J. of Ramanujan Society of Mathematics and Mathematical Sciences
Vol. 8, No. 1 (2020), pp. 31-36

ISSN (Online): 2582-5461

ISSN (Print): 2319-1023

SOME DEFINITE INTEGRAL ASSOCIATED TO ERROR FUNCTION AND HYPERGEOMETRIC FUNCTION

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(**Received:** *Oct. 02, 2020* **Accepted:** *Oct. 23, 2020* **Published:** *Dec. 30, 2020)*

Abstract: In this paper we have developed some definite integral involving error function in association with Hypergeometric and first kind of Modified Bessel function.

Keywords and Phrases: Bessel Function, Hypergeometric Function, Error Function.

2010 Mathematics Subject Classification: 33B20, 33C05, 33C10, 33C20, 33D15.

1. Introduction

Yurry A. Brychkov [Brychkov p. 188 (4.4.5.1, 4.4.5.2)] has derived the following formulae

$$\int_0^1 cos^{-1}x \ \mathrm{erf}(ax)\ dx = \frac{\sqrt{\pi}}{2a}\Big[-1+e^{\frac{-a^2}{2}}\Big\{(a^2+1)I_0\Big(\frac{a^2}{2}\Big)+a^2I_1\Big(\frac{a^2}{2}\Big)\Big\}\Big]. \quad (1.1)$$

$$\int_0^1 x^2\ cos^{-1}x \ \mathrm{erf}(ax)\ dx$$
$$= \frac{\sqrt{\pi}}{36a^3}\Big[(4a^4+3a^2+6)e^{\frac{-a^2}{2}}I_0\Big(\frac{a^2}{2}\Big)+a^2(4a^2-1)e^{\frac{a^2}{2}}I_1\Big(\frac{a^2}{2}\Big)-6\Big]. \quad (1.2)$$

The error function is defined as:

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k \ z^{2k+1}}{k! \ (2k+1)}. \tag{1.3}$$

Bessel functions of the first kind, denoted as $J_\alpha(x)$, are solutions of Bessel's differential equation that are finite at the origin ($x = 0$) for integer or positive α, and diverge as x approaches zero for negative non-integer α ([12]). It is possible to define the function by its Taylor series expansion around $x = 0$.

$$J_\alpha(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \ \Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.4}$$

where $\Gamma(z)$ is the gamma function, a shifted generalization of the factorial function to non-integer values. The Bessel function of the first kind is an entire function if α is an integer.

The Bessel functions are valid even for complex arguments x, and an important special case is that of a purely imaginary argument (See[12]). In this case, the solutions to the Bessel equation are called the modified Bessel functions (or occasionally the hyperbolic Bessel functions) of the first and second kind. The first kind of modified Bessel function is defined as

$$I_\alpha(x) = \iota^{-\alpha} J_\alpha(\iota x) = \sum_{m=0}^{\infty} \frac{1}{m! \ \Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{1.5}$$

A generalized hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

$$\frac{c_{k+1}}{c_k} = \frac{P(k)}{Q(k)} = \frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(K+b_2)...(k+b_q)(k+1)} \ z. \tag{1.6}$$

Where $k+1$ in the denominator is present for historical reasons of notation [Koepf p. 12 (2.9)], and the resulting generalized hypergeometric function is written

$${}_pF_q \left[\begin{array}{ccc} a_1, a_2, \cdots, a_p & ; & \\ & & z \\ b_1, b_2, \cdots, b_q & ; & \end{array} \right] = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \cdots (a_p)_k z^k}{(b_1)_k (b_2)_k \cdots (b_q)_k k!} \tag{1.7}$$

where the parameters $b_1, b_2, \cdots, b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p \leq q$, converges for $\mid z \mid < 1$ if $p = q+1$, diverges for all z , $z \neq 0$ if $p > q+1$ [Luke p. 156 (3)].

