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ON DOUBLE-FRAMED SOFT SETS OF NON-ASSOCIATIVE ORDERED SEMIGROUPS

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Abstract: In this note, the authors introduce the notion of double-framed soft sets (briefly, DFS-sets) in an ordered $\mathcal{AG}$-groupoid. An ordered $\mathcal{AG}$-groupoid can be referred to as a non-associative ordered semigroup, as the main difference between an ordered semigroup and an ordered $\mathcal{AG}$-groupoid is the switching of an associative law. We define and give the examples of DFS l-ideals, DFS r-ideals and DFS bi-ideals in an ordered $\mathcal{AG}$-groupoid and also investigate the relationship between them. We give an alternate definition for a strongly regular element of a unitary ordered $\mathcal{AG}$-groupoid and show that how a strongly regular ordered $\mathcal{AG}$-groupoid becomes an ordered $\mathcal{AG}^{**}$-groupoid and a completely inverse ordered $\mathcal{AG}$-groupoid. As an application of our results we get characterizations of a strongly regular ordered $\mathcal{AG}$-groupoid in terms of DFS one-sided (two-sided) ideals and DFS bi-ideals. These concepts will help in verifying the existing characterizations and will

help in achieving new and generalized results in future works.

Keywords and Phrases: DFS-sets, ordered $\mathcal{AG}$-groupoid, pseudo-inverse, left invertive law and DFS ideals.

2010 Mathematics Subject Classification: 20M10 and 20N99.

1. Introduction

The concept of soft set theory was introduced by Molodtsov in [17]. This theory can be used as a generic mathematical tool for dealing with uncertainties. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields [1, 2, 5, 6, 7, 8, 9]. At present, the research work on soft set theory in algebraic fields is progressing rapidly [20, 22, 23, 24]. A soft set is a parameterized family of subsets of the universe set. In the real world, the parameters of this family arise from the view point of fuzzy set theory. Most of the researchers of algebraic structures have worked on the fuzzy aspect of soft sets. Soft set theory is applied in the field of optimization by Kovkov in [13]. Several similarity measures have been discussed in [16], decision making problems have been studied in [22], reduction of fuzzy soft sets and its applications in decision making problems have been analyzed in [14]. The notions of soft numbers, soft derivatives, soft integrals and many more have been formulated in [15]. This concept have been used for forecasting the export and import volumes in international trade [26].

Recently, Jun et al. further extended the notion of softs set into double-framed soft sets and defined double-framed soft subalgebra of BCK/BCI algebra and studied the related properties in [8]. Jun et al. also defined the concept of a double-framed soft ideal (briefly, DFS ideal) of a BCK/BCI-algebra and gave many valuable results for this theory. In [12], Khan et al. have applied the idea of double-framed soft set to ordered semigroups and defined prime and irreducible DFS ideals of an ordered semigroup over a universe set U. Khan et al. have also characterized different classes of an ordered semigroup by using different DFS ideals.

In the present paper, we apply the idea given by Jun et al. in [8], to ordered $\mathcal{AG}$-groupoids. We introduce and investigate the notions of DFS l-ideals, DFS r-ideals and DFS bi-ideals. We study the relationship between these DFS ideals in detail. We give a necessary and sufficient condition for a strongly regular ordered $\mathcal{AG}$-groupoid to become an ordered $\mathcal{AG}^{**}$-groupoid and completely inverse ordered $\mathcal{AG}$-groupoid. Further we show that the strongly regular, intra-regular and weakly regular classes of a unitary ordered $\mathcal{AG}$-groupoid coincide. Finally we characterize a strongly regular class of an ordered $\mathcal{AG}$-groupoid by one-sided (two-sided) ideals and bi-ideals based on DFS-sets.

2. Preliminaries

An $\mathcal{AG}$-groupoid is a non-associative and a non-commutative algebraic structure lying in a grey area between a groupoid and a commutative semigroup. Commutative law is given by $abc = cba$ in ternary operations. By putting brackets on the left of this equation, i.e. $(ab)c = (cb)a$, in 1972, M. A. Kazim and M. Naseeruddin introduced a new algebraic structure called a left almost semigroup abbreviated as an $\mathcal{LA}$-semigroup [10]. This identity is called the left invertive law. P. V. Protic and N. Stevanovic called the same structure an Abel-Grassmann's groupoid abbreviated as an AG-groupoid [21].

This structure is closely related to a commutative semigroup because a commutative $\mathcal{AG}$-groupoid is a semigroup [18]. It was proved in [10] that an $\mathcal{AG}$-groupoid S is medial, that is, $ab \cdot cd = ac \cdot bd$ holds for all $a,b,c,d \in S$. An $\mathcal{AG}$-groupoid may or may not contain a left identity. The left identity of an $\mathcal{AG}$-groupoid permits the inverses of elements in the structure. If an $\mathcal{AG}$-groupoid contains a left identity, then this left identity is unique [18]. In an $\mathcal{AG}$-groupoid S with left identity (unitary $\mathcal{AG}$-groupoid), the paramedial law $ab \cdot cd = dc \cdot ba$ holds for all $a,b,c,d \in S$. By using medial law with left identity, we get $a \cdot bc = b \cdot ac$ for all $a,b,c \in S$. We should genuinely acknowledge that much of the ground work has been done by M. A. Kazim, M. Naseeruddin, Q. Mushtaq, M. S. Kamran, P. V. Protic, N. Stevanovic, M. Khan, W. A. Dudek and R. S. Gigon. One can be referred to [3, 4, 11, 18, 19, 21, 25] in this regard.

An $\mathcal{AG}$-groupoid $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with an $\mathcal{AG}$-groupoid operation, meaning that for $x, y, z \in S$, $x \leq y \Rightarrow zx \leq zy$ and $xz \leq yz$, is called an ordered $\mathcal{AG}$-groupoid [28].

Let us define a binary operation "$\circ_e$" (e-sandwich operation) on an ordered $\mathcal{AG}$-groupoid $(S, \cdot, \leq)$ with left identity e as follows:

$$a \circ_e b = ae \cdot b, \ \forall \ a, b \in S.$$

Then $(\mathcal{S}, \circ_e, \leq)$ becomes an ordered semigroup [28].

Note that an ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup because if an ordered $\mathcal{AG}$-groupoid has a right identity then it becomes an ordered semigroup.

Let $\emptyset \neq A \subseteq S$, we denote $(A]$ by $(A] := \{x \in S / x \leq a$ for some $a \in A\}$. If $A = \{a\}$, then we write $(\{a\}]$. For $\emptyset \neq A, B \subseteq S$, we denote $AB =: \{ab / a \in A, b \in B\}$.

• A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if:

(i) $SA \subseteq A$ $(AS \subseteq A)$;

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently: A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a left (right) ideal of S if $(SA] \subseteq A$ $((AS] \subseteq A)$.

• By two-sided ideal or simply ideal, we mean a nonempty subset of an ordered $\mathcal{AG}$-groupoid S which is both left and right ideal of S.

• Let S be an ordered $\mathcal{AG}$-groupoid. By an ordered $\mathcal{AG}$-subgroupoid of S, we means a nonempty subset A of S such that $(A^2] \subseteq A$.

• A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a generalized bi-ideal of S if:

(i) $AS \cdot A \subseteq A$;

(ii) if $a \in A$ and $b \in S$ such that $b \leq a$, then $b \in A$.

Equivalently: A nonempty subset A of an ordered $\mathcal{AG}$-groupoid S is called a generalized bi-ideal of S if $(AS \cdot A] \subseteq A$.

• An ordered AG-subgroupoid A of an ordered AG-groupoid S is called a bi-ideal of S if $(AS \cdot A] \subseteq A$.

Lemma 2.1. [28] *Let S be an ordered $\mathcal{AG}$-groupoid and $\emptyset \neq A, B \subseteq S$. Then the followings hold*:
(i) $A \subseteq (A]$;
(ii) *If* $A \subseteq B$, *then* $(A] \subseteq (B]$;
(iii) $(A]\,(B] \subseteq (AB]$;
(iv) $(A] = ((A]]$;
(vi) $((A]\,(B]] = (AB]$;
(vii) *Also for every ideal* T *of* S, $(T] = T$.

3. Soft Sets

In [24], Sezgin and Atagun introduced some new operations on soft set theory and defined soft sets in the following way.

Let U be an initial universe set, E a set of parameters, $P(U)$ the power set of U and $A \subseteq E$. Then a *soft set* f_A over U is a function defined by:

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset, \text{ if } x \notin A.$$

Here f_A is called an *approximate function.* A soft set over U can be represented by the set of ordered pairs

$$f_A = \{(x,\ f_A(x)) :\ x \in E,\ f_A(x) \in P(U)\}\,.$$

It is clear that a soft set is a parameterized family of subsets of U. The set of all soft sets is denoted by $S(U)$.

• Let f_A, $f_B \in S(U)$. Then f_A is a soft subset of f_B, denoted by $f_A \mathrel{\widetilde{\subseteq}} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in S$. Two soft sets f_A, f_B are said to be equal soft sets if

$f_A \mathrel{\widetilde{\subseteq}} f_B$ and $f_B \mathrel{\widetilde{\subseteq}} f_A$ and is denoted by $f_A \mathrel{\widetilde{=}} f_B$. The union of f_A and f_B, denoted by $f_A \mathrel{\widetilde{\cup}} f_B$, is defined by $f_A \mathrel{\widetilde{\cup}} f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \cup f_B(x)$, $\forall\, x \in E$. In a similar way, we can define the intersection of f_A and f_B.

