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Solutions of the Pell Equation $x^2 - (a^2b^2c^2 + 2ab)y^2 = N$ when $N \in \pm 1, \pm 4$.

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Abstract

Let a, b and c be natural numbers and $d = a^2b^2c^2 + 2ab$. In this paper, by using continued fraction expansion of $\sqrt{d}$. We find fundamental solution of the equations $x^2 - (a^2b^2c^2 + 2ab)y^2 = \pm 1$ and we get all positive integer solutions of the equations $x^2 - (a^2b^2c^2 + 2ab)y^2 = \pm 1$ in terms of generalized Fibonacci and Lucas sequences. Moreover, we find all positive integer solutions of the equations $x^2 - (a^2b^2c^2 + 2ab)y^2 = \pm 4$ in terms of generalized Fibonacci and Lucas sequences.

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1 Introduction

Let $d \neq 1$ be a positive square free integer and N be any fixed positive integer. Then the equation $x^2 - dy^2 = \pm$ N is known as Pell equation and is named after John Pell(1611-1685), a mathematician who searched for integer solutions to equations of this type in the seventeenth century. For $N = 1$, the Pell equation $x^2 - dy^2 = \pm 1$ is known as classical Pell equation and was studied by Brahmagupta(598-670) and Bhaskara(1114-1185). The Pell equation $x^2 - dy^2 = \pm 1$ has infinitely many solutions (x_n, y_n) for $n \geq 1$. There are several methods for finding the fundamental solutions of Pell's equation $x^2 - dy^2 = 1$ for a positive non square integer "d", e.g. the cyclic method[4] known in India in the 12^{th} century, or the slightly less less efficient but more regular English method (17^{th} century) which produce all solution is based on the simple finite continued fraction expansion of $\sqrt{d}$.

Let $\frac{p_i}{q_i}$ be the sequence of convergence to the continued fraction for $\sqrt{d}$. Then the pair (x_1, y_1) solving Pell's equation and minimizing x satisfies $x_1 = p_i$ and $y_1 = q_i$

for some i. This pair is called the fundamental solution. Thus the fundamental solution may be found by performing the continued fraction expansion and testing each successive convergent until a solution to Pell's equation is found. Continued fraction plays an important role in solutions of the Pell equations $x^2 - dy^2 = \pm 1$. Whether or not there exist a positive integer solution to the equation $x^2 - dy^2 = -1$ depends on the period length of the continued fraction expansion of $\sqrt{d}$. It can be seen that the equation $x^2 - dy^2 = -1$ has no positive integer solutions. To find all positive integer solutions of the equations $x^2 - dy^2 = \pm 1$. One first determines a fundamental solution.

In this paper, after the Pell's equations are described briefly, the fundamental solution to the Pell equations, $x^2 - (a^2b^2c^2 + 2ab)y^2 = \pm 1$ are calculated, by means of the generalized Fibonacci and Lucas sequences. Especially, all positive integer solutions of the equations $x^2 - (k^2 - 2k)y^2 = \pm 1$ and $x^2 - (k^2 - 2k)y^2 = \pm 4$ are discovered. Now, we briefly mention the generalized Fibonacci and Lucas sequences $(U_n(k,s)) and (V_n(k,s))$. Let k and s be two nonzero integers with $k^2 + 4s > 0$.

Generalized Fibonacci sequence is defined by $U_0(k,s) = 0, U_1(k,s) = 1$ and $U_{(n+1)} = kU_n(k,s) + sU_{(n-1)}(k,s)$ for $n \geq 1$ and generalized Lucas sequence is defined by $V_0(k,s) = 2$, $V_1(k,s) = k$ and $V_{(n+1)} = kV_n(k,s) + sV_{(n-1)}(k,s)$ for $n \geq 1$, respectively. It is well known that $U_n(k,s) = \alpha^n - \beta^n/\alpha - \beta$ and $V_n(k,s) = \alpha^n + \beta^n$ where, $\alpha = (k + \sqrt{k^2 + 4s})/2$ and $\beta = (k - \sqrt{k^2 + 4s})/2$. The above identities are known as Binet's formula. Clearly, $\alpha + \beta = k$, $\alpha - \beta\sqrt{k^2 + 4s}$ and $\alpha\beta = -s$. For more information about generalized Fibonacci and Lucas sequences one can refer [1]-[6], [11]-[18].

