J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 8, No. 1 (2020), pp. 37-44

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

INTEGRAL REPRESENTATIONS OF EULER-TYPE FOR THE QUADRUPLE HYPERGEOMETRIC FUNCTIONS $F_{20}^{(4)}$

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(**Received:** *Nov. 14, 2020* **Accepted:** *Dec. 18, 2020* **Published:** *Dec. 30, 2020)*

Abstract: The authors establish a set of fifteen new integral representations of Euler-type for the Sharma and Parihar hypergeometric function in four variables $F_{20}^{(4)}$; whose kernels include the quadruple functions $K_1, K_{10}, F_{14}^{(4)}$ and $X_8^{(4)}$; the Exton hypergeometric functions of three variables X_4; the Lauricella functions of three variable F_E, F_N and F_R; the Appell's series of two variables F_2, F_3 and F_4; and the Gauss hypergeometric function.

Keywords and Phrases: Beta function, Eulerian integrals, Quadruple hypergeometric series.

2010 Mathematics Subject Classification: 33C20, 33C65.

1. Introduction

In diverse areas in engineering and mathematical physics, integral representations play an important role in the view point of application. A number of integral representations involving various hypergeometric functions have been investigated by many authors (see, for example [2-3, 9, 10, 11, 17]).

Sharma and Parihar [11] defined the function $F_{20}^{(4)}$, as;

$$F_{20}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_3,c_1;x,y,z,u)$$

$$=\sum_{m,n,p,q=0}^{\infty}\frac{(a_1)_{m+n+p}(a_2)_q(a_3)_{m+n}(a_4)_p(a_5)_q}{(c_1)_{m+q}(c_2)_n(c_3)_p}\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^p}{p!}\frac{u^q}{q!} \tag{1}$$

where $(a)_n$ is the Pochhammer symbol given by;

$$(a)_n=\frac{\Gamma(a+n)}{\Gamma(a)}=a(a+1)\cdots(a+n-1);\qquad (n\in N:=1,2,3,\cdots),(a)_0=1.$$

2. Integral Representation of Euler-type

In this section, we establish a set of fifteen new integral representations of Euler-type for the hypergeometric function of four variables $F_{20}^{(4)}$ defined in (1).

Theorem 2.1. *Each of the following relationships for integral representation $F_{20}^{(4)}$ holds true*:

$$F_{20}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_3,c_1;x,y,z,u)=\frac{aM_1^{a_1}M_2^{a_1+a_2}\Gamma(a_1+a_2+a_3)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)}$$

$$\cdot\int_0^{\infty}\int_0^{\infty}\frac{cosh\alpha(sinh^2\alpha)^{a_1-\frac{1}{2}}cosh\beta(sinh^2\beta)^{a_1+a_2-\frac{1}{2}}}{(1+M_1sinh^2\alpha)^{a_1+a_2}(1+M_2sinh^2\beta)^{a_1+a_2+a_3}}$$

$$\cdot X_8^{(4)}(a_1+a_2+a_3,a_1+a_2+a_3,a_1+a_2+a_3,a_1+a_2+a_3,a_1+a_2+a_3,$$

$$a_5,a_4,a_1+a_2+a_3;c_1,c_1,c_3,c_2;\lambda_1x,\lambda_2u,\lambda_3z,\lambda_1y)d\alpha d\beta \tag{2}$$

where $\lambda_1=\frac{M_1M_2sinh^2\alpha sinh^2\beta}{(1+M_1sinh^2\alpha)(1+M_2sinh^2\beta)^2};\lambda_2=\frac{M_2sinh^2\alpha}{(1+M_1sinh^2\alpha)(1+M_2sinh^2\beta)};$
$\lambda_1=\frac{M_1M_2sinh^2\alpha sinh^2\beta}{(1+M_1sinh^2\alpha)(1+M_2sinh^2\beta)}$ *and* $Re(a_i>0,(i=1,2,3),M_1>0,M_2>0.$

$$F_{20}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_3,c_1;x,y,z,u)$$

$$=\frac{\Gamma(a_3+a_4)}{\Gamma(a_3)\Gamma(a_4)(S-R)^{a_3+a_4-1}}\int_R^S(\alpha-R)^{a_3-1}(S-\alpha)^{a_4-1}$$