• Let S be an ordered $\mathcal{AG}$-groupoid. Let f_A, $f_B \in S(U)$. Then the *soft product* [24] of f_A and f_B, denoted by $f_A \mathrel{\widetilde{\circ}} f_B$, is defined as follows:

$$(f_A \mathrel{\widetilde{\circ}} f_B)(x) = \begin{cases} \bigcup\limits_{(y,z)\in A_x} \{f_A(y) \cap g_B(z)\} & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset \end{cases}$$

where $A_x = \{(y, z) \in S \times S / x \leq yz\}$.

• A double-framed soft pair $\left\langle (f_A^+,\ f_A^-; A\right\rangle$ is called a *double-framed soft set* (briefly, DFS-set of A) [8] of A over U, where f_A^+ and f_A^- are mappings from A to $P(U)$. The set of all DFS-sets of A over U will be denoted by $DFS(U)$.

• Let $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ and $g_A = \left\langle (g_A^+,\ g_A^-); A\right\rangle$ be two double-framed soft sets of an ordered $\mathcal{AG}$-groupoid S over U. Then the *uni-int soft product* [12], denoted by $f_A \diamond g_A = \left\langle (f_A^+ \mathrel{\widetilde{\circ}} g_A^+,\ f_A^- \mathrel{\widetilde{\star}} g_A^-); A\right\rangle$ is defined to be a double-framed soft set of S over U, in which $f_A^+ \mathrel{\widetilde{\circ}} g_A^+$ and $f_A^- \mathrel{\widetilde{\star}} g_A^-$ are mapping from S to $P(U)$, given as follows:

$$f_A^+ \mathrel{\widetilde{\circ}} g_A^+ : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcup\limits_{(y,z)\in A_x} \{f_A^+(y) \cap g_A^+(z)\} & \text{if } A_x \neq \emptyset \\ \emptyset & \text{if } A_x = \emptyset, \end{cases}$$

$$f_A^- \mathrel{\widetilde{\star}} g_A^- : S \longrightarrow P(U), x \longmapsto \begin{cases} \bigcap\limits_{(y,z)\in A_x} \{f_A^-(y) \cup g_A^-(z)\} & \text{if } A_x \neq \emptyset \\ U & \text{if } A_x = \emptyset. \end{cases}$$

• Let $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ and $g_A = \left\langle (g_A^+,\ g_A^-); A\right\rangle$ be two double-framed soft sets over a common universe set U. Then $\left\langle (f_A^+,\ f_A^-); A\right\rangle$ is called a *double-framed soft subset* (*briefly, DFS-subset*) [12] of $\left\langle (g_A^+,\ g_A^-); A\right\rangle$, denote by $\left\langle (f_A^+,\ f_A^-); A\right\rangle \sqsubseteq \left\langle (g_A^+,\ g_A^-); A\right\rangle$ if:

(i) $A \subseteq B$;

(ii) $(\forall e \in A) \begin{pmatrix} f_A^+ \text{ and } g_A^+ \text{ are identical approximations } (f_A^+(e) \subseteq g_A^+(e)) \\ f_A^- \text{ and } g_A^- \text{ are identical approximations } (f_A^-(e) \subseteq g_A^-(e)) \end{pmatrix}$.

• For two DFS-sets $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ and $g_A = \left\langle (g_A^+,\ g_A^-); A\right\rangle$ over U are said to be *equal*, denoted by $\left\langle (f_A^+,\ f_A^-); A\right\rangle = \left\langle (g_A^+,\ g_A^-); A\right\rangle$, if $\left\langle (f_A^+,\ f_A^-); A\right\rangle \sqsubseteq \left\langle (g_A^+,\ g_A^-); A\right\rangle$ and $\left\langle (g_A^+,\ g_A^-); A\right\rangle \sqsubseteq \left\langle (f_A^+,\ f_A^-); A\right\rangle$.

• For two DFS-sets $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ and $g_A = \left\langle (g_A^+,\ g_A^-); A\right\rangle$ over U, the *DFS int-uni* set [12] of $\left\langle (f_A^+,\ f_A^-); A\right\rangle$ and $\left\langle (g_A^+,\ g_A^-); A\right\rangle$, is defined to be a DFS-set

$\left\langle (f_A^+ \cap g_A^+,\ f_A^- \cup g_A^-); A \right\rangle$, where $f_A^+ \cap g_A^+$ and $f_A^- \cup g_A^-$ are mapping given as follows:

$$\begin{aligned} f_A^+ \cap g_A^+ &: A \longrightarrow P(U), x \longmapsto f_A^+(x) \cap g_A^+(x); \\ f_A^- \cup g_A^- &: A \longrightarrow P(U), x \longmapsto f_A^-(x) \cup g_A^-(x). \end{aligned}$$

It is denoted by $\left\langle (f_A^+,\ f_A^-); A \right\rangle \sqcap \left\langle (g_A^+,\ g_A^-); A \right\rangle = \left\langle (f_A^+ \cap g_A^+,\ f_A^- \cup g_A^-); A \right\rangle$.

• A double-framed soft set $f_A = \left\langle (f_A^+,\ f_A^-); A \right\rangle$ of S over U is called a *double-framed soft $\mathcal{AG}$-subgroupoid* (briefly, DFS $\mathcal{AG}$-subgroupoid) of S over U if it satisfies $f_A^+(xy) \supseteq f_A^+(x) \cap f_A^+(y)$, $f_A^-(xy) \subseteq f_A^-(x) \cup f_A^-(y)$, $\forall\ x, y \in S$.

• A double-framed soft set $f_A = \left\langle (f_A^+,\ f_A^-); A \right\rangle$ of S over U is called

(i) a *double-framed soft left ideal* (briefly, DFS l-ideal) of S over U if it satisfies:

(a) $f_A^+(xy) \supseteq f_A^+(y)$ and $f_A^-(xy) \subseteq f_A^-(y)$;

(b) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x, y \in S$.

(ii) a *double-framed soft right ideal* (briefly, DFS r-ideal) of S over U if it satisfies:

(a) $f_A^+(xy) \supseteq f_A^+(x)$ and $f_A^-(xy) \subseteq f_A^-(x)$;

(b) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x, y \in S$.

(iii) a *double-framed soft ideal* (briefly, DFS ideal) of S over U, if it is both DFS l-ideal and DFS r-ideal of S over U.

• A double-framed soft set $f_A = \left\langle (f_A^+,\ f_A^-); A \right\rangle$ of S over U is called a *double-framed bi-ideal* (briefly, DFS bi-ideal) of S over U if it satisfies:

(a) $\left\langle (f_A^+,\ f_A^-); A \right\rangle$ is a DFS $\mathcal{AG}$-subgroupoid of S over U;

(b) $f_A^+(xy \cdot z) \supseteq f_A^+(x) \cap f_A^+(z)$ and $f_A^-(xy \cdot z) \subseteq f_A^-(x) \cup f_A^-(z)$;

(c) $x \leq y \Longrightarrow f_A^+(x) \supseteq f_A^+(y)$ and $f_A^-(x) \subseteq f_A^-(y)$, $\forall\ x, y, z \in S$.

• A double-framed soft set $f_A = \left\langle (f_A^+,\ f_A^-); A \right\rangle$ of S over U is called a *double-framed generalized bi-ideal* (briefly, DFS generalized bi-ideal) of S over U if it satisfies (b) and (c).

• Let A be a nonempty subset of S. Then the *characteristic double-framed soft mapping* of A, denoted by $\left\langle (\mathcal{X}_A^+,\ \mathcal{X}_A^-); A \right\rangle = \mathcal{X}_A$ is defined to be a double-framed soft set, in which $\mathcal{X}_A^+$ and $\mathcal{X}_A^-$ are soft mappings over U, given as follows:

$$\begin{aligned} \mathcal{X}_A^+ &: S \longrightarrow P(U), x \longmapsto \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A, \end{cases} \\ \mathcal{X}_A^- &: S \longrightarrow P(U), x \longmapsto \begin{cases} \emptyset & \text{if } x \in A \\ U & \text{if } x \notin A. \end{cases} \end{aligned}$$

Note that the characteristic mapping of the whole set S, denoted by $\mathcal{X}_S = \left\langle (\mathcal{X}_S^+,\ \mathcal{X}_S^-); S \right\rangle$, is called the *identity double-framed soft mapping*, where $\mathcal{X}_S^+(x) = U$ and $\mathcal{X}_S^-(x) = \emptyset$, $\forall\ x \in S$.

The following result holds for an ordered semigroup [6] just because of the closure property which makes very clear for an ordered $\mathcal{AG}$-groupoid to hold the same Lemma.

Lemma 3.1. *For a nonempty subset A of an ordered $\mathcal{AG}$-groupoid S, the following conditions are equivalent*:
(i) *A is a left ideal (right ideal or bi-ideal) of S*;
(ii) *The DFS set $\mathcal{X}_A$ of S over U is a DFS l-ideal (DFS r-ideal or DFS bi-ideal) of S over U.*

The following result holds for an ordered smemigroup [12] just because of the closure property which makes very clear for an ordered $\mathcal{AG}$-groupoid to hold the same Lemma.