2 Preliminary notes

Let d be a positive integer which is not a perfect square and N be any nonzero fixed integer in the Pell equation $x^2 - dy^2 = N$. If $a^2 - db^2 = N$, we say that (a,b) is a solution to the Pell equation $x^2 - dy^2 = N$. We use the notations (a,b) and $a + b\sqrt{d}$ interchangeably to denote solutions of the equation $x^2 - dy^2 = N$. Also if a and b are both positive, we say that $a + b\sqrt{d}$ is a positive solution to the equation $x^2 - dy^2 = N$. There is a continued fraction expansion of $\sqrt{d}$ such that $\sqrt{d} = [a_0; \overline{a_1, a_2, ..., a_{l-2}, 2a_0}]$, where l is period length and the a_j's are given by the recursion formula:

$$\alpha_0 = \sqrt{d}, \quad a_k = [\alpha_k] \text{and} \quad \alpha_(k+1) = 1/\alpha_k - \beta_k, \quad k = 0, 1, 2, 3, ...$$

Recall that $a_l = 2a_0$ and $a_(i + k) = a_k$ for $k \geq 1$. The n^th convergent of $\sqrt{d}$ for

$n \geq 0$.

$$\frac{p_n}{q_n}=[a_0,a_1,...,a_n]=a_o+\frac{1}{a_1+\frac{1}{a_2+...\frac{1}{a_{n-1}+\frac{1}{a_n}}}}.$$

Let $x_1+y_1\sqrt{d}$ be a positive solution to the equation $x^2-dy^2=N$. We say that $x_1+y_1\sqrt{d}$ is the fundamental solution of the equation $x^2-dy^2=N$, if $x_2+y_2\sqrt{d}$ is different solution to the equation $x^2-dy^2=N$, then $x_1+y_1\sqrt{d}<x_2+y_2\sqrt{d}$. Recall that if $a+b\sqrt{d}<r+s\sqrt{d}$ if and only if $a<r$ and $b<s$. The following lemma and theorems can be found many elementary text books [1], [3], [4], [9], [10], [13], [16], [17].

Lemma 2.1. *If $x_1+y_1\sqrt{d}$ is the fundamental solution to the equation $x^2-dy^2=-1$, then $(x_1+y_1\sqrt{d})^2$ is the fundamental solution to the equation $x^2-dy^2=-1$.*

Lemma 2.2. *Let l be the period length of continued fraction expansion of $\sqrt{d}$. If l is even, then the fundamental solution to the equation $x^2-dy^2=1$ is given by,*

$$x_1+y_1\sqrt{d}=p_{l-1}+q_{l-1}\sqrt{d}$$

and the equation $x^2-dy^2=-1$ has no integer solutions. If l is odd, then the fundamental solution to the equation $x^2-dy^2=1$ is given by $x_1+y_1\sqrt{d}=p_{2l-1}+q_{2l-1}\sqrt{d}$ and the fundamental solution to the equation $x^2-dy^2=-1$ is given by,

$$x_1+y_1\sqrt{d}=p_{l-1}+q_{l-1}\sqrt{d}.$$

Theorem 2.1. *Let $x_1+y_1\sqrt{d}$ be the fundamental solution to the equation $x^2-dy^2=1$. Then all positive integer solutions of the equation $x^2-dy^2=1$ are given* by,

$$x_n+y_n\sqrt{d}=(x_n+y_n\sqrt{d})^n, with\ n\geq 1.$$

Theorem 2.2. *Let $x_1+y_1\sqrt{d}$ be the fundamental solution to the equation $x^2-dy^2=-1$. Then all positive integer solutions of the equation $x^2-dy^2=-1$ are* given by,

$$x_n+y_n\sqrt{d}=(x_n+y_n\sqrt{d})^{2n-1}, with\ n\geq 1.$$

Theorem 2.3. *Let $x_1+y_1\sqrt{d}$ be the fundamental solution to the equation $x^2-dy^2=4$. Then all positive integer solutions of the equation $x^2-dy^2=4$ are given* by,