$$\cdot F_{14}^{(4)}(a_1,a_1,a_1,a_2,a_3+a_4,a_3+a_4,a_3+a_4,a_5;c_1,c_2,c_3,c_1;\lambda x,\lambda y,\lambda z,u)d\alpha \tag{3}$$

where, $\lambda=(S-R)^{-1};(Re(a_i)>0,(i=3,4),R<S).$

$$F_{20}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_3,c_1;x,y,z,u)$$

$$= \frac{\Gamma(a_1+a_2)\Gamma(a_3+a_4)}{2^{a_1+a_2+a_3+a_4-2}\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(a_4)}$$

$$\cdot \int_{-1}^{+1}\int_{-1}^{+1}(1+\alpha)^{a_1-1}(1-\alpha)^{a_2-1}(1+\beta)^{a_3-1}(1-\beta)^{a_4-1}\cdot K_1(a_1+a_2, a_1+a_2, a_1+a_2,$$

$$a_1+a_2, a_3+a_4, a_3+a_4, a_3+a_4, a_5; c_1, c_2, c_3, c_1; \lambda_1 x, \lambda_1 y, \lambda_2 z, \lambda_3 u)d\alpha d\beta \qquad (4)$$

where, $\lambda_1 = \frac{(1+\alpha)(1+\beta)}{4}, \lambda_2 = \frac{(1+\alpha)(1-\beta)}{4}, \lambda_3 = \frac{(1-\alpha)}{2}$*; and* $Re(a_i) > 0; i = 1, 2, 3.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$

$$= \frac{\Gamma(a_1+a_2)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_3)\Gamma(c_1-a)(S_1-R_1)^{a_1+a_2-1}(S_2-R_2)^{c_1-1}}$$

$$\cdot \int_{R_1}^{S_1}\int_{R_2}^{S_2}(\alpha-R_1)^{a_1-1}(S_1-\alpha)^{a_2-1}(\beta-R_2)^{a-1}(S_2-\beta)^{c_1-a-1}$$

$$\cdot K_{10}(a_1+a_2, a_1+a_2, a_1+a_2, a_1+a_2, a_2, a_2, a_4, a_5;$$

$$a, c_2, c_3, c_1-a; \lambda_1 x, \lambda_2 y, \lambda_2 z, \lambda_4 u)d\alpha d\beta \qquad (5)$$

where, $\lambda_1 = \frac{(\alpha-R_1)(\beta-R_2)}{(S_1-R_1)(S_2-R_2)}, \lambda_2 = \frac{(\alpha-R_1)}{(S_1-R_1)}, \lambda_3 = \frac{(S_1-\alpha)(S_2-\beta)}{(S_1-R_1)(S_2-R_2)}$;
and $Re(a_1) > 0, Re(a_2) > 0, Re(a) > 0, Re(c_1-a) > 0, R_1 < S_1, R_2 < S_2.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u) = \frac{\Gamma(a_1+a_2)\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)\Gamma(c_1-a_2)}$$

$$\cdot \int_0^1\int_0^1(\alpha)^{a_1-1}(1-\alpha)^{a_4-1}\beta^{a_2-1}(1-\beta)^{a_1-a_2-1}(1-\beta u)^{a_5}$$

$$\cdot X_4(a_1+a_4; c_3, c_1-a_2, c_2; \alpha(1-\alpha)z, \alpha(1-\beta)x, \alpha y)d\alpha d\beta \qquad (6)$$

where, $Re(a_i) > 0, i = 1, 2, 4; Re(c_1-a_2) > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$

$$= \frac{M_1^{a_1}M_2^{a_3}\Gamma(a_1+a_2)\Gamma(a_3+a_5)}{\Gamma(a_1)\Gamma(a_2)\Gamma(a_4)\Gamma(c_1-a_2)}\cdot\int_0^\infty\int_0^\infty\frac{\alpha^{a_1-1}\beta^{a_3-1}}{(1+M_1\alpha)^{a_1+a_2}(1+M_2\beta)^{a_3+a_5}}$$