The function ${}_2F_1(a, b; c; z)$ corresponding to $p = 2, q = 1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p. 123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial, falling sequential product, or lower factorial) is defined as the polynomial [Steffensen p.8]

$$(x)_n = x(x-1)(x-2)...(x-n+1) = \prod_{k=1}^{n}(x-k+1) = \prod_{k=0}^{n-1}(x-k) \tag{1.8}$$

2. Main Formulae of the Integration

$$\int_0^1 x\ cos^{-1}x\ \text{erf}(ax)\ dx = \frac{4}{9\sqrt{\pi}}\ a\ {}_2F_2\Big(\frac{1}{2}, 2; \frac{5}{2}, \frac{5}{2}; -a^2\Big) \tag{2.1}$$

$$\int_0^1 x^3\ cos^{-1}x\ \text{erf}(ax)\ dx$$
$$= -\frac{4}{75\sqrt{\pi}}\ a\ \Big[{}_2F_2\Big(\frac{5}{2}, 3; \frac{7}{2}, \frac{7}{2}; -a^2\Big) - 5\ {}_2F_2\Big(\frac{1}{2}, 3; \frac{3}{2}, \frac{7}{2}; -a^2\Big)\Big] \tag{2.2}$$

$$\int_0^1 x^4\ cos^{-1}x\ \text{erf}(ax)\ dx = \frac{\sqrt{\pi}}{300a^5}\Big[e^{-a^2/2}\Big\{(-33+4(4a^4-a^2))a^2\ I_1\Big(\frac{a^2}{2}\Big)+$$
$$+2(8a^6+6a^4+15a^2+30)I_0\Big(\frac{a^2}{2}\Big)\Big\} - 60\Big] \tag{2.3}$$

$$\int_0^1 x^5\ cos^{-1}x\ \text{erf}(ax)\ dx$$
$$= -\frac{16}{735\sqrt{\pi}}\ a\ \Big[{}_2F_2\Big(\frac{7}{2}, 4; \frac{9}{2}, \frac{9}{2}; -a^2\Big) - 7\ {}_2F_2\Big(\frac{1}{2}, 4; \frac{3}{2}, \frac{9}{2}; -a^2\Big)\Big] \tag{2.4}$$

$$\int_0^1 x^6\ cos^{-1}x\ \text{erf}(ax)\ dx = \frac{\sqrt{\pi}}{1960a^7}\Big[e^{-a^2/2}\Big\{(64a^8+48a^6+215a^4+420a^2+840)$$

$$I_0\Big(\frac{a^2}{2}\Big) + a^2(64a^6 - 16a^4 - 227a^2 - 650)I_1\Big(\frac{a^2}{2}\Big)\Big\} - 840\Big] \tag{2.5}$$

$$\int_0^1 x^7\ cos^{-1}x\ \mathrm{erf}(ax)\ dx$$
$$= -\frac{32}{2835\sqrt{\pi}}\ a\ \Big[{}_2F_2\Big(\frac{9}{2}, 5; \frac{11}{2}, \frac{11}{2}; -a^2\Big) - 9\ {}_2F_2\Big(\frac{1}{2}, 5; \frac{3}{2}, \frac{11}{2}; -a^2\Big)\Big] \tag{2.6}$$

$$\int_0^1 x^9\ cos^{-1}x\ \mathrm{erf}(ax)\ dx$$
$$= -\frac{256}{38115\sqrt{\pi}}\ a\ \Big[{}_2F_2\Big(\frac{11}{2}, 6; \frac{13}{2}, \frac{13}{2}; -a^2\Big) - 11\ {}_2F_2\Big(\frac{1}{2}, 6; \frac{3}{2}, \frac{13}{2}; -a^2\Big)\Big] \tag{2.7}$$

3. Derivation of the Integrals

Derivation of integral(2.1)

$$\int_0^1 x\ cos^{-1}x\ \mathrm{erf}(ax)\ dx = \Big[x\ cos^{-1}x\Big(\frac{e^{-a^2x^2}}{a\sqrt{\pi}} + x\ \mathrm{erf}(ax)\Big)\Big]_0^1$$
$$- \int_0^1 \Big(-\frac{x}{\sqrt{1-x^2}} + cos^{-1}x\Big)\Big(\frac{e^{-a^2x^2}}{a\sqrt{\pi}} + x\ \mathrm{erf}(ax)\Big)\ dx$$

After this applying computational method,we have proved the integral (2.1). On this same way, we have established the remaining integral .

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