Lemma 3.2. *Let $f_A = \langle (f_A^+, f_A^-); A\rangle$ be any DFS-set of an ordered $\mathcal{AG}$-groupoid S over U. Then the following assertions hold:*
(i) *f_A is a DFS r-ideal (l-ideal) of S over U if and only if $f_A \diamond \mathcal{X}_S \sqsubseteq f_A$ ($\mathcal{X}_S \diamond f_A \sqsubseteq f_A$)*;
(ii) *f_A is a DFS bi-ideal of S over U if and only if $f_A \diamond f_A \sqsubseteq f_A$ and $(f_A \diamond \mathcal{X}_S) \diamond f_A \sqsubseteq f_A$.*

- A double-framed soft set $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ of S over U is called *DFS idempotent* if $f_A \diamond f_A = f_A$.
- A double-framed soft set $f_A = \left\langle (f_A^+,\ f_A^-); A\right\rangle$ of S over U is called *DFS semiprime* if $f_A(x) \sqsupseteq f_A(x^2)$, $\forall\ x \in A$.

Lemma 3.3. *Let A be any right (left, bi-) ideal of an ordered $\mathcal{AG}$-groupoid S. Then A is semiprime (idempotent) if and only if $\mathcal{X}_A$ is DFS semiprime (DFS idempotent).*
Proof. Let A be a right (left, bi-) ideal of S, then by Lemma 3.1, $\mathcal{X}_A$ is a DFS r- (DFS l-, DFS bi-) ideal of S over U. Let $a^2 \in A$, then $\mathcal{X}_A^+(a) \supseteq \mathcal{X}_A^+(a^2)$, therefore $\mathcal{X}_A^+(a^2) = U \subseteq \mathcal{X}_A^+(a)$, this implies $\mathcal{X}_A^+(a) = U$ and similarly $\mathcal{X}_A^-(a) = \emptyset$. Thus $a \in A$ and therefore A is semiprime. Converse is simple. Similarly we can show that the required result holds for the case of idempotent condition.

Remark 3.4. *The set $(DFS(U), \diamond, \sqsubseteq)$ forms an ordered $\mathcal{AG}$-groupoid and satisfies all the basic laws.*

Remark 3.5. *If S is an ordered $\mathcal{AG}$-groupoid, then $\mathcal{X}_S \diamond \mathcal{X}_S = \mathcal{X}_S$.*

The following result also holds for an ordered smemigroup [12] just because of the closure property which is very trivial for an ordered $\mathcal{AG}$-groupoid to hold the same Lemma.

Lemma 3.6. *Let S be an ordered $\mathcal{AG}$-groupoid. For $\emptyset \neq A, B \subseteq S$, the following assertions hold*:

(i) $A \subseteq B \Leftrightarrow \mathcal{X}_A \sqsubseteq \mathcal{X}_B$;
(ii) $\mathcal{X}_A \sqcap \mathcal{X}_B = \mathcal{X}_{A\cap B}$;
(iii) $\mathcal{X}_A \sqcup \mathcal{X}_B = \mathcal{X}_{A\cup B}$;
(iv) $\mathcal{X}_A \diamond \mathcal{X}_B = \mathcal{X}_{(AB]}$.

4. On DFS strongly regular ordered $\mathcal{AG}$-groupoids

Throughout this paper, let $E = S$, where S is an ordered $\mathcal{AG}$-groupoid, unless otherwise stated.

4.1. Basic Results

Example 4.1. There are six students in the initial universe set U given by

$$U = \{s_1, s_2, s_3, s_4, s_5, s_6\}.$$

Let a set of parameters $E = \{e_0, e_1, e_2, e_3, e_4\}$ be a set of status of each student in U with the following type of grades:

e_0 stands for the parameter "A-grade",
e_1 stands for the parameter "B$^+$-grade",
e_2 stands for the parameter "B-grade",
e_3 stands for the parameter "C$^+$-*grade*",
e_4 stands for the parameter "C-grade",
with the following binary operation and order given below.

$*$	e_0	e_1	e_2	e_3	e_4
e_0	e_1	e_1	e_3	e_3	e_4
e_1	e_1	e_1	e_1	e_1	e_4
e_2	e_0	e_1	e_2	e_3	e_4
e_3	e_0	e_1	e_0	e_1	e_4
e_4	e_0	e_4	e_4	e_4	e_4

$$\leq = \{(e_0, e_0), (e_0, e_1), (e_2, e_2), (e_0, e_2), (e_3, e_3), (e_0, e_4), (e_4, e_4), (e_1, e_1)\}.$$

Then $(E, *, \leq)$ is an ordered $\mathcal{AG}$-groupoid with left identity e_2. Let $A = \{e_0, e_1, e_2, e_3\}$ *and define a DFS-set* $\langle (f_A^+,\ f_A^-); A\rangle$ *of* S over U as follows:

$$f_A^+(x) = \left\{ \begin{array}{c} \{s_1, s_2, s_3\} \text{ if } x = e_0 \\ \{s_1, s_2, s_3, s_4\} \text{ if } x = e_1 \\ \{s_2, s_3\} \text{ if } x = e_2 \\ \{s_1, s_2, s_3, s_4\} \text{ if } x = e_3 \end{array} \right\} \text{ and } f_A^-(x) = \left\{ \begin{array}{c} \{s_1, s_2, s_4, s_5\} \text{ if } x = e_0 \\ \{s_1, s_2, s_4\} \text{ if } x = e_1 \\ U \text{ if } x = e_2 \\ \{s_1, s_2, s_4\} \text{ if } x = e_3 \end{array} \right\}.$$

Then it is easy to verify that $\left\langle (f_A^+,\ f_A^-); A \right\rangle$ is a DFS l-ideal of S over U.

Let $B = \{e_0, e_1, e_3, e_4\}$ and define a DFS-set $\left\langle (g_B^+,\ g_B^-); B \right\rangle$ of S *over* U *as follows*:

$$g_B^+(x) = \left\{ \begin{array}{c} \{s_1, s_2, s_3, s_4\} \text{ if } x = e_0 \\ U \text{ if } x = e_1 \\ \{s_2, s_3, s_4, s_5\} \text{ if } x = e_3 \\ \{s_3, s_4, s_5, s_6\} \text{ if } x = e_4 \end{array} \right\} \text{and}$$

$$g_B^-(x) = \left\{ \begin{array}{c} \{s_2, s_3\} \text{ if } x = e_0 \\ \{s_3\} \text{ if } x = e_1 \\ \{s_3, s_4\} \text{ if } x = e_3 \\ U \text{ if } x = e_4 \end{array} \right\}.$$

Then it is easy to verify that $\left\langle (g_B^+,\ g_B^-); B \right\rangle$ is a DFS r-ideal of S over U.

Let us explore the relationship between DFS idempotent subsets of a unitary ordered $\mathcal{AG}$-groupoid S and its DFS bi-ideals, explicitly, when will a DFS idempotent subset of S be a DFS bi-ideal. We answer this question in the following Proposition.

Proposition 4.2. *Let f_A be a DFS idempotent subset of a unitary ordered $\mathcal{AG}$-groupoid S over U, and let $f_A = g_B \diamond h_C$ for a DFS l-ideal h_C and a DFS r-ideal g_B of S over U. Then f_A is a DFS bi-ideal of S over U.*
Proof. By using Lemma 3.2, we have

$$(f_A \diamond \mathcal{X}_S) \diamond f_A = (f_A \diamond \mathcal{X}_S) \diamond (f_A \diamond f_A) \sqsubseteq (g_B \diamond \mathcal{X}_S) \diamond (\mathcal{X}_S \diamond h_C) \sqsubseteq g_B \diamond h_C = f_A.$$

Another question is the realization of DFS-subsets f_A of an ordered $\mathcal{AG}$-groupoid which are both DFS idempotent and DFS bi-ideal. This is given in the following Proposition.

Proposition 4.3. *Let f_A be a DFS idempotent subset and DFS bi-ideal of a unitary ordered $\mathcal{AG}$-groupoid S over U. Then there exist a DFS l-ideal h_C and a DFS r-ideal g_B of S over U such that $f_A = g_B \diamond h_C$.*
Proof. Necessity. Assume that f_A is a DFS bi-ideal of S over U such that f_A is DFS idempotent. Setting $h_C = \mathcal{X}_S \diamond f_A$ and $g_B = \mathcal{X}_S \diamond f_A^2$, then by using Lemma

3.2, we have

$$\begin{aligned}
g_B \diamond h_C &= (\mathcal{X}_S \diamond f_A^2) \diamond (\mathcal{X}_S \diamond f_A) = (f_A^2 \diamond \mathcal{X}_S) \diamond (\mathcal{X}_S \diamond f_A) \\
&= ((\mathcal{X}_S \diamond f_A) \diamond (\mathcal{X}_S \diamond \mathcal{X}_S)) \diamond f_A^2 = ((\mathcal{X}_S \diamond \mathcal{X}_S) \diamond (f_A \diamond \mathcal{X}_S)) \diamond f_A^2 \\
&= (\mathcal{X}_S \diamond ((f_A \diamond f_A) \diamond (\mathcal{X}_S \diamond \mathcal{X}_S))) \diamond f_A^2 \\
&= (\mathcal{X}_S \diamond ((\mathcal{X}_S \diamond \mathcal{X}_S) \diamond (f_A \diamond f_A))) \diamond f_A^2 \\
&= (\mathcal{X}_S \diamond (f_A \diamond (\mathcal{X}_S \diamond f_A))) \diamond f_A^2 = (f_A \diamond (\mathcal{X}_S \diamond (\mathcal{X}_S \diamond f_A))) \diamond f_A^2 \\
&\sqsubseteq (f_A \diamond \mathcal{X}_S) \diamond f_A^2 \sqsubseteq f_A,
\end{aligned}$$

and $f_A = f_A^2 \sqsubseteq (\mathcal{X}_S \diamond f_A^2) \diamond (\mathcal{X}_S \diamond f_A) = g_B \diamond h_C$.