$$\cdot F_E(a_1+a_2, a_1+a_2, a_1+a_2, a_4, a_3+a_5, a_3+a_5; c_3, c_1, c_2; \lambda_1 z, \lambda_2 x+\lambda_3 u, \lambda_2 y)d\alpha d\beta \quad (7)$$

where, $\lambda_1 = \frac{M_1\alpha}{(1+M_1\alpha)}, \lambda_2 = \frac{M_1M_2\alpha\beta}{(1+M_1\alpha)(1+M_2\beta)}, \lambda_3 = \frac{1}{(1+M_1\alpha)(1+M_2\beta)}$;
and $Re(a_i) > 0, (i = 1, 2, 3, 5), M_1 > 0, M_2 > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$

$$= \frac{\Gamma(c_1)}{\Gamma(a_1)\Gamma(a_2)\Gamma(c_1-a_2)(S-R)^{c_1-a_5-1}}$$
$$\cdot \int_R^S (\alpha-R)^{a_2-1}(S-\alpha)^{c_1-a_2-1}[(S-R)-(\alpha-R)u]^{-a_5}$$
$$\cdot F_E\left(a_1, a_1, a_1, a_4, a_3, a_3; c_3, c_2, c_1-a_2, c_2; z, \frac{(S-\alpha)x}{(S-R)}, y\right)d\alpha \tag{8}$$

where, $Re(a_2) > 0, Re(c_1 - a_2) > 0, R < S.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$
$$= \frac{\Gamma(c_2)}{2^{c_2-1}\Gamma(a_3)\Gamma(c_2-a_3)} \cdot \int_{-1}^{+1} (1+\alpha)^{a_3-1}(1-\alpha)^{c_2-a_3-1}[1-(1+\alpha)y]^{-a_1}$$
$$\cdot F_N\left(a_4, a_5, 1+a_3-c_2, a_1, a_2, a_1; c_3, c_1, c_1, c_2; \frac{z}{[1-(1+\alpha)y]}, u, \frac{(1+\alpha)x}{(\alpha-1)[1-(1+\alpha)y]}\right)d\alpha \tag{9}$$

where, $Re(a_3) > 0, Re(c_2 - a_3) > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$
$$= \frac{2M^{a_1}\Gamma(c_2)}{\Gamma(a_1)\Gamma(c_2-a_1)} \int_0^\infty \frac{cosh\alpha (sinh^2\alpha)^{a_1-\frac{1}{2}}}{(1+Msinh^2\alpha)^{c_3-a_3}} \left[(1+Msinh^2\alpha) - Mysinh^2\alpha\right]^{-a_3}$$
$$\cdot F_N\Bigg(a_4, a_5, a_3, 1+a_1-c_2, a_2, 1+a_1-c_2; c_3, c_1, c_1; -Mzsinh^2\alpha, u,$$
$$-\frac{Msinh^2\alpha(1+Msinh^2\alpha)x}{(1+Msinh^2\alpha) - Mysinh^2\alpha}\Bigg)d\alpha \tag{10}$$

where, $Re(a_1) > 0, Re(c_2 - a_1) > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$
$$= \frac{\Gamma(c_3)}{2^{c_3-2}\Gamma(a_1)\Gamma(c_3-a_1)} \int_{-1}^{+1} \frac{(1+\alpha)^{2a_1-1}(1-\alpha)^{2(c_3-a_1)-1}}{(1+\alpha^2)^{c_3-a_4}} \left[(1+\alpha^2) - \frac{1}{2}(1+\alpha)^2 z\right]^{-a_4}$$
$$\cdot F_R\left(1+a_1-c_3, a_2, 1+a_1-c_3, a_3, a_5, a_3; c_2, c_1, c_1; -\frac{(1+\alpha)^2 y}{(1-\alpha)^2}, u, -\frac{(1+\alpha)^2 x}{(1-\alpha)^2}\right)d\alpha \tag{11}$$

where, $Re(a_1) > 0, Re(c_3 - a_1) > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u) = \frac{2(1+M)^{a_4}\Gamma(c_4)}{\Gamma(c_4)\Gamma(c_3 - a_4)}$$