$$x_n+y_n\sqrt{d}=\frac{(x_1+y_1\sqrt{d})^n}{2^{n-1}}, with\ n\geq 1.$$

Theorem 2.4. *Let $x_1+y_1\sqrt{d}$ be the fundamental solution to the equation $x^2-dy^2=-4$. Then all positive integer solutions of the equation $x^2-dy^2=-4$ are* given by,

$$x_n+y_n\sqrt{d}=\frac{(x_1+y_1\sqrt{d})^{2n-1}}{4^{n-1}}, with\ n\geq 1.$$

Now, we will assume that k, a and b are positive integers. We give continued fraction expansion of $\sqrt{d}$ for $d = a^2b^2c^2 + 2ab$ and $d = a^2b^2c^2 + ab$

Theorem 2.5. *Let $d = a^2b^2c^2 + 2ab$. Then $\sqrt{d} = [abc; \overline{c, 2abc}]$.*

Proof

$$\begin{aligned}
\sqrt{d} &= \sqrt{a^2b^2c^2 + 2ab} \\
&= abc + \sqrt{a^2b^2c^2 + 2ab} - abc \\
&= abc + \cfrac{1}{\frac{1}{\sqrt{a^2b^2c^2+2ab}-abc}} \\
&= abc + \cfrac{1}{\frac{\sqrt{a^2b^2c^2+2ab}+abc}{a^2b^2c^2+2ab-a^2b^2c^2}} \\
&= abc + \cfrac{1}{\frac{\sqrt{a^2b^2c^2+2ab}+2abc-abc}{2ab}} \\
&= abc + \cfrac{1}{\frac{2abc}{2ab} + \frac{\sqrt{a^2b^2c^2+2ab}-abc}{2ab}} \\
&= abc + \cfrac{1}{c + \cfrac{1}{\frac{2ab}{\sqrt{a^2b^2c^2+2ab}-abc}}} \\
&= abc + \cfrac{1}{c + \cfrac{1}{\frac{2ab(\sqrt{a^2b^2c^2+2ab}+abc)}{a^2b^2c^2+2ab-a^2b^2c^2}}} \\
&= abc + \cfrac{1}{c + \cfrac{1}{\sqrt{a^2b^2c^2+2ab}+abc}} \\
&= abc + \cfrac{1}{c + \cfrac{1}{2abc + \cfrac{1}{\frac{1}{\sqrt{a^2b^2c^2+2ab}+abc}}}} \\
&= abc + \cfrac{1}{c + \cfrac{1}{2abc + \cfrac{1}{c + \frac{2ab}{\sqrt{a^2b^2c^2+2ab}-abc}}}}
\end{aligned}$$

Therefore, $\sqrt{d} = [abc; \overline{c, 2abc}]$.

Example 2.1. *Let $d = a^2b^2c^2 + 2ab$, $\sqrt{d} = [abc; \overline{c, 2abc}]$ and $a = 2, b = 2$ and $c = 1$ then the equation becomes $x^2 - 24y^2 = 1$. The continued fraction* expansion of $\sqrt{24}$ is $[4; \overline{1, 8}]$.

Theorem 2.6. *Let $d = a^2b^2c^2 + ab$. Then $\sqrt{d} = [abc; \overline{2c, 2abc}]$.*

Proof Proof of this theorem same as the theorem 2.5.

Hence the continued fraction expansion of $\sqrt{d} = [abc; \overline{2c, 2abc}]$.

Example 2.2. *Let* $d = a^2b^2c^2 + ab$, $\sqrt{d} = [abc; \overline{2c, 2abc}]$ *and* $a = 3, b = 2$ *and* $c = 1$ *then the equation becomes* $x^2 - 42y^2 = 1$. *The continued fraction expansion of* $\sqrt{42}$ *is* $[6; \overline{2, 12}]$.