The Propositions 4.2 and 4.3 combined together give us a characterization Theorem which we state in the following.

Theorem 4.4. *Let S be a unitary ordered $\mathcal{AG}$-groupoid and f_A be a DFS idempotent subset of S over U. Then f_A is a DFS bi-ideal of S over U if and only if there exist a DFS l-ideal h_C and a DFS r-ideal g_B of S over U such that $g_B \diamond h_C = f_A$.*

Remark 4.5. *Every DFS r-ideal of a unitary ordered $\mathcal{AG}$-groupoid S over U is a DFS l-ideal of S over U but the converse inclusion is not true in general which can be followed from Example 4.1.*

Note that if S is a unitary ordered $\mathcal{AG}$-groupoid, then $(SS] = S$.

Lemma 4.6. *Let R be a right ideal and L be a left ideal of a unitary ordered $\mathcal{AG}$-groupoid S. Then $(RL]$ is a left ideal of S.*

Proof. Let R and L be any left and right ideals of S respectively. Then by using Lemma 2.1, we get

$$\begin{aligned}
S(RL] &= (SS](RL] \subseteq (SS \cdot RL] = (SR \cdot SL] \subseteq (SR \cdot (SL]] = (SR \cdot L] \\
&= ((SS]R \cdot L] \subseteq ((SS)R \cdot L] = ((RS)S \cdot L] \subseteq ((RS]S \cdot L] \subseteq (RL],
\end{aligned}$$

which shows that $(RL]$ is a left ideal of S.

*An element a of an ordered $\mathcal{AG}$-*groupoid *S is called a strongly regular element of S, if there exists some x in S such that $a \leq ax \cdot a$ and $ax = xa$, where x is called a pseudo-inverse of a. S is called strongly regular ordered $\mathcal{AG}$-groupoid if all elements of S are strongly regular.*

*A completely inverse ordered $\mathcal{AG}$-*groupoid *S is an ordered $\mathcal{AG}$-*groupoid *satisfying the identity $ax = xa$, where x is a strong inverse of a, that is, $a \leq ax \cdot a$ and $x \leq xa \cdot x$, $\forall\ a \in S$.*

Lemma 4.7. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. Then E is a semilattice, where E is the set of all idempotents of S.*

Proof. It is simple.

Theorem 4.8. *A strongly regular unitary ordered $\mathcal{AG}$-groupoid S is completely inverse if and only if E is a semilattice.*

Proof. Necessity. It can be followed from Lemma 4.7.

Sufficiency. Let $a \in S$ and Suppose $a^{'}, a^{''} \in S$ are inverses of a. Then $a \leq aa^{'} \cdot a$, $a^{'} \leq a^{'}a \cdot a^{'}$, $a^{'}a = aa^{'}$ and $a \leq aa^{''} \cdot a$, $a^{''} \leq a^{''}a \cdot a^{''}$, $a^{''}a = aa^{''}$. Clearly $aa^{'}, aa^{''} \in E$. Thus $aa^{'} \leq (aa^{''} \cdot a)a^{'} = a^{'}a \cdot aa^{''} = a^{''}a \cdot aa^{'} = (aa^{'})a \cdot a^{''} \leq aa^{''}$. Therefore $a^{'} \leq a^{'}a \cdot a^{'} \leq a^{''}a \cdot a^{'} \leq a^{'}a \cdot a^{''} = a^{''}a \cdot a^{''} \leq a^{''}$.

An ordered $\mathcal{AG}$-groupoid S is called an ordered $\mathcal{AG}^{**}$-groupoid if it satisfies the identity $a \cdot bc = b \cdot ac$, for all $a,b,c \in S$ [3].

Note that every unitary ordered $\mathcal{AG}$-groupoid is an ordered $\mathcal{AG}^{**}$-groupoid but the converse is not true in general [27].

Corollary 4.9. *A strongly regular ordered $\mathcal{AG}^{**}$-groupoid S is completely inverse if and only if E is a semilattice.*

Theorem 4.10. *A strongly regular ordered $\mathcal{AG}$-groupoid is an ordered $\mathcal{AG}^{**}$-groupoid if and only if E is a semilattice.*

Proof. Necessity. It can be followed from Lemma 4.7.

Sufficiency. Let $a,b,c \in S$, then there exist $a^{'},b^{'},c^{'} \in S$ such that $a \cdot bc \leq (aa^{'} \cdot bb^{'} \cdot cc^{'})(a \cdot bc)$, as clearly $aa^{'}, bb^{'}, cc^{'} \in E$. Therefore

$$\begin{aligned}
a \cdot bc &\leq (aa^{'} \cdot a)(bc) = (bc \cdot a)(aa^{'}) = (ac \cdot b)(aa^{'}) = (aa^{'} \cdot b)(ac) \\
&\leq (aa^{'} \cdot b) \cdot (aa^{'} \cdot a)(cc^{'} \cdot c) = (aa^{'} \cdot b) \cdot (aa^{'} \cdot cc^{'})(ac) \\
&= (aa^{'} \cdot aa^{'} \cdot cc^{'}) \cdot (b \cdot ac) \leq (aa^{'} \cdot aa^{'} \cdot cc^{'}) \cdot (bb^{'} \cdot aa^{'} \cdot cc^{'})(b \cdot ac) \\
&= (aa^{'} \cdot bb^{'} \cdot aa^{'} \cdot cc^{'}) \cdot (aa^{'} \cdot cc^{'})(b \cdot ac) \\
&= (bb^{'} \cdot bb^{'} \cdot aa^{'} \cdot cc^{'}) \cdot (aa^{'} \cdot cc^{'})(b \cdot ac) \\
&= (bb^{'} \cdot aa^{'} \cdot cc^{'}) \cdot (bb^{'} \cdot aa^{'} \cdot cc^{'})(b \cdot ac) \\
&= (bb^{'} \cdot aa^{'} \cdot cc^{'})(b \cdot ac) \leq b \cdot ac.
\end{aligned}$$

Hence S is an ordered $\mathcal{AG}^{**}$-groupoid.

Theorem 4.11. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. An element a of S is strongly regular if and only if $a \leq ax \cdot ay$ for some $x, y \in S$ ($a \leq ba^2 \cdot c$ for some $b, c \in S$).*

Proof. Necessity. Let $a \in S$ is strongly regular, then $a \leq ax \cdot a \leq (ax) \cdot (xa)(ax \cdot a) = (ax) \cdot (a \cdot ax)(ax) = (ax) \cdot a((a \cdot ax)x) = ax \cdot ay$, where $(a \cdot ax)x = y \in S$. Thus $a \leq ax \cdot ay$ for some $x, y \in S$. Also, $a \leq ax \cdot a \leq (ax)(ax \cdot ay) = (ax)(a^2 \cdot xy) = (xy \cdot a^2)(xa) = ba^2 \cdot c$, where $xy = b \in S$ and $xa = c \in S$.

Sufficiency. Let $a \in S$ such that $a \leq ax \cdot ay$ for some $x, y \in S$, then $a \leq ax \cdot ay = (ay \cdot x)a = (xy \cdot a)a = ua \cdot a$, where $xy = u \in S$. Thus $au \leq (ua \cdot a)u = ua \cdot ua =$

$u(ua \cdot a) \leq ua$, and $a \leq ua \cdot a = au \cdot a$. Also if $a \in S$ such that $a \leq ba^2 \cdot c$ for some $b, c \in S$, then it is easy to show that $av \leq va$ and $a \leq av \cdot a$ for some $v \in S$. Thus S is strongly regular.

Corollary 4.12. *The strongly regular, weakly regular and intra-regular classes of a unitary ordered $\mathcal{AG}$-groupoid coincide.*

Lemma 4.13. *Let $f_A = \left\langle (f_A^+,\ f_A^-); A \right\rangle$ be any DFS r-ideal (DFS l-ideal, DFS bi-ideal) of a strongly regular unitary ordered $\mathcal{AG}$-groupoid S over U. Then the following assertions hold:*
(i) $f_A = f_A \diamond S$ ($f_A = S \diamond f_A$, $f_A = (f_A \diamond S) \diamond f_A$);
(ii) f_A is DFS idempotent (DFS semiprime).

Proof. It is simple.

4.2. Characterization Problems

In this section, we generalize the results of an ordered semigroup and get some interesting characterizations which we usually do not find in an ordered semigroup.

Theorem 4.14. *Let R (resp. L) be any right (resp. left) ideal and f_A, g_B be any DFS l-ideals of a unitary ordered $\mathcal{AG}$-groupoid S. Then the following conditions are equivalent:*
(i) S is strongly regular;
(ii) $(RL] \cap L = (R \cdot RL^3]$ and R is idempotent;
(iii) $f_A \sqcap g_B = (f_A \diamond g_B) \diamond f_A$ and f_A is DFS idempotent.

