quad |d_3|\leq\frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{4m^2(x)}{25}+\frac{|m(x)|}{7}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{7(1+2\beta)^k \phi(s)} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \leq \frac{1}{14}\left|6 + 25\left(\frac{n(x)}{m^2(x)}\right)\right| \\ \frac{2|m(x)|^3 \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k \phi(s)|6m^2(x)+25n(x)|} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \geq \frac{1}{14}\left|6 + 25\left(\frac{n(x)}{m^2(x)}\right)\right|. \end{cases}$$

Corollary 3.4. *Let* $g(z)$ *be in the family* $\mathscr{P}_{\sum}(x, \mu, \beta, k)$. *Then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k \phi(s)\sqrt{[2|2\mu^2 m^2(x) + (1+2\mu)^2 n(x)|]}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k \phi(s)}\left[\frac{m^2(x)}{(1+2\mu)^2} + \frac{|m(x)|}{2(1+3\mu)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{2(1+2\beta)^k \phi(s)(1+3\mu)} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \leq J_1 \\ \frac{|m(x)|^3 \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right|}{2(1+2\beta)^k \phi(s)[2\mu^2 m^2(x)+(1+2\mu)^2 n(x)]} & ; \left|1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}\right| \geq J_1, \end{cases}$$

where $J_1 = \frac{1}{(1+3\mu)}\left|2\mu^2 + (1+2\mu)^2\left(\frac{n(x)}{m^2(x)}\right)\right|$.

Corollary 3.5. *Let* $0 \leq \gamma \leq 1$, $\mu \geq 0$, $\mu \geq \gamma$ *and* $g(z)$ *of the form (1) be in* $\mathfrak{S}_{\sum}(x, \gamma, \mu, 0, \phi(s))$. *Then*

$$|d_3 - d_2^2| \leq \frac{|m(x)|}{2(1-\gamma+3\mu)}.$$

Remark 3.1. *Corollary 3.5 reduces to Corollary 5, Corollary 6 and Corollary 7 of [3] when* $\mu = \gamma$, $\mu = \gamma = 0$, $k = 0$, $\phi(s) = 1$ *and* $\mu = \gamma = 1, k = 0$, $\phi(s) = 1$, *respectively.*

Corollary 3.6. *Let* $g(z)$ *be in the family* $\mathfrak{K}_{\sum}(x, \mu, \beta, k, \phi(s))$, $\mu \geq 0$. *Then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k \phi(s)\sqrt{|(4\mu^2+2\mu+1)m^2(x)+8(1+\mu)^2 n(x)|}}, \quad |d_3| \leq \frac{1}{(1+2\beta)^k \phi(s)}\left[\frac{m^2(x)}{4(1+\mu)^2} + \frac{|m(x)|}{3(1+2\mu)}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3 - \delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{3(1+2\beta)^k \phi(s)(1+2\mu)} & ; |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}| \leq M_1 \\ \frac{|m(x)|^3 \, |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}|}{(1+2\beta)^k |(4\mu^2+2\mu+1)m^2(x)+8(1+\mu)^2 n(x)|} & ; |1 - \frac{(1+2\beta)^k \delta}{(1+\beta)^{2k}\phi(s)}| \geq M_1, \end{cases}$$

where $M_1 = \frac{1}{3(1+2\mu)}\left|(4\mu^2+2\mu+1)+8(1+\mu)^2\left(\frac{n(x)}{m^2(x)}\right)\right|$.

Corollary 3.7. *Let $g(z)$ be in the family $\mathfrak{L}_{\sum}(x,\mu,\,\beta,k,\,\phi(s))$, $\mu \geq 1$. Then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{2\mu|(2\mu-1)m^2(x)+4\mu\,n(x)|}},$$

$$|d_3| \leq \frac{1}{2\mu(1+2\beta)^k\phi(s)}\left[\frac{m^2(x)}{2\mu}+\frac{|m(x)|}{3}\right]$$

and for $\delta \in \mathbb{R}$,

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{6\mu(1+2\beta)^k\phi(s)} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq\ M_2 \\ \frac{|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{2\mu(1+2\beta)^k\phi(s)|(2\mu-1)m^2(x)+4\mu\,n(x)|} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq\ M_2, \end{cases}$$

where $M_2 = \frac{1}{3}\left|(2\mu-1)+4\mu\left(\frac{n(x)}{m^2(x)}\right)\right|$.

Corollary 3.8. *Let $g(z)$ be in the family $\mathscr{M}_{\sum}(x,\xi,\,\beta,k,\,\phi(s))$, $\xi \geq 1$. Then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{|(\xi+1-4(2\xi-1)^2)m^2(x)-8(2\xi-1)^2n(x)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{m^2(x)}{4(2\xi-1)^2}+\frac{|m(x)|}{3(3\xi-1)}\right]$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{3(1+2\beta)^k\phi(s)(3\xi-1)} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq\ \Omega_1 \\ \frac{2|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|(\xi+1-4(2\xi-1)^2)m^2(x)-8(2\xi-1)^2n(x)|} & ;\ \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq\ \Omega_1, \end{cases}$$

where $\Omega_1 = \frac{1}{3(3\xi-1)}\left|(\xi+1-4(2\xi-1)^2)-8(2\xi-1)^2\left(\frac{n(x)}{m^2(x)}\right)\right|$.

Corollary 3.9. *Let $g(z)$ be in the family $\mathfrak{N}_{\sum}(x,\tau,\,\beta,k,\,\phi(s))$, $\tau \geq 1$. Then*

$$|d_2| \leq \frac{|m(x)|\sqrt{|m(x)|}}{(1+\beta)^k\phi(s)\sqrt{|(8\tau^2-7\tau+1-4(2\tau-1)^2)m^2(x)-8(2\tau-1)^2n(x)|}},$$

$$|d_3| \leq \frac{1}{(1+2\beta)^k\phi(s)}\left[\frac{m^2(x)}{4(2\tau-1)^2}+\frac{|m(x)|}{3(3\tau-1)}\right]$$

and for $\delta \in \mathbb{R}$

$$|d_3-\delta d_2^2| \leq \begin{cases} \frac{|m(x)|}{3(1+2\beta)^k\phi(s)(3\tau-1)} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \leq \Omega_2 \\ \frac{|m(x)|^3\left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right|}{(1+2\beta)^k\phi(s)|((8\tau^2-7\tau+1)-4(2\tau-1)^2)m^2(x)-8(2\tau-1)^2n(x)|} & ; \left|1-\frac{(1+2\beta)^k\delta}{(1+\beta)^{2k}\phi(s)}\right| \geq \Omega_2, \end{cases}$$

where $\Omega_2 = \frac{1}{3(3\tau-1)}\left|(8\tau^2-7\tau+1-4(2\tau-1)^2)-8(2\tau-1)^2\left(\frac{n(x)}{m^2(x)}\right)\right|.$

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