$$\cdot \int_0^{\frac{\pi}{2}} \frac{(sin^2\alpha)^{a_4 - \frac{1}{2}}(cos^2\alpha)^{c_3 - a_4 - \frac{1}{2}}}{(1 + Msin^2\alpha)^{c_3 - a_1}} \left[(1 + Msin^2\alpha) - (1+M)zsin^2\alpha\right]^{-a_1}$$

$$\cdot F_R\Bigg(a_1, a_2, a_1, a_3, a_5, a_3; c_2, c_1, c_1; \frac{(1 + Msin^2\alpha)y}{[(1 + Msin^2\alpha) - (1+M)zsin^2\alpha]},$$

$$u, \frac{(1 + Msin^2\alpha)x}{[(1 + Msin^2\alpha) - (1+M)zsin^2\alpha]}\Bigg) d\alpha \tag{12}$$

where, $Re(a_4) > 0, Re(c_3 - a_4) > 0, M > -1.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u) = \frac{\Gamma(c_1)\Gamma(c_3)}{\Gamma(a_4)\Gamma(a)\Gamma(c_1 - a)\Gamma(c_3 - a_4)}$$

$$\cdot \int_0^1 \int_0^1 \alpha^{a_1 - 1}(1-\alpha)^{c_1 - a - 1}\beta^{a_4 - 1}(1-\beta)^{c_3 - a_4 - 1}(1-\beta z)^{-a_1}\ {}_2F_1(a_2, a_5; c_1 - a; (1-\alpha)u)$$

$$\cdot F_4\left(a_1, a_3; a, c_2; \frac{\alpha x}{(1-\beta z)}, \frac{y}{(1-\beta z)}\right) d\alpha d\beta \tag{13}$$

where, $Re(a_4) > 0, Re(a) > 0, Re(c_1 - a) > 0, Re(c_3 - a_4) > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u) = \frac{4\Gamma(c_1)\Gamma(c_2)}{\Gamma(a_1)\Gamma(a)\Gamma(c_1 - a)\Gamma(c_2 - a_1)}$$

$$\cdot \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (sin^2\alpha)^{a_1 - \frac{1}{2}}(cos^2\alpha)^{c_2 - a_1 - \frac{1}{2}}(sin^2\beta)^{c_1 - a - \frac{1}{2}}(cos^2\beta)^{a - \frac{1}{2}}(1 - ysin^2\alpha)^{-a_3}$$

$$\cdot\ {}_2F_1(a_2, a_5; a; ucos^2\beta) F_2\left(1 + a_1 - c_2, a_3, a_4; c_1 - a, c_3; -\frac{xtan^2\alpha tan^2\beta}{(1 - ysin^2\alpha)}, -ztan^2\alpha\right) d\alpha d\beta \tag{14}$$

where, $Re(a_1) > 0, Re(a) > 0, Re(c_1 - a) > 0, Re(c_2 - a_1) > 0.$

$$F_{20}^{(4)}(a_1, a_1, a_1, a_2, a_3, a_3, a_4, a_5; c_1, c_2, c_3, c_1; x, y, z, u)$$

$$= \frac{\Gamma(c_1)\Gamma(c_2)}{2^{c_1 + c_2 - 4}\Gamma(a_1)\Gamma(a_2)\Gamma(c_1 - a_2)\Gamma(c_2 - a_1)} \int_{-1}^{+1} \int_{-1}^{+1} \frac{(1+\alpha)^{2a_1 - 1}(1-\alpha)^{2(c_2 - a_1) - 1}}{(1+\alpha^2)^{c_2 - a_3}}$$

$$\cdot\frac{(1+\beta)^{2a_2-1}(1-\beta)^{2(c_1-a_2)-1}}{(1+\beta^2)^{c_1-a_5}}\left[(1+\alpha^2)-\frac{1}{2}(1+\alpha)^2y\right]^{-a_3}\left[(1+\beta^2)-\frac{1}{2}(1+\beta)^2u\right]^{-a_5}$$
$$\cdot F_2\Bigg(1+a_1-c_2,a_3,a_4;c_1-a_2,c_3;-\frac{(1+\alpha)^2(1+\alpha^2)(1-\beta)^2x}{2(1-\alpha)^2(1+\beta^2)[(1+\alpha^2)-\frac{1}{2}(1+\alpha)^2y]};$$
$$-\frac{(1+\alpha)^2z}{(1-\alpha)^2}\Bigg)d\alpha d\beta \tag{15}$$

