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WP BAILEY PAIRS AND DOUBLE SERIES IDENTITIES

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Abstract: In this paper, using WP Bailey pairs and conjugate WP Bailey pairs, certain double series identities have been established.

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1. Introduction

We begin by recalling some standard notations and terminology. Let a and q be complex numbers with $|q| < 1$. Then the q- shifted factorial is defined by

$$(a;q)_0 = 1,\ (a;q)_n = (1-a)(1-aq)...(1-aq^{n-1}),\ \ n \in N \ \text{ and } \ (a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r).$$

Also, for the sake of brevity, we often write

$$(a_1, a_2, ..., a_r; q)_n = (a_1; q)_n (a_2; q)_n ... (a_r; q)_n.$$

The basic q- hypergeometric series is defined by

$${}_r\Phi_s \left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array} \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left\{(-)^n q^{n(n-1)/2}\right\}^{1+s-r}$$

In an attempt to clarify Rogers second proof of the Rogers-Ramanujan identities Bailey [4] made the following simple but very useful observation,
if

$$\beta_n = \sum_{r=0}^{n} \alpha_r u_{n-r} v_{n+r} \tag{1.1}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \delta_{r+n} u_r v_{r+2n} \tag{1.2}$$

then under suitable convergence conditions,

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.3}$$

where u_r, v_r, α_r and δ_r are arbitrary chosen sequences of r alone.
Above statement (1.1) – (1.3) is called Bailey lemma or Bailey transform. In order to apply above transform, Bailey chose $u_r = \frac{1}{(q;q)_r}$ and $v_r = \frac{1}{(aq;q)_r}$ and converted the transform as,
if

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}} \tag{1.4}$$

and

$$\gamma_n = \sum_{r=0}^{\infty} \frac{\delta_{r+n}}{(q;q)_r(aq;q)_{r+2n}} \tag{1.5}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n \gamma_n = \sum_{n=0}^{\infty} \beta_n \delta_n \tag{1.6}$$

where α_n and β_n are arbitrary chosen sequences of r alone.
The sequence $\langle \alpha_n, \beta_n \rangle$ is called Bailey pair where as sequence $\langle \gamma_n, \delta_n \rangle$ is called conjugate Bailey pair. Andrews [2, 3] generalized the concept of Bailey Pair and conjugate Bailey Pair by defining WP Bailey Pair and WP Conjugate Bailey pair as,
if

$$\beta_n(a,k) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r(a,k) \tag{1.7}$$

and

$$\gamma_n(a,k) = \sum_{r=0}^{n} \frac{(k/a;q)_r (k;q)_{r+2n}}{(q;q)_r (aq;q)_{r+2n}} \delta_{r+n}(a,k) \tag{1.8}$$

then under suitable convergence conditions

$$\sum_{n=0}^{\infty} \alpha_n(a,k)\gamma_n(a,k) = \sum_{n=0}^{\infty} \beta_n(a,k)\delta_n(a,k) \tag{1.9}$$

A pair of sequences $\langle \alpha_n(a,k), \beta_n(a,k) \rangle$ satisfying (1.7) is called WP Bailey Pair and a pair of sequences $\langle \gamma_n(a,k), \delta_n(a,k) \rangle$ satisfying (1.8) is called WP Conjugate Bailey Pair.

Now, putting the value of $\beta_n(a,k)$ from (1.7) in (1.9), we get

$$\sum_{n=0}^{\infty} \delta_n(a,k) \sum_{r=0}^{n} \frac{(k/a;q)_{n-r} (k;q)_{n+r}}{(q;q)_{n-r} (aq;q)_{n+r}} \alpha_r(a,k) = \sum_{n=0}^{\infty} \alpha_n(a,k)\gamma_n(a,k) \tag{1.10}$$

Applying the following rearrangement technique

$$\sum_{n=0}^{\infty} A(r,n) = \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} A(r,n+r)$$

[6; (2.1.1), p. 100] in (1.10), we finally get

$$\sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \delta_{n+r}(a,k) \frac{(k/a;q)_n (k;q)_{n+2r}}{(q;q)_n (aq;q)_{n+2r}} \alpha_r(a,k) = \sum_{n=0}^{\infty} \alpha_n(a,k)\gamma_n(a,k) \tag{1.11}$$

where $\alpha_n(a,k), \gamma_n(a,k)$ and $\delta_n(a,k)$ are arbitrary sequences.