Proof. $(i) \Longrightarrow (iii)$: Let f_A and g_B be any DFS l-ideals of a strongly regular S over U. Now for $a \in S$, there exist some $x, y \in S$ such that $a \leq ax \cdot ay = ya \cdot xa \leq y(ax \cdot ay) \cdot xa = (ax)(y \cdot ay) \cdot xa = (ay \cdot y)(xa) \cdot xa = (y^2a \cdot xa)(xa)$. Thus $(y^2a \cdot xa, xa) \in A_a$. Therefore

$$\begin{aligned} ((f_A^+ \tilde{\circ}\, g_B^+) \tilde{\circ}\, f_A^+)(a) &= \bigcup_{(y^2a \cdot xa, xa) \in A_a} \left\{ (f_A^+ \tilde{\circ}\, g_B^+)(y^2a \cdot xa) \cap f_A^+(xa) \right\} \\ &\supseteq \bigcup_{(y^2a, xa) \leq (y^2a, xa)} \{ f_A^+(y^2a) \cap g_B^+(xa) \} \cap f_A^+(xa) \\ &\supseteq f_A^+(y^2a) \cap g_B^+(xa) \cap f_A^+(xa) \supseteq f_A^+(a) \cap g_B^+(a), \end{aligned}$$

and

$$\begin{aligned} ((f_A^- \tilde{\star}\, g_B^-) \tilde{\star}\, f_A^-)(a) &= \bigcap_{(y^2a \cdot xa, xa) \in A_a} \left\{ (f_A^- \tilde{\star}\, g_B^-)(y^2a \cdot xa) \cup f_A^-(xa) \right\} \\ &\subseteq \bigcap_{(y^2a, xa) \leq (y^2a, xa)} \{ f_A^-(y^2a) \cup g_B^-(xa) \} \cup f_A^-(xa) \\ &\subseteq f_A^-(y^2a) \cup g_B^-(xa) \cup f_A^-(xa) \subseteq f_A^-(a) \cup g_B^-(a), \end{aligned}$$

which shows that $(f_A \diamond g_B) \diamond f_A \sqsupseteq f_A \sqcap g_B$. By using Lemmas 3.2 and 4.13, it is easy to show that $(f_A \diamond g_B) \diamond f_A \sqsubseteq f_A \sqcap g_B$. Thus $f_A \sqcap g_B = (f_A \diamond g_B) \diamond f_A$ and by Lemma 4.13f=fg, f_A is *DFS* idempotent.

$(iii) \Longrightarrow (ii)$: Let R and L be any right and left ideals of S respectively. Then by using Lemmas 3.1 and 4.6, $\mathcal{X}_{(RL]}$ and $\mathcal{X}_L$ are DFS l-ideals of S over U. Now by using Lemma 3.6, we get

$$\mathcal{X}_{(RL]\cap L} = \mathcal{X}_{(RL]} \sqcap \mathcal{X}_L = (\mathcal{X}_{(RL]} \diamond \mathcal{X}_L) \diamond \mathcal{X}_{(RL]} = \mathcal{X}_{((RL]L\cdot(RL]]},$$

which give us $(RL] \cap L = ((RL]L \cdot (RL]]$. Now by using Lemma 2.1applied, we get

$$\begin{aligned} ((RL]L \cdot (RL]] &= ((RL)L \cdot RL] = (L^2R \cdot RL] = (LR \cdot RL^2] = (R(LR \cdot L^2)] \\ &= (R(L^2 \cdot RL)] = (R(R \cdot L^2L)] = (R \cdot RL^3], \end{aligned}$$

which implies that $(RL] \cap L = (R \cdot RL^3]$. Since $\mathcal{X}_R$ is a DFS r-ideal of S over U, so it is also a DFS l-ideal of S over U Remark 4.5. Thus by using the given assumption and Lemma 3.3 semidem, R is idempotent.

$(ii) \Longrightarrow (i)$: It is easy to see that $(Sa^2]$ and $(Sa]$ are the right and left ideals of S respectively. Setting $R = (Sa^2]$ and $L = (Sa]$, then by using the given assumption and Lemma 2.1, we have $R = (Sa^2] = (SS \cdot aa] = (Sa \cdot Sa] = ((Sa](Sa]] = (Sa]$, therefore $RL = (Sa^2](Sa] = (Sa](Sa] = (Sa]$, and clearly $a \in (Sa]$. Thus $a \in ((Sa^2](Sa]] \cap (Sa] = ((Sa^2] \cdot (Sa^2](Sa]^3] = ((Sa] \cdot (Sa](Sa]^3] = ((Sa](Sa]] = (Sa \cdot Sa] = (aS \cdot aS]$. Hence S is strongly regular.

Theorem 4.15. *Let f_A, g_B and h_C be any DFS r-ideal, DFS bi-ideal and DFS l-ideal of a unitary ordered $\mathcal{AG}$-groupoid S respectively. Then S is strongly regular if and only if $f_A \sqcap g_B \sqcap h_C = (f_A^2 \diamond g_B^2) \diamond h_C^2$ and f_A is DFS semiprime.*

Proof. Necessity: Assume that f_A, g_B and h_C be any DFS r-ideal, DFS bi-ideal and DFS l-ideal of S over U respectively. Now by using Lemmas 3.2 and 4.13, we have

$$(f_A^{+^2} \widetilde{\circ} g_B^{+^2}) \widetilde{\circ} h_C^{+^2} = (h_C^{+^2} \widetilde{\circ} g_B^{+^2}) \widetilde{\circ} (f_A^{+} \widetilde{\circ} f_A^{+}) = f_A^{+} \widetilde{\circ} ((h_C^{+^2} \widetilde{\circ} g_B^{+^2}) \widetilde{\circ} f_A^{+}) \subseteq f_A^{+} \widetilde{\circ} \mathcal{X}_S^{+} \subseteq f_A^{+},$$

$$\begin{aligned} (f_A^{+^2} \widetilde{\circ} g_B^{+^2}) \widetilde{\circ} h_C^{+^2} &= (f_A^{+^2} \widetilde{\circ} (g_B^{+} \widetilde{\circ} g_B^{+})) \widetilde{\circ} h_C^{+^2} = (g_B^{+} \widetilde{\circ} (f_A^{+^2} \widetilde{\circ} g_B^{+})) \widetilde{\circ} h_C^{+^2} \\ &= (h_C^{+^2} \widetilde{\circ} (f_A^{+^2} \widetilde{\circ} g_B^{+})) \widetilde{\circ} g_B^{+} = (h_C^{+^2} \widetilde{\circ} ((g_B^{+} \widetilde{\circ} f_A^{+}) \widetilde{\circ} f_A^{+})) \widetilde{\circ} g_B^{+} \\ &= ((g_B^{+} \widetilde{\circ} f_A^{+}) \widetilde{\circ} (h_C^{+^2} \widetilde{\circ} f_A^{+})) \widetilde{\circ} g_B^{+} \\ &= ((f_A^{+} \widetilde{\circ} h_C^{+^2}) \widetilde{\circ} (f_A^{+} \widetilde{\circ} g_B^{+})) \widetilde{\circ} g_B^{+} \end{aligned}$$

$$\begin{aligned}
&= ((f_A^+ \,\widetilde{\circ}\, h_C^{+^2}) \,\widetilde{\circ}\, (f_A^+ \,\widetilde{\circ}\, (g_B^+ \,\widetilde{\circ}\, g_B^+))) \,\widetilde{\circ}\, g_B^+ \\
&= ((f_A^+ \,\widetilde{\circ}\, h_C^{+^2}) \,\widetilde{\circ}\, (g_B^+ \,\widetilde{\circ}\, (f_A^+ \,\widetilde{\circ}\, g_B^+))) \,\widetilde{\circ}\, g_B^+ \\
&= (g_B^+ \,\widetilde{\circ}\, ((f_A^+ \,\widetilde{\circ}\, h_C^{+^2}) \,\widetilde{\circ}\, (f_A^+ \,\widetilde{\circ}\, g_B^+))) \,\widetilde{\circ}\, g_B^+ \\
&\subseteq (g_B^+ \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, g_B^+ \subseteq g_B^+,
\end{aligned}$$

and

$$\begin{aligned}
(f_A^{+^2} \,\widetilde{\circ}\, g_B^{+^2}) \,\widetilde{\circ}\, h_C^{+^2} &= (f_A^{+^2} \,\widetilde{\circ}\, g_B^{+^2}) \,\widetilde{\circ}\, (h_C^+ \,\widetilde{\circ}\, h_C^+) = (h_C^+ \,\widetilde{\circ}\, h_C^+) \,\widetilde{\circ}\, (g_B^{+^2} \,\widetilde{\circ}\, f_A^{+^2}) \\
&= ((g_B^{+^2} \,\widetilde{\circ}\, f_A^{+^2}) \,\widetilde{\circ}\, h_C^+) \,\widetilde{\circ}\, h_C^+ \subseteq \mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^+ \subseteq h_C^+.
\end{aligned}$$

Therefore $(f_A^{+^2} \,\widetilde{\circ}\, g_B^{+^2}) \,\widetilde{\circ}\, h_C^{+^2} \subseteq f_A^+ \cap g_B^+ \cap h_C^+$, and similarly $(f_A^{-^2} \,\widetilde{\star}\, g_B^{-^2}) \,\widetilde{\star}\, h_C^{-^2} \supseteq f_A^- \cup g_B^- \cup h_C^-$.