Corollary 2.1. *Let* $d = a^2b^2c^2 + 2ab$. *Then the fundamental solution to the equation* $x^2 - dy^2 = 1$ *is* $x_1 + y_1\sqrt{d} = abc^2 + 1 + c\sqrt{d}$ *and the equation* $x^2 - dy^2 = -1$ *has no integer solutions.*

Proof The continued fraction expansion of $\sqrt{d}$ is $[abc; \overline{c, 2abc}]$. Let $a_0 = abc, a_1 = c$ and $a_2 = 2abc$.

$$\frac{p_1}{q_1} = \frac{1 + a_0a_1}{a_1} = \frac{1 + abc^2}{c} \tag{1}$$

Therefore the fundamental solution of the equation $x^2 - dy^2 = 1$ is $x_1 + y_1\sqrt{d} = abc^2 + 1 + c\sqrt{d}$. The continued fraction expansion of $\sqrt{d}$ is even by Lemma 2.2 and the equation $x^2 - dy^2 = -1$ has no integer solution.

Example 2.3. *Let* $a = 3, b = 2$ *and* $c = 1$ *then* $d = a^2b^2c^2 + 2ab = 48$ *then the continued fraction of* $\sqrt{48}$ *is* $[6; \overline{1, 12}]$. *The fundamental solution of the equation* $x^2 - 48y^2 = 1$ *is* $x_1 + y_1\sqrt{d} = 7 + \sqrt{48}$. *The period length of* $\sqrt{48}$ *is always even. Therefore the equation* $x^2 - 48y^2 = -1$ *has no positive integer solution.*

Corollary 2.2. *Let* $d = a^2b^2c^2 + ab$. *Then the fundamental solution to the equation* $x^2 - dy^2 = 1$ *is* $x_1 + y_1\sqrt{d} = abc^2 + 1 + 2c\sqrt{d}$ *and the equation* $x^2 - dy^2 = -1$ *has no integer solutions.*

Example 2.4. *Let* $x^2 - dy^2 = 1$, *where* $d = a^2b^2c^2 + ab, a = 3, b = 2$ *and* $c = 1$ *then the equation becomes* $x^2 - 42y^2 = 1$. *The continued fraction expansion of* $\sqrt{42} = [6; \overline{2, 12}]$ *and the fundamental solution of* $x^2 - 42y^2 = 1$ *is* $x_1 + y_1\sqrt{d} = 7 + 2\sqrt{42}$.

3 Main Results

Theorem 3.1. *Let* $d = a^2b^2c^2 + 2ab$. *Then all positive integer solutions of the equation* $x^2 - dy^2 = 1$ *are given by,*

$$(x, y) = (\frac{V_n(2abc^2 + 2, -1)}{2}, cU_n(2abc^2 + 2, -1))$$

with $n \geq 1$.

Proof The fundamental solution of the equation $x^2 - dy^2 = 1$ is,

$$x_1 + y_1\sqrt{d} = abc^2 + 1 + c\sqrt{d}$$

. Let

$$\alpha = abc^2 + 1 + c\sqrt{d}, \beta = 2abc^2 + 1 - c\sqrt{d}$$

,

$$\alpha+\beta=2abc^2+2, \alpha-\beta=2c\sqrt{d}, \alpha\beta=1.$$

Therefore,

$$x_n+y_nd\sqrt{d}=(x_1+y_1\sqrt{d})^n, \quad x_n+y_nd\sqrt{d}=\alpha^n, \quad x_n-y_nd\sqrt{d}=\beta^n,$$

$$x_n=\frac{1}{2}(V_n(2abc^2+2,-1)), \quad y_n=cU_n(2abc^2+2,-1).$$

Therefore, all positive integer solutions of the equation $x^2-dy^2=1$ is,

$$(x,y)=(\frac{V_n(2abc^2+2,-1)}{2}, cU_n(2abc^2+2,-1))$$

with $n\geq 1$.

Example 3.1. *Let $x^2-dy^2=1$, where $d=a^2b^2c^2+2ab, a=3, b=2$ and $c=1$ then the equation becomes $x^2-46y^2=1$. Then the fundamental solution of the equation is*

$$x_1+y_1\sqrt{46}=5+\sqrt{46}$$

. Let

$$\alpha=7+\sqrt{46}, \beta=13-\sqrt{46}$$

,

$$\alpha+\beta=18, \alpha-\beta=-8+2\sqrt{46}, \alpha\beta=1$$

and

$$x_n+y_n\sqrt{46}=(x_1+y_1\sqrt{46})^n$$

then

$$(x_n,y_n)=(V_n(14,-1),V_n(14,-1))$$

.