We shall make use of following summation formulas in our analysis

$${}_2\Phi_1 \left[\begin{array}{l} a, b; q; c/ab \\ c \end{array} \right] = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \quad \text{where} \quad |c/ab| < 1. \tag{1.12}$$

[5, App. II (II.8)]

$${}_2\Phi_1 \left[\begin{array}{l} a, b; q; c/ab \\ cq \end{array} \right] = \frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty} \left\{ \frac{ab(1+c) - c(a+b)}{ab - c} \right\}, \text{ where } |c/ab| < 1. \tag{1.13}$$

[7, (1.4), p. 77]

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; q; q^{\frac{1}{2}}/b \\ a^2q/b \end{matrix}\right] = \frac{1}{2}\frac{(a^2, q^{\frac{1}{2}}; q)_\infty}{(a^2q/b, q^{\frac{1}{2}}/b; q)_\infty}\left\{\frac{(aq^{\frac{1}{2}}/b; q^{\frac{1}{2}})_\infty}{(a; q^{\frac{1}{2}})_\infty} + \frac{(-aq^{\frac{1}{2}}/b; q^{\frac{1}{2}})_\infty}{(-a; q^{\frac{1}{2}})_\infty}\right\}, \tag{1.14} $$

[8, (3.5), p. 74]

where $|q^{1/2}/b| < 1$.

$$ {}_2\Phi_1\left[\begin{matrix} a^2, b; q; q^{\frac{3}{2}}/b \\ a^2q/b \end{matrix}\right] = \frac{1}{2a}\frac{(a^2, q^{\frac{1}{2}}; q)_\infty}{(a^2q/b, q^{\frac{1}{2}}/b; q)_\infty}\left\{\frac{(aq^{\frac{1}{2}}/b; q^{\frac{1}{2}})_\infty}{(a; q^{\frac{1}{2}})_\infty} - \frac{(-aq^{\frac{1}{2}}/b; q^{\frac{1}{2}})_\infty}{(-a; q^{\frac{1}{2}})_\infty}\right\}, \tag{1.15} $$

[8, (3.6), p. 75]

where $|q^{3/2}/b| < 1$.

Theorem 1.1. *Choosing $\delta_r(a,k) = \left(\frac{a^2q}{k^2}\right)^r$ in (1.8) and using the summation formula (1.12), we get*

$$ \gamma_n(a,k) = \frac{(a^2q/k, aq/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}\frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}(a^2q/k^2)^n \tag{1.16} $$

Putting these values of $\gamma_n(a,k)$ and $\delta_r(a,k)$ in (1.11), we get

$$ \sum_{n,r=0}^{\infty}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}(a^2q/k^2)^{n+r}\alpha_r(a,k) $$

$$ = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}(a^2q/k^2)^n\alpha_n(a,k)}{(a^2q/k;q)_{2n}} \tag{1.17} $$

where $\alpha_n(a,k)$ is an arbitrary sequence.

Theorem 1.2. *Choosing $\delta_r(a,k) = \left(\frac{a^2}{k^2}\right)^r$ in (1.8) and using the summation formula (1.13), we get*

$$ \gamma_n(a,k) = \frac{(a^2q/k, aq/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}\frac{k}{k+a}\frac{(k;q)_{2n}}{(a^2q/k;q)_{2n}}(a^2q/k^2)^n(1+aq^{2n}) \tag{1.18} $$

Putting these values of $\gamma_n(a,k)$ and $\delta_r(a,k)$ in (1.11), we get

$$ \sum_{n,r=0}^{\infty}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}(a^2/k^2)^{n+r}\alpha_r(a,k) $$

$$= \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty} \frac{k}{k+a} \sum_{n=0}^{\infty} \frac{(k;q)_{2n}(1+aq^{2n})(a^2/k^2)^n \alpha_n(a,k)}{(a^2q/k;q)_{2n}} \tag{1.19}$$

where $\alpha_n(a,k)$ *is an arbitrary sequence.*

Theorem 1.3. *Choosing* $\delta_r(a,k) = \left(\frac{aq^{1/2}}{k}\right)^r$ *in (1.8) and using the summation formula (1.14), we get*