Thus $(\overset{2}{f_A} \diamond \overset{2}{g_B}) \diamond \overset{2}{h_C} \sqsubseteq f_A \sqcap g_B \sqcap h_C$. Now for $a \in S$, there exist $x, y \in S$ such that

$$\begin{aligned}
a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = (ax \cdot ay) \cdot ((ax \cdot ay)x)y \\
&= (aa \cdot xy) \cdot ((ax \cdot ay)x)y = (((ax \cdot ay)x)y \cdot xy)(aa) \\
&= ((yx)(ax \cdot ay) \cdot xy)(aa) = ((yx)(yx \cdot aa) \cdot xy)(aa) \\
&= ((yx)(a \cdot (yx)a) \cdot xy)(aa) = (a(yx \cdot (yx)a) \cdot xy)(aa) \\
&= ((xy)(yx \cdot (yx)a) \cdot a)(aa) = ((x \cdot yx)(y \cdot (yx)a) \cdot a)(aa) \\
&= ((yx^2)(yx \cdot ya) \cdot a)(aa) = ((yx^2)(yy \cdot xa) \cdot a)(aa) \\
&= ((yx^2)(ax \cdot y^2) \cdot a)(aa) = ((ax)(yx^2 \cdot y^2) \cdot a)(aa) \\
&\leq (((ax \cdot ay)x)(yx^2 \cdot y^2) \cdot a)(aa) = (((yx^2 \cdot y^2)x)(ax \cdot ay) \cdot a)(aa) \\
&= ((xy^2 \cdot yx^2)(ax \cdot ay) \cdot a)(aa) = ((ax)((xy^2 \cdot yx^2)(ay)) \cdot a)(aa) \\
&= ((((xy^2 \cdot yx^2)(ay))x)a \cdot a)(aa) = (((x \cdot ay)(xy^2 \cdot yx^2))a \cdot a)(aa) \\
&= (((a \cdot xy)(xy^2 \cdot yx^2))a \cdot a)(aa) = (((yx^2 \cdot xy^2)(xy \cdot ea))a \cdot a)(aa) \\
&= (((yx^2 \cdot xy^2)(ae \cdot yx))a \cdot a)(aa) = ((ae \cdot (yx^2 \cdot xy^2)(yx))a \cdot a)(aa) \\
&= ((ae \cdot b)a \cdot a)(aa), \text{ where } b = (yx^2 \cdot xy^2)(yx).
\end{aligned}$$

This showing that $((ae \cdot b)a \cdot a, aa) \in A_a$. Therefore

$$\begin{aligned}
f_A^{+2}((ae \cdot b)a) &= (f_A^+ \,\widetilde{\circ}\, f_A^+)((ae \cdot b)a) = \bigcup_{(ae \cdot b, a) \leq (ae \cdot b, a)} f_A^+(ae \cdot b) \cap f_A^+(a) \\
&\supseteq f_A^+(ae \cdot (yx^2 \cdot xy^2)(yx)) \cap f_A^+(a) \supseteq f_A^+(a) \cap f_A^+(a) = f_A^+(a),
\end{aligned}$$

Also $h_C^{+^2}(aa) = (h_C^+ \,\widetilde{\circ}\, h_C^+)(ea \cdot ea) \supseteq h_C^+(a)$. Similarly we can show that $g_B^{+^2}(a) \supseteq g_B^+(a)$. Therefore

$$\begin{aligned}((f_A^{+^2} \,\widetilde{\circ}\, g_B^{+^2}) \,\widetilde{\circ}\, h_C^{+^2})(a) &= \bigcup_{a \leq ((ae \cdot b)a \cdot a)(aa)} \{f_A^{+^2}((ae \cdot b)a) \cap g_B^{+^2}(a) \cap h_C^{+^2}(aa)\} \\ &\supseteq f_A^+(a) \cap g_B^+(a) \cap h_C^+(a),\end{aligned}$$

which shows that $(f_A^{+^2} \widetilde{\circ} g_B^{+^2}) \widetilde{\circ} h_C^{+^2} \supseteq f_A^+ \cap g_B^+ \cap h_C^+$ and similarly $(f_A^{-^2} \widetilde{\star} g_B^{-^2}) \widetilde{\star} h_C^{-^2} \subseteq f_A^- \cup g_B^- \cup h_C^-$. Thus $(f_A^2 \diamond g_B^2) \diamond h_C^2 \sqsupseteq f_A \sqcap g_B \sqcap h_C$. Hence $f_A \sqcap g_B \sqcap h_C = (f_A^2 \widetilde{\circ} g_B^{+^2}) \widetilde{\circ} h_C^2$.

Sufficiency: Let f_A be any DFS r-ideal and h_A be any DFS l-ideal of S over U respectively. Since $\mathcal{X}_S$ is a DFS bi-ideal of S over U, we get

$$\begin{aligned}f_A^+ \cap h_C^+ &= f_A^+ \cap \mathcal{X}_S^+ \cap h_C^+ = (f_A^{+^2} \,\widetilde{\circ}\, \mathcal{X}_S^{+^2}) \,\widetilde{\circ}\, h_C^{+^2} = (\mathcal{X}_S^{+^2} \,\widetilde{\circ}\, f_A^{+^2}) \,\widetilde{\circ}\, h_C^{+^2} \\ &= ((f_A^{+^2} \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, h_C^{+^2} = ((\mathcal{X}_S^+ \,\widetilde{\circ}\, f_A^{+^2}) \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, h_C^{+^2} \\ &= (h_C^{+^2} \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, (\mathcal{X}_S^+ \,\widetilde{\circ}\, f_A^{+^2}) = (f_A^{+^2} \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, (\mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^{+^2}) \\ &= ((f_A^+ \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, (f_A^+ \,\widetilde{\circ}\, \mathcal{X}_S^+)) \,\widetilde{\circ}\, ((\mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^+) \,\widetilde{\circ}\, (\mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^+)) \\ &= ((f_A^+ \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, (\mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^+)) \,\widetilde{\circ}\, ((f_A^+ \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, (\mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^+)) \\ &\subseteq (f_A^+ \,\widetilde{\circ}\, h_C^+) \,\widetilde{\circ}\, (f_A^+ \,\widetilde{\circ}\, h_C^+),\end{aligned}$$

which shows that $f_A^+ \cap h_C^+ \subseteq (f_A^+ \widetilde{\circ} h_C^+) \widetilde{\circ} (f_A^+ \widetilde{\circ} h_C^+)$. Now for any $a \in S$, if $a \nleq bc$, for some $b, c \in S$, then the proof is straightforward. Let $a \leq bc$, for some $b, c \in S$. Then

$$\begin{aligned}((f_A^+ \,\widetilde{\circ}\, h_C^+) \,\widetilde{\circ}\, (f_A^+ \,\widetilde{\circ}\, h_C^+))(a) &= ((f_A^+ \,\widetilde{\circ}\, f_A^+) \,\widetilde{\circ}\, (h_C^+ \,\widetilde{\circ}\, h_C^+))(a) \\ &\subseteq ((f_A^+ \,\widetilde{\circ}\, \mathcal{X}_S^+) \,\widetilde{\circ}\, (S \,\widetilde{\circ}\, h_C^+))(a) \\ &= \bigcup_{a \leq bc} \{(f_A^+ \,\widetilde{\circ}\, \mathcal{X}_S^+)(b) \cap (\mathcal{X}_S^+ \,\widetilde{\circ}\, h_C^+)(c)\} \\ &= \bigcup_{a \leq bc} \left\{ \bigcup_{b \leq lm} \{f_A^+(l) \cap \mathcal{X}_S^+(m)\} \cap \bigcup_{c \leq op} \{\mathcal{X}_S^+(o) \cap h_C^+(p)\} \right\} \\ &\subseteq \bigcup_{a \leq bc} \left\{ \bigcup_{b \leq lm} \{f_A^+(lm)\} \cap \bigcup_{c \leq op} \{h_C^+(op)\} \right\} \\ &= \bigcup_{a \leq bc} \{f_A^+(b) \cap h_C^+(c)\} \subseteq \bigcup_{a \leq bc} \{f_A^+(bc) \cap h_C^+(bc)\} \\ &= f_A^+(a) \cap h_C^+(a),\end{aligned}$$

which implies that $(f_A^+ \widetilde{\circ}\, h_C^+) \widetilde{\circ}\, (f_A^+ \widetilde{\circ}\, h_C^+) \subseteq f_A^+ \cap h_C^+$ and therefore $f_A^+ \cap h_C^+ = (f_A^+ \widetilde{\circ}\, h_C^+) \widetilde{\circ}\, (f_A^+ \widetilde{\circ}\, h_C^+)$. Similarly $f_A^- \cup h_C^- = (f_A^- \widetilde{\star}\, h_C^-) \widetilde{\star}\, (f_A^- \widetilde{\star}\, h_C^-)$. Thus $f_A \sqcap h_C = (f_A \diamond h_C) \diamond (f_A \diamond h_C)$. Let R and L be any right and left ideals of S. Then by using Lemma 3.1, $\mathcal{X}_R$ and $\mathcal{X}_L$ are the DFS r-ideal and DFS l-ideal of S over U respectively. Now by using Lemma 3.6, we get

$$\begin{aligned} \mathcal{X}_{R\cap L} &= \mathcal{X}_R \sqcap \mathcal{X}_L = (\mathcal{X}_R \diamond \mathcal{X}_L) \diamond (\mathcal{X}_R \diamond \mathcal{X}_L) = (\mathcal{X}_R \diamond \mathcal{X}_R) \diamond (\mathcal{X}_L \diamond \mathcal{X}_L) \\ &= \mathcal{X}_{R^2} \diamond \mathcal{X}_{L^2} = \mathcal{X}_{(R^2 L^2]} = \mathcal{X}_{(L^2 R^2]}, \end{aligned}$$

which implies that $R \cap L = (L^2 R^2]$. Now let $a^2 \in R$, then $a \leq ax \cdot ay = a^2 \cdot xy \in RS \subseteq R$. Hence R is semiprime. Since $(Sa^2]$ and $(Sa]$ are the right and left ideals of S such that $a^2 \in (Sa^2]$ and $a \in (Sa]$, therefore

$$\begin{aligned} a &\in (Sa^2] \cap (Sa] = ((Sa^2](Sa^2])(Sa^2] \cdot (Sa] \subseteq (SS](Sa^2] \cdot (Sa] \\ &\subseteq ((SS)(Sa^2) \cdot (Sa)] = ((a^2 S)S \cdot Sa] = ((SS)(aa) \cdot Sa] \\ &= ((aa)(SS) \cdot Sa] = ((Sa)a \cdot Sa] \subseteq (Sa \cdot Sa] = (aS \cdot aS]. \end{aligned}$$

This implies that S is strongly regular.