Theorem 3.2. *Let $d\equiv 2(mod4)$ or $d\equiv 3(mod4)$. Then the equation $x^2-dy^2=-4$ has positive integer solution if and only if the equation $x^2-dy^2=-1$ has positive integer solutions.*

Theorem 3.3. Let $d\equiv 0(mod4)$.*If fundamental solution to the equation $x^2-(d/4)y^2=1$ is $x_1+y_1\sqrt{d/4}$, then the fundamental solution to the equation $x^2-dy^2=4$ is $(2x_1,y_1)$.*

Theorem 3.4. *Let $d\equiv 1(mod4)$ or $d\equiv 2(mod4)$ or $d\equiv 3(mod4)$. If fundamental solution to the equation $x^2-dy^2=1$ is $x_1+y_1\sqrt{d}$, then fundamental solution to the equation $x^2-dy^2=4$ is $(2x_1,2y_1)$.*

Theorem 3.5. *Let* $d = a^2b^2c^2 + 2ab$. *Then the fundamental solution of the equation*

$$x_1 + y_1\sqrt{d} = 2abc^2 + 2 + 2c\sqrt{d}.$$

Proof

(i) Assume that b is even, and if a is even, or if a is odd, then $d \equiv 0(mod4)$. Let $b = 2k$, for some $k \in Z$. Then

$$\frac{d}{4} = \frac{a^2 4k^2c^2 + 4ak}{4} = a^2k^2c^2 + ak$$

Then

$$\sqrt{a^2k^2c^2 + ak} = [akc; \overline{2c, 2akc}].$$

Therefore, the fundamental solution to the equation $x^2 - dy^2 = 4$ is

$$\frac{p_1}{q_1} = \frac{1 + 2akc^2}{2c}$$

,

$$x_1 + y_1\sqrt{d} = 2akc^2 + 1 + 2c\sqrt{d}$$

. Since $b = 2k$, $k = b/2$ then

$$x_1 + y_1\sqrt{d} = abc^2 + 1 + 2c\sqrt{d}$$

. By Theorem 3.3, $x^2 - (d/4)y^2 = 1$ is $x_1 + y_1\sqrt{d/4}$, then the solution of $x^2 - dy^2 = 4$ is $(2x_1, y_1)$. The fundamental solution of

$$x^2 - (a^2b^2c^2 + 2ab)y^2 = 4$$

is $2(abc^2 + 1) + 2c\sqrt{d}$.

(ii) Assume that b is odd, and if a is odd, and if c is odd (or)
If b is odd and if a is odd and if c is even (or)
If b is odd and if a is odd, then Theorem3.4, $d \equiv 1(mod4)$ or $d \equiv 2(mod4)$ or $d \equiv 3(mod4)$. If fundamental solution of $x^2 - dy^2 = 4$ is $x_1 + y_1\sqrt{d}$, then the fundamental solution of $x^2 - dy^2 = 4$ is $(2x_1, y_1)$. Therefore, the fundamental solution of $x^2 - dy^2 = 4$ is $(2(abc^2 + 1), 2c)$. Therefore,

$$x_1 + y_1\sqrt{d} = 2(abc^2 + 1) + 2c\sqrt{d}.$$

Example 3.2. *Let* $x^2 - dy^2 = 4$, *where* $d = a^2b^2c^2 + 2ab$, $a = 3$,$b = 2$ *and* $c = 1$ *then the equation becomes* $x^2 - 48y^2 = 4$ *then by theorem 3.3,* $x^2 - 12y^2 = 1$. *Then the fundamental solution is* $x^2 - 12y^2 = 1$ *is* $x_1 + y_1\sqrt{12} = 10 + 2\sqrt{12}$, *Therefore the fundamental solution of* $x^2 - 48y^2 = 4$ *is* $x_1 + y_1\sqrt{48} = 10 + 2\sqrt{48}$.