$$\gamma_n(a,k) = \frac{1}{2} \frac{(k, q^{1/2}; q)_\infty}{(aq, aq^{1/2}/k; q)_\infty} (aq^{1/2}/k)^n$$

$$\times \left\{ \frac{(aq^{1/2}/k^{1/2}; q^{1/2})_\infty (k^{1/2}; q^{1/2})_n}{(k^{1/2}; q^{1/2})_\infty (aq^{1/2}/k^{1/2}; q^{1/2})_n} + \frac{(-aq^{1/2}/k^{1/2}; q^{1/2})_\infty (-k^{1/2}; q^{1/2})_n}{(-k^{1/2}; q^{1/2})_\infty (-aq^{1/2}/k^{1/2}; q^{1/2})_n} \right\} \tag{1.20}$$

Putting these values of $\gamma_n(a,k)$ *and* $\delta_r(a,k)$ *in (1.11), we get*

$$\frac{1}{2} \frac{(k, q^{\frac{1}{2}}; q)_\infty}{(aq, aq^{\frac{1}{2}}/k; q)_\infty} \frac{(aq^{\frac{1}{2}}/k^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty}{(q^{\frac{1}{2}}k^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty} \sum_{n=0}^{\infty} \frac{(q^{\frac{1}{2}}k^{\frac{1}{2}}; q^{\frac{1}{2}})_n}{(aq^{\frac{1}{2}}/k^{\frac{1}{2}}; q^{\frac{1}{2}})_n} (aq^{\frac{1}{2}}/k)^n \alpha_n(a,k)$$

$$+ \frac{1}{2} \frac{(k, q^{\frac{1}{2}}; q)_\infty}{(aq, aq^{\frac{1}{2}}/k; q)_\infty} \frac{(-aq^{\frac{1}{2}}/k^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty}{(-q^{\frac{1}{2}}k^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty} \sum_{n=0}^{\infty} \frac{(-q^{\frac{1}{2}}k^{\frac{1}{2}}; q^{\frac{1}{2}})_n}{(-aq^{\frac{1}{2}}/k^{\frac{1}{2}}; q^{\frac{1}{2}})_n} (aq^{\frac{1}{2}}/k)^n \alpha_n(a,k)$$

$$= \sum_{n,r=0}^{\infty} (aq^{1/2}/k)^{n+r} \frac{(k/a;q)_n (k;q)_{n+2r}}{(q;q)_n (aq;q)_{n+2r}} \alpha_r(a,k) \tag{1.21}$$

where $\alpha_n(a,k)$ *is an arbitrary sequence.*

Theorem 1.4. *Choosing* $\delta_r(a,k) = \left(\frac{aq^{3/2}}{k}\right)^r$ *in (1.8) and using the summation formula (1.15), we get*

$$\gamma_n(a,k) = \frac{1}{2k^{1/2}} \frac{(k, q^{1/2}; q)_\infty}{(aq, aq^{1/2}/k; q)_\infty} (aq^{1/2}/k)^n$$

$$\times \left\{ \frac{(aq^{1/2}/k^{1/2}; q^{1/2})_\infty (k^{1/2}; q^{1/2})_n}{(k^{1/2}; q^{1/2})_\infty (aq^{1/2}/k^{1/2}; q^{1/2})_n} - \frac{(-aq^{1/2}/k^{1/2}; q^{1/2})_\infty (-k^{1/2}; q^{1/2})_n}{(-k^{1/2}; q^{1/2})_\infty (-aq^{1/2}/k^{1/2}; q^{1/2})_n} \right\} \tag{1.22}$$