Theorem 4.16. *Let R (resp. L) be any right ideal (resp. left ideal) and f_A (resp. g_B) be any DFS r-ideal over U (resp. DFS l-ideal over U) of a unitary ordered $\mathcal{AG}$-groupoid S. Then the following conditions are equivalent:*

(i) S is strongly regular;

(ii) $R \cap L = (R^3 L] = (L^3 R]$ and R is semiprime;

(iii) $f_A \sqcap g_B = f_A^3 \diamond g_B = g_B^3 \diamond f_A$ and f_A is DFS semiprime.

Proof. $(i) \Longrightarrow (iii)$: Let f_A and g_B be any DFS r-ideal and DFS l-ideal of a strongly regular S over U respectively. From Lemma 3.2, it is easy to show that $f_A^{+3} \diamond g_B^+ \sqsubseteq f_A^+ \sqcap g_B^+$. Now for $a \in S$, there exist some x,$y \in S$ such that

$$\begin{aligned} a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = y(ax \cdot ay) \cdot x(ax \cdot ay) \\ &= (ax)(y \cdot ay) \cdot (ax)(x \cdot ay) = (ax)(ay^2) \cdot (ax)(a \cdot xy) \\ &= (y^2 a)(xa) \cdot (ax)(a \cdot xy) = ((ax)(a \cdot xy))(xa) \cdot y^2 a \\ &= ((ax)(a \cdot xy))(ex \cdot a) \cdot y^2 a = ((ax)(a \cdot xy))(ax \cdot e) \cdot y^2 a \\ &= bc \cdot y^2 a = d \cdot y^2 a, \text{ where } d = bc = ((ax)(a \cdot xy))(ax \cdot e). \end{aligned}$$

Thus

$$\begin{aligned} ((f_A^+ \widetilde{\circ}\, f_A^+) \widetilde{\circ}\, f_A^+)(d) &= \bigcup_{d \leq bc} \{(f_A^+ \widetilde{\circ}\, f_A^+)(b) \cap f_A^+(c)\} \supseteq (f_A^+ \widetilde{\circ}\, f_A^+)(b) \cap f_A^+(c) \\ &= \bigcup_{b \leq (ax)(a \cdot xy)} \{f_A^+(ax) \cap f_A^+(a \cdot xy)\} \cap f_A^+(ax \cdot e) \\ &\supseteq f_A^+(ax) \cap f_A^+(a \cdot xy) \cap f_A^+(ax \cdot e) \supseteq f_A^+(a). \end{aligned}$$

Therefore

$$(f_A^{+3} \tilde{\circ} g_B^+)(a) = \bigcup_{a \leq d \cdot y^2 a} \{((f_A^+ \tilde{\circ} f_A^+) \tilde{\circ} f_A^+)(d) \cap g_B^+(y^2 a)\} \supseteq f_A^+(a) \cap g_B^+(a),$$

which shows that $f_A^+ \cap g_B^+ \subseteq f_A^{+3} \tilde{\circ} g_B^+$, and similarly $f_A^- \cup g_B^- \supseteq f_A^{-3} \tilde{\star} g_B^-$. Thus $f_A \sqcap g_B = f_A^3 \diamond g_B$, and by Lemma 4.13, f_A^+ is DFS semiprime. It is easy to show that $f_A^3 \diamond g_B = g_A^3 \diamond f_A$, hence the proof is omitted.

$(iii) \Longrightarrow (ii)$: Let R and L be any right and left ideals of S. Then by using Lemma 3.1, $\mathcal{X}_R$ and $\mathcal{X}_L$ are the DFS r-ideal and DFS l-ideal of S over U respectively. Now by using Lemma 3.6, we get

$$\mathcal{X}_{R\cap L} = \mathcal{X}_R \sqcap \mathcal{X}_L = ((\mathcal{X}_R \diamond \mathcal{X}_R) \diamond \mathcal{X}_R) \diamond \mathcal{X}_L = \mathcal{X}_{(R^3]} \diamond \mathcal{X}_L = \mathcal{X}_{((R^3]L]} = \mathcal{X}_{(R^3 L]},$$

which implies that $R \cap L = (R^3 L]$ and by Lemma 3.3, R is semiprime. Also it is easy to see that $(R^3 L] = (L^3 R]$, hence the proof is omitted.

$(ii) \Longrightarrow (i)$: Since $(Sa^2]$ and $(Sa]$ are the right and left ideals of S such that $a^2 \in (Sa^2]$ and $a \in (Sa]$, therefore by given assumption and Lemma 2.1, we have

$$\begin{aligned} a &\in (Sa^2] \cap (Sa] = ((Sa^2](Sa^2])(Sa^2] \cdot (Sa] \subseteq (SS](Sa^2] \cdot (Sa] \\ &\subseteq ((SS)(Sa^2) \cdot (Sa)] = ((a^2 S)S \cdot Sa] = ((SS)(aa) \cdot Sa] \\ &= ((aa)(SS) \cdot Sa] = ((Sa)a \cdot Sa] \subseteq (Sa \cdot Sa] = (aS \cdot aS]. \end{aligned}$$

Thus S is strongly regular.

Theorem 4.17. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. Then the following conditions are equivalent:*
(i) S is strongly regular;
(ii) $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3)$ and f_A is DFS semiprime (for any DFS r-ideal f_A and DFS l-ideal g_B of S over U);
(iii) $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3))$ and f_A is DFS semiprime (for any DFS r-ideal f_A and DFS bi-ideal g_B of S over U);
(iv) $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3)$ and f_A is DFS semiprime (for any DFS r-ideal f_A and DFS generalized bi-ideal g_B of S over U);
(v) $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3)$ and f_A, g_B are DFS semiprime (for DFS bi-ideals f_A, g_B of S over U);
(vi) $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3)$ and f_A, g_B are DFS semiprime (for DFS generalized bi-ideals f_A, g_B of S over U).

Proof. $(i) \Longrightarrow (vi)$: Let f_A and g_B be DFS generalized bi-ideals of S over U.

Now for $a \in S$, there exist $x,y \in S$ such that

$$\begin{aligned}
a &\leq ax \cdot ay \leq (ax \cdot ay)x \cdot (ax \cdot ay)y = (ax \cdot ay)(ax \cdot ay) \cdot xy \\
&= (ax \cdot ay)(aa \cdot xy) \cdot xy = (aa)((ax \cdot ay)(xy)) \cdot xy \\
&= ((xy)(ax \cdot ay))(aa) \cdot xy = a(((xy)(ax \cdot ay))a) \cdot xy \\
&= (xy)(((xy)(ax \cdot ay))a) \cdot a = (xy)(a(ax \cdot ay) \cdot xy) \cdot a \\
&= (xy)(ax \cdot (ax \cdot ay)y) \cdot a = (xy)(ax \cdot (y \cdot ay)(ax)) \cdot a \\
&= (ax)(xy \cdot (y \cdot ay)(ax)) \cdot a = ((xy \cdot (ay^2)(ax))x)a \cdot a \\
&= ((x \cdot (ay^2)(ax)) \cdot xy)a \cdot a = ((x \cdot (aa)(y^2x)) \cdot xy)a \cdot a \\
&= ((aa \cdot x(y^2x)) \cdot xy)a \cdot a = ((aa \cdot y^2x^2) \cdot xy)a \cdot a \\
&= ((xy \cdot y^2x^2) \cdot aa)a \cdot a = ((xy \cdot x^2y^2) \cdot aa)a \cdot a \\
&= ((y^2x^2 \cdot yx) \cdot aa)a \cdot a = (y^3x^3 \cdot aa)a \cdot a = (aa \cdot x^3y^3)a \cdot a \\
&= ((x^3y^3 \cdot a)a)a \cdot a = ((x^3y^3 \cdot (ax \cdot ay))a)a \cdot a = ((x^3y^3 \cdot (aa \cdot xy))a)a \cdot a \\
&= ((aa \cdot x^4y^4)a)a \cdot a = ((x^4y^4 \cdot a)a \cdot a)a \cdot a = ((x^4y^4 \cdot (ax \cdot ay))a \cdot a)a \cdot a \\
&= ((aa \cdot x^5y^5)a \cdot a)a \cdot a = ((y^5x^5 \cdot aa)a \cdot a)a \cdot a = ((a \cdot (y^5x^5)a)a \cdot a)a \cdot a \\
&= ba \cdot a, \text{ where } b = (a \cdot (y^5x^5)a)a \cdot a.
\end{aligned}$$