Theorem 3.6. *Let* $d = a^2b^2c^2 + 2ab$. *Then the equation* $x^2 - dy^2 = -4$ *has no positive integer solutions.*

Proof Assume that, a is odd, and if b is odd and c is odd, then $d \equiv 3(mod 4)$.
If a is odd and b is odd and c is even then $d \equiv 2(mod 4)$.
If a is odd and b is even and c is odd then $d \equiv 0(mod 4)$.
By Theorem 3.2, and Corollary 2.2, $x^2 - dy^2 = -4$ has no positive integer solutions.
Assume that a is even and $m^2 - dn^2 = -4$, for some positive integer m, n.
Then d is even and therefore m is even.
Let $a = 2k$ then,

$$m^2 - (4k^2b^2c^2 + 4kb)n^2 = -4$$

$$(m^2/4) - (k^2b^2c^2 + kb)n^2 = -1$$

. This is impossible. Therefore, $x^2 - dy^2 = -4$ has no positive integer solutions.

Example 3.3. *Let* $x^2 - dy^2 = -4$, *where* $d = a^2b^2c^2 + 2ab$, $a = 3$, $b = 2$ *and* $c = 1$ *then the equation becomes* $x^2 - 48y^2 = -4$ *has no positive integer solutions.*

Theorem 3.7. *Let* $d = a^2b^2c^2 + 2ab$. *Then all positive integer solutions of the equation* $x^2 - dy^2 = 1$ *are given by,*

$$(x, y) = (V_n(2abc^2 + 2, -1)/2, cU_n(2abc^2 + 2, -1)),$$

with $n \geq 1$.

Proof The fundamental solution of the equation $x^2 - dy^2 = 1$ is, $x_1 + y_1\sqrt{d}$=$abc^2 + 2 + 2c\sqrt{d}$. Let

$$\alpha = abc^2 + 2 + c\sqrt{d}, \beta = abc^2 + 2 - c\sqrt{d}$$

$$\alpha + \beta = 2abc^2 + 4, \alpha - \beta = 2c\sqrt{d}, \alpha\beta = 1$$

Therefore,

$$x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$$

$$x_n + y_n\sqrt{d} = \alpha^n, x_n - y_n\sqrt{d} = \beta^n\beta$$

$$x_n = \frac{1}{2}(V_n(2abc^2 + 2, -1)) and y_n = cU_n(2abc^2 + 2, -1)$$

. Therefore, all positive integer solutions of the equation $x^2 - dy^2 = 1$ is,

$$(x, y) = (V_n(2abc^2 + 2, -1)/2, cU_n(2abc^2 + 2, -1)),$$

with $n \geq 1$.

Corollary 3.1. *Let* $d=k^2+2k$, *then the continued fraction of* $\sqrt{k^2+2k}$ *is* $[k;\overline{1,2k}]$ *for* $k \geq 3$.

Corollary 3.2. *Let* $d=k^2+2k$. *Then all positive integer solutions of the equation* $x^2-dy^2=1$ *are given by,*

$$(x,y)=(V_n(2k+2,-1)/2, cU_n(2k+2,-1)),$$

with $n \geq 1$ *and the equation* $x^2-(k^2+2k)y^2=-1$ *has no positive integer solution.*

Corollary 3.3. *All positive integer solutions of the equation* $x^2-(k^2+2k)y^2=4$ *are given by,*

$$(x,y)=(V_n(k+1,-1), cU_n(k+1,-1)),$$

with $n \geq 1$ *and the equation* $x^2-(k^2+2k)y^2=-4$ *has no positive integer solution.*

4 Conclusion

In this paper, by using continued fraction expansion of $\sqrt{d}$, we find fundamental solution of the $x^2-dy^2=\pm1$, where a, b, and c are natural numbers and $d=a^2b^2c^2+2ab$. Moreover, we investigate Pell equations of the form $x^2-dy^2=\pm N$ when $N=\pm1,\pm4$ and we are looking for positive integer solutions in x and y. We get all positive integer solutions of the Pell equations $x^2-dy^2=N$ in terms of generalized Fibonacci and Lucas sequences when $N=\pm1,\pm4$ and $d=a^2b^2c^2+2ab$. Finally, all positive integer solutions of the equations $x^2-dy^2=\pm1$ and $x^2-dy^2=\pm4$ are given in terms of Fibonacci and Lucas sequences.

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