Putting these values of $\gamma_n(a,k)$ *and* $\delta_r(a,k)$ *in (1.11), we get*

$$\frac{1}{2k^{1/2}} \frac{(k, q^{\frac{1}{2}}; q)_\infty}{(aq, aq^{\frac{1}{2}}/k; q)_\infty} \frac{(aq^{\frac{1}{2}}/k^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty}{(k^{\frac{1}{2}}; q^{\frac{1}{2}})_\infty} \sum_{n=0}^{\infty} \frac{(k^{\frac{1}{2}}; q^{\frac{1}{2}})_n}{(aq^{\frac{1}{2}}/k^{\frac{1}{2}}; q^{\frac{1}{2}})_n} (aq^{\frac{1}{2}}/k)^n \alpha_n(a,k)$$

$$-\frac{1}{2k^{1/2}}\frac{(k,q^{\frac{1}{2}};q)_\infty}{(aq,aq^{\frac{1}{2}}/k;q)_\infty}\frac{(-aq^{\frac{1}{2}}/k^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}{(-k^{\frac{1}{2}};q^{\frac{1}{2}})_\infty}\sum_{n=0}^{\infty}\frac{(-k^{\frac{1}{2}};q^{\frac{1}{2}})_n}{(-aq^{\frac{1}{2}}/k^{\frac{1}{2}};q^{\frac{1}{2}})_n}(aq^{\frac{1}{2}}/k)^n\alpha_n(a,k)$$

$$=\sum_{n,r=0}^{\infty}(aq^{3/2}/k)^{n+r}\frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}\alpha_r(a,k) \tag{1.23}$$

where $\alpha_n(a,k)$ is an arbitrary sequence.

2. Special Cases of Theorems (1.1) - (1.4)

Taking $\alpha_n(a,k)=1$ in (1.17), we find

$$\sum_{n,r=0}^{\infty}\frac{\left(\frac{k}{a};q\right)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}\left(\frac{a^2q}{k^2}\right)^{n+r}=\frac{\left(\frac{aq}{k},\frac{a^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}\left(\frac{a^2q}{k^2}\right)^n}{\left(\frac{a^2q}{k};q\right)_{2n}} \tag{2.1}$$

As $k\to\infty$, (2.1) yields

$$\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2+r^2}a^{n+2r}}{(q;q)_n(aq;q)_{n+2r}}=\frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}q^{2n^2}a^{2n} \tag{2.2}$$

Taking $a=1$ in (2.2), we get

$$\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2+r^2}}{(q;q)_n(q;q)_{n+2r}}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{2n^2}$$

$$\Rightarrow\quad\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2+r^2}}{(q;q)_n(q;q)_{n+2r}}=\frac{1}{2(q;q)_\infty}\left\{1+\sum_{n=-\infty}^{\infty}q^{2n^2}\right\} \tag{2.3}$$

Applying Jacobi's triple product identity [5, App. II (II 28)] on the right hand side of (2.3), we get

$$\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2+r^2}}{(q;q)_n(q;q)_{n+2r}}=\frac{1}{2(q;q)_\infty}\left\{1+(q^4;q^4)_\infty(-q^2;q^4)_\infty^2\right\} \tag{2.4}$$

Taking $a=q$ in (2.2), we get

$$\sum_{n,r=0}^{\infty}\frac{q^{n^2+2nr+r^2+2r+n}}{(q;q)_n(q;q)_{n+2r+1}}=\frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{2n(n+1)} \tag{2.5}$$

Making use of [1, (1.17) p.11] on the right hand side of (2.5), we get

$$\sum_{n,r=0}^{\infty} \frac{q^{(n+r)^2+r(r+2)+n}}{(q;q)_n(q;q)_{n+2r+1}} = \frac{1}{(q;q)_\infty} \frac{(q^8;q^8)_\infty}{(q^4;q^4)_\infty} \tag{2.6}$$

Taking $\alpha_r(a,k) = \dfrac{\left(\frac{a^2q}{k};q\right)_{2r}}{(k;q)_{2r}} \left(\dfrac{k^2}{a^2q}\right)^r \dfrac{q^{r^2}}{(q^4;q^4)_r}$ in (1.17), we find

$$\sum_{n,r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_n (k;q)_{n+2r} \left(\frac{a^2q}{k};q\right)_{2r}}{(q;q)_n(aq;q)_{n+2r}(k;q)_{2r}} \left(\frac{a^2q}{k^2}\right) \frac{q^{r^2}}{(q^4;q^4)_r} = \frac{\left(\frac{aq}{k},\frac{a^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} \tag{2.7}$$