where, $Re(a_1)>0, Re(a_2)>0, Re(c_1-a_2)>0, Re(c_2-a_1)>0.$

$$F_{20}^{(4)}(a_1,a_1,a_1,a_2,a_3,a_3,a_4,a_5;c_1,c_2,c_3,c_1;x,y,z,u)$$
$$=\frac{\Gamma(c_2)\Gamma(c_3)}{2^{c_2+c_3-2}\Gamma(a_3)\Gamma(a_4)\Gamma(c_2-a_3)\Gamma(c_3-a_4)}\int_{-1}^{+1}\int_{-1}^{+1}(1+\alpha)^{a_3-1}(1-\alpha)^{c_2-a_3-1}$$
$$\cdot(1+\beta)^{a_4-1}(1-\beta)^{c_3-a_4-1}[1-\frac{1}{2}(1+\alpha)y-\frac{1}{2}(1+\beta)z]^{-a_1}$$
$$\cdot F_3\left(1+a_3-c_2,a_2,a_3,a_5;c_1;-\frac{(1+\alpha)x}{(1-\alpha)[1-\frac{1}{2}(1+\alpha)y-\frac{1}{2}(1+\beta)z]},u\right)d\alpha d\beta \tag{16}$$

where, $Re(a_3)>0, Re(a_4)>0, Re(c_2-a_3)>0, Re(c_3-a_4)>0.$

Where $X_8^{(4)}$ *is the quadruple function (see [2]);* $F_{14}^{(4)}$ *is Sharma's and Parihar's function [12];* K_1 *and* K_{10} *are Exton functions of four variables (see [8]);* X_4 *is the Exton function [9] of three variables;* F_E, F_N *and* F_R *are Lauricella functions(see [10] and [15]) of three variables;* F_2, F_3 *and* F_4 *are Appell's functions of two variables (see [1], [15]).*

Proof. Once substituting the series definition of the special function in each integrand, then changing the order of the integral and the summation; finally taking into account the following integral representations of the Beta function and their various associated Eulerian integrals (see, [7, 13, 14, 17] we derive each of the integral representations (2) to (16).

$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt=(S-R)^{1-a-b}\int R^S(t-R)^{a-1}(S-t)^{b-1}dt$$

where, $R<S; Re(a)>0, Re(b)>0$).

$$B(a,b)=\frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\qquad (a,b\in C-Z_0^-).$$

$$B(a,b)=2\int_0^{\frac{\pi}{2}}(sin\alpha)^{2a-1}(cos\alpha)^{2b-1}d\alpha=2^{2-a-b}\int -1^1\frac{(1+\alpha)^{2a-1}(1-\alpha)^{2b-1}}{(1+\alpha^2)^{a+b}}d\alpha$$

where, $Re(a) > 0, Re(b) > 0$.
and

$$B(a,b) = M^a \int_0^\infty \frac{\alpha^{a-1}}{(1+M\alpha)^{a+b}} d\alpha = 2M^a \int_0^\infty \frac{cosh\alpha (sinh^2\alpha)^{2a-1}}{(1+Msinh^2\alpha)^{a+b}} d\alpha$$

where, $M > 0; Re(a) > 0, Re(b) > 0$.

3. An Open Problem

Based upon the work presented in this paper, we find it to be worthwhile to motivate the interested reader to consider the following related open problem.

Open Problem: Recently, Srivastva *et al* (see [16]) has introduced new concepts namely $R_\alpha = R(2,1,1,1,2,2); R_\beta = R(2,2,1,1,2,2); R_m = R(m,m,1,1,1,2)(m \in \mathbb{N})$. Interested researchers may establish relationships between R_α, R_β and R_m; and results presented in this paper.

4. Acknowledgements

The research work of the first author is supported through a major research project of National Board of Higher Mathematics (NBHM) of the Department of Atomic Energy (DAE), Government of India by its sanction letter Ref. No 02011/12/2020 NBHM(R.P.)/R & D II/7867, dated 19th October 2020.

Conflicts of Interest: Both the authors declare that they have no conflict of interest.

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