Therefore

$$\begin{aligned}
(f_A^{+^3} \tilde{\circ} g_B^+)(a) &= \bigcup_{a \leq ba \cdot a} \left\{ f_A^{+^3}(ba) \cap g_B^+(a) \right\} \\
&\supseteq \bigcup_{ba \leq ba} \{ f_A^{+^2}(((a \cdot (y^5x^5)a)a \cdot a)) \cap f_A^+(a) \} \cap g_B^+(a) \\
&\supseteq \bigcup_{b \leq (a \cdot (y^5x^5)a)a \cdot a} \{ f_A^+((a \cdot (y^5x^5)a)a) \cap f_A^+(a) \} \cap f_A^+(a) \cap g_B^+(a) \\
&\supseteq f_A^+((a \cdot (y^5x^5)a)a) \cap f_A^+(a) \cap f_A^+(a) \cap g_B^+(a) \\
&\supseteq f_A^+(a) \cap f_A^+(a) \cap f_A^+(a) \cap g_B^+(a) = f_A^+(a) \cap g_B^+(a),
\end{aligned}$$

which shows that $f_A^{+^3} \tilde{\circ} g_B^+ \supseteq f_A^+ \cap g_B^+$ and similarly we can show that $g_B^+ \tilde{\circ} f_A^{+^3} \supseteq f_A^+ \cap g_B^+$. Therefore $(f_A^{+^3} \tilde{\circ} g_B^+) \cap (g_B^+ \tilde{\circ} f_A^{+^3}) \supseteq f_A^+ \cap g_B^+$. Similarly $(f_A^{-^3} \tilde{\star} g_B^-) \cup (g_B^- \tilde{\star} f_A^{-^3}) \subseteq f_A^- \cup g_B^-$. It is easy to show that $(f_A^{+^3} \tilde{\circ} g_B^+) \cap (g_B^+ \tilde{\circ} f_A^{+^3}) \subseteq f_A^+ \cap g_B^+$ and $(f_A^{-^3} \tilde{\star} g_B^-) \cup (g_B^- \tilde{\star} f_A^{-^3}) \supseteq f_A^- \cup g_B^-$. Thus $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3)$.

$(vi) \Longrightarrow (v) \Longrightarrow (iv) \Rightarrow (iii) \Longrightarrow (ii)$ are obvious cases.

$(ii) \Longrightarrow (i)$: Let f_A be any DFS r-ideal and g_B be any DFS l-ideal of S over U. Since $f_A \sqcap g_B = (f_A^3 \diamond g_B) \sqcap (g_B \diamond f_A^3)$, therefore $f_A^+ \cap g_B^+ \subseteq f_A^{+^3} \tilde{\circ} g_B^+$ and

$f_A^+ \cap g_B^+ \subseteq g_B^+ \,\widetilde{\circ}\, f_A^{+^3}$. Let $f_A^+ \cap g_B^+ \subseteq f_A^{+^3} \,\widetilde{\circ}\, g_B^+$, but from Theorem 4.16, $f_A^{+^3} \,\widetilde{\circ}\, g_B^+ \subseteq f_A^+ \cap g_B^+$. Therefore $f_A^+ \cap g_B^+ = f_A^{+^3} \,\widetilde{\circ}\, g_B^+$ and similarly $f_A^- \cup g_B^- = f_A^{-^3} \,\widetilde{\star}\, g_B^-$. Thus $f_A \sqcap g_B = f_A^3 \diamond g_B$ therefore by using Theorem 4.16, S is strongly regular. Now let $f_A^+ \cap g_B^+ \subseteq g_B^+ \,\widetilde{\circ}\, f_A^{+^3}$. It is easy to show that $g_B^+ \,\widetilde{\circ}\, f_A^{+^3} \subseteq f_A^+ \cap g_B^+$, therefore $f_A^+ \cap g_B^+ = g_B^+ \,\widetilde{\circ}\, f_A^{+^3}$ and similarly $f_A^- \cup g_B^- = g_B^- \,\widetilde{\star}\, f_A^{-^3}$.Thus $f_A \sqcap g_B = g_B \diamond f_A^{+^3}$ and therefore by using Theorem 4.16, S is strongly regular.

Theorem 4.18. *Let S be a unitary ordered $\mathcal{AG}$-groupoid. Then the following conditions are equivalent*:
(i) S is strongly regular;
(ii) Every ideal of S is semiprime;
(iii) Every bi-ideal of S is semiprime;
(iv) Every DFS bi-ideal of S is DFS semiprime;
(v) Every DFS generalized bi-ideal of S is DFS semiprime;
(vi) For every DFS bi-ideal f_A of S over U, $f_A(a) = f_A(a^2)$, $\forall\ a \in S$;
(vii) For every DFS generalized bi-ideal f_A of S over U, $f_A(a) = f_A(a^2)$, $\forall\ a \in S$.
Proof. $(i) \implies (vii)$: Let S be strongly regular and f_A be a DFS generalized bi-ideal of S. Let $a \in S$, then there exist $b, c \in S$ such that $a \leq (ba^2)c$, therefore

$$\begin{aligned} a &\leq ba^2 \cdot c = (b \cdot aa)c = (a \cdot ba)c = (c \cdot ba)a \leq c(b(ba^2 \cdot c)) \cdot a = b(c(ba^2 \cdot c)) \cdot a \\ &= b(ba^2 \cdot c^2) \cdot a = (ba^2 \cdot bc^2)a = (b^2 \cdot a^2c^2)a = (a^2 \cdot b^2c^2)a = (a \cdot b^2c^2)a^2 \\ &\leq (ba^2 \cdot c)(b^2c^2) \cdot a^2 = (c^2c)(b^2 \cdot ba^2) \cdot a^2 \\ &= (c^2b^2)(c \cdot ba^2) \cdot a^2 \leq (c^2b^2)(uv \cdot ba^2) \cdot a^2 \\ &= (c^2b^2)(a^2v \cdot bu) \cdot a^2 = (c^2b^2)(a^2b \cdot vu) \cdot a^2 = ((c^2b^2 \cdot vu)b)(aa) \cdot a^2 \\ &= (ab)(a(c^2b^2 \cdot vu)) \cdot a^2 = (aa)(b(b^2c^2 \cdot vu)) \cdot a^2 = a^2(b(b^2c^2 \cdot vu) \cdot a^2. \end{aligned}$$

Thus, we have $f_A^+(a) \supseteq f_A^+(a^2(b(b^2c^2 \cdot vu)) \cdot a^2) \supseteq f_A^+(a^2) \cap f_A^+(a^2) = f_A^+(a^2)$, and similarly $f_A^-(a) \subseteq f_A^-(a^2)$. Thus $f_A(a) \sqsupseteq f_A(a^2)$. Again

$$\begin{aligned} a^2 &= aa \leq (ba^2 \cdot c)(ba^2 \cdot c) = (ba^2 \cdot ba^2)(cc) = (bb \cdot a^2a^2)c^2 = b^2(a^2)^2 \cdot c^2 \\ &= b^2(a^2a^2) \cdot c^2 = a^2(b^2a^2) \cdot c^2 = c^2(b^2a^2) \cdot a^2 = (c^2 \cdot b^2a^2)(aa) \\ &= (a \cdot b^2a^2)(ac^2) = (aa)(b^2a^2 \cdot c^2) = (b^2a^2 \cdot c^2)a \cdot a = (c^2a^2 \cdot b^2)a \cdot a \\ &= ((c^2 \cdot aa)b^2 \cdot a)a = ((a \cdot c^2a)b^2 \cdot a)a = (ab^2)(a \cdot c^2a) \cdot a = a^2(b^2 \cdot c^2a) \cdot a \\ &= a^2(c^2 \cdot b^2a) \cdot a = (aa)(c^2 \cdot b^2a) \cdot a = (b^2a \cdot c^2)(aa) \cdot a = a((b^2a \cdot c)a) \cdot a, \end{aligned}$$

therefore $f_A^+(a^2) \supseteq f_A^+(a((b^2a \cdot c)a) \cdot a) \supseteq f_A^+(a) \cap f_A^+(a) = f_A^+(a)$ and similarly $f_A^-(a^2) \subseteq f_A^-(a)$. Thus $f_A(a^2) \sqsupseteq f_A(a)$. Hence $f_A(a^2) = f_A(a)$.

$(vii) \implies (vi)$ and $(vii) \implies (v)$ are obvious.

$(iv) \implies (iii)$: It can be followed from Lemma 3.3.

$(iii) \implies (ii)$: It is obvious.

$(ii) \implies (i)$: Since $(Sa^2]$ is an ideal of a unitary ordered $\mathcal{AG}$-groupoid S containing a^2, thus by using Lemma 2.1 applied, we have $a \in (Sa^2] = (SS \cdot a^2] = (a^2S \cdot S] = ((aa \cdot SS)S] = ((SS \cdot aa)S] = (Sa^2 \cdot S]$. Therefore S is strongly regular.

5. Conclusions

We have considered the following problems in detail:

i) Define and compare DFS left/right and bi-ideals of an ordered $\mathcal{AG}$-groupoid and respective examples are provided.

ii) Discuss the structural properties of a strongly regular ordered $\mathcal{AG}$-groupoid in terms of DFS left/right and bi-ideals.

iii) Compare a strongly regular class of an ordered $\mathcal{AG}$-groupoid with other important classes of an ordered $\mathcal{AG}$-groupoid, which will provide us a way to study DFS-sets in more generalized form in future.

This paper generalized the theory of an $\mathcal{AG}$-groupoid in the following ways:

i) In an $\mathcal{AG}$-groupoid (without order) by using the DFS-sets.

ii) In an $\mathcal{AG}$-groupoid (with and without order) by using fuzzy sets instead of DFS-sets.

Some important issues for future work are:

i) To develop strategies for obtaining more valuable results in related areas.

ii) To apply these notions and results for studying DFS expert sets and applications in decision making problems.

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