Using [1,(3.14) p.86] and [1,(3.2.1) p.87], we have

$$\sum_{n,r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_n (k;q)_{n+2r} \left(\frac{a^2q}{k};q\right)_{2r}}{(q;q)_n(aq;q)_{n+2r}(k;q)_{2r}} \left(\frac{a^2q}{k^2}\right) \frac{q^{r^2}}{(q^4;q^4)_r} = \frac{\left(\frac{aq}{k},\frac{a^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty} \frac{(q^2;q^4)_\infty}{(q,q^4;q^5)_n} \tag{2.8}$$

Again, choosing $\alpha_n(a,k) = \dfrac{\left(\frac{a^2q}{k};q\right)_{2n}}{(k;q)_{2n}} \left(\dfrac{k^2}{a^2}\right)^n \dfrac{q^{n^2+n}}{(q^4;q^4)_n}$ in (1.17), we find

$$\sum_{n,r=0}^{\infty} \frac{\left(\frac{k}{a};q\right)_n (k;q)_{n+2r} \left(\frac{a^2q}{k};q\right)_{2r}}{(q;q)_n(aq;q)_{n+2r}(k;q)_{2r}} \left(\frac{a^2q}{k^2}\right)^n \frac{q^{r^2+2r}}{(q^4;q^4)_r} = \frac{\left(\frac{aq}{k},\frac{a^2q}{k};q\right)_\infty}{\left(aq,\frac{a^2q}{k^2};q\right)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^4;q^4)_n}$$

$$= \frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty} \frac{(q^2;q^4)_\infty}{(q^2,q^3;q^5)_\infty} \tag{2.9}$$

For $a=k$, (2.8) yields

$$\sum_{r=0}^{\infty} \frac{q^{r^2}}{(q^4;q^4)_r} = \frac{(q^2;q^4)_\infty}{(q,q^4;q^5)_\infty} \tag{2.10}$$

Similarly, if we take $a=k$ in (2.9), we get

$$\sum_{r=0}^{\infty} \frac{q^{r^2+2r}}{(q^4;q^4)_r} = \frac{(q^2;q^4)_\infty}{(q^2,q^3;q^4)_\infty} \tag{2.11}$$

Choosing $\alpha_n(a,k) = \frac{\left(\frac{a^2q}{k};q\right)_{2n}}{(k;q)_{2n}} \frac{1}{(q;q)_n}$ in (1.17), we get

$$\sum_{n,r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}(a^2q/k^2)^{n+r}\frac{(a^2q/k;q)_{2r}}{(k;q)_{2r}(q;q)_r} = \frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty}\sum_{n=0}^{\infty}\frac{(a^2q/k^2)^n}{(q;q)_n}$$

$$= \frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty}\frac{1}{(a^2q/k^2;q)_\infty} \tag{2.12}$$

Choosing $\alpha_n(a,k) = 1$ in (1.19), we have

$$\sum_{n,r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}}(a^2/k^2)^n(a^2/k^2)^r$$

$$= \frac{(aq/k,a^2q/k;q)_\infty}{(aq,a^2q/k^2;q)_\infty}\frac{k}{k+a}\sum_{n=0}^{\infty}\frac{(k;q)_{2n}(a^2/k^2)^n(1+aq^{2n})}{(a^2q/k;q)_{2n}} \tag{2.13}$$

As $k \to \infty$, (2.13) becomes

$$\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2-n-r+r^2}a^{n+2r}}{(q;q)_n(aq;q)_{n+2r}} = \frac{1}{(aq;q)_\infty}\sum_{n=0}^{\infty}q^{n(2n-1)}a^{2n}(1+aq^{2n}) \tag{2.14}$$

Taking $a = 1$ in (2.14), we find

$$\sum_{n,r=0}^{\infty}\frac{q^{(n+r)^2-n-r+r^2}}{(q;q)_n(q;q)_{n+2r}} = \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}q^{2n^2-n}(1+q^{2n})$$

$$= \frac{1}{(q;q)_\infty}\sum_{n=0}^{\infty}(q^{2n^2-n}+q^{2n^2+n})$$

$$= \frac{1}{(q;q)_\infty} + \frac{(q^4;q^4)_\infty(-q;q^4)_\infty(-q^3;q^4)_\infty}{(q;q)_\infty} \tag{2.15}$$

Similar other results can be deduced from other theorems too.

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