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LINEAR RECURSIVE RELATIONS FOR BERNOULLI NUMBERS AND APPLICATIONS

H. Belbachir, E. V. P. Spreafico* and M. Rachidi*

U S T H B, Faculté de Mathématiques,
RECITS Laboratory, BP 32,
El-Alia, 16111 Bab-Ezzouar, Algiers, ALGERIA

E-mail : hbelbachir@usthb.dz

*Instituto de Matemática - INMA, UFMS
Av. Costa e Silva/N, Cidade Universitária
Campo Grande, MS - BRASIL

E-mail : elen.spreafico@ufms.br, mustapha.rachidi@ufms.br

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Abstract: This study concerns a new approach of Bernoulli numbers B_n and Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$, using properties of some linear recursive relations of infinite order. Linear recursive relations for generating B_n and $B_n^{(k)}$ are established and some identities are provided. Moreover, linear, combinatorial and analytic approaching processes of B_n and $B_n^{(k)}$ are proposed. The closed connection with partial Bell polynomials is considered. Finally, applications to Genocchi numbers G_n, Euler numbers E_n and zeta function $\zeta(n)$ are discussed.

Keywords and Phrases: Linear recursive relations of infinite order, ∞-generalized Fibonacci sequences, Bernoulli numbers, Combinatorial formula, Approximation processes, Genocchi numbers, Euler numbers, zeta function, partial Bell polynomials.

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1. Introduction

Bernoulli numbers have been extensively studied in the literature, since they arise in many area of theoretical and applied mathematics, and also have wide-ranging of applications in applied sciences and engineering. These numbers continue to attract much attention, especially, several properties of recurrence relations for generating Bernoulli numbers are considered in many studies (see for example [4, 5, 6, 7, 8, 9, 10, 11, 12, 16, 17], and reference therein). In general, the Bernoulli numbers B_n and Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$ are commonly defined through their associated generating function, namely,

$$\frac{t}{e^t-1} = \sum_{n=0}^{+\infty} B_n \frac{t^n}{n!} \tag{1}$$

and

$$\left(\frac{t}{e^t-1}\right)^k = \sum_{n=0}^{+\infty} B_n^{(k)} \frac{t^n}{n!}. \tag{2}$$

For reason of convenience, in the sequel $B_n^{(k)}$ will be referred to as Bernoulli numbers of order $k \geq 2$. The generating functions defining some usual known sequences of numbers, as Fibonacci and Lucas numbers, are rational functions. For generalized Fibonacci sequences, the generating function is also a rational function (see [15]). Moreover, each kind of these sequences of numbers satisfies a specific linear recursive relation of finite order. Conversely, the generating function defined by a linear recursive relation of finite order is a rational function. Taking into account the preceding facts and that the generating functions (1)-(2) of B_n and $B_n^{(k)}$ are not rational functions, comes the following natural question: are there some kind of linear relations for generating recursively the Bernoulli numbers B_n and Bernoulli numbers $B_n^{(k)}$? Recall that, in the literature, there are several recursive relationships linked to Bernoulli numbers B_n and Bernoulli numbers $B_n^{((k)}$, but they are not linear with constant coefficients, (see, for example, [10, 12] and references therein). However, observing (1)-(2) the denominators in the quotient of generating functions are infinite series. Comparing with known sequences of numbers whose generating functions are rational, the recurrence relationship for Bernoulli numbers B_n and $B_n^{(k)}$, if they exist, must be of infinite order.

In this context, we are interested to establish linear recursive relations of Fibonacci type of infinite order, for generating the Bernoulli numbers B_n and $B_n^{(k)}$. This recursive process is based on the so called generalized Fibonacci sequences of infinite order. Such sequences have been introduced and developed in [3, 13]. These sequences are defined as follows. Let $\{a_i\}_{i\geq 0}$ and $\{\alpha_i\}_{i\geq 0}$ be two sequences of real or complex numbers. Suppose that, for every $N \in \mathbb{N}$, there exists $i > N$ such that $a_i \neq 0$. The associated sequence of infinite order or the so-called ∞-generalized Fibonacci sequence $\{w_n\}_{n\in\mathbb{Z}}$ is defined by,

$$\begin{aligned} w_n &= \alpha_{-n} \qquad \text{for} \qquad n \leq 0, \\ w_n &= \sum_{i=0}^{\infty} a_i w_{n-i-1} \qquad \text{for} \qquad n \geq 1. \end{aligned} \tag{3}$$

The sequences $\{a_i\}_{i\geq 0}$ and $\{\alpha_i\}_{i\geq 0}$ are called the *coefficients sequence* and *initial sequence,* respectively. The right side of Expression (3) represents a series, therefore, for the existence of w_n it is important to worry about the convergence of this series (for more details see [3, 13], and references therein). In this study, we are also concerned in the connection between sequences (3) and a specific family of sequences defined by linear recursive relation of finite order. In fact, it was shown in [3] that sequences (3) can be studied using properties of a family of linear difference equations of finite order $r \geq 2$ defined by,

$$u_{n+1} = b_0 u_n + \ldots + b_{r-1} u_{n-r+1}, \qquad \text{for} \quad n \geq r-1, \tag{4}$$

where $b_0, b_1, \ldots, b_{r-1}$ are the constant coefficients and $u_0 = \alpha_0, \ldots, u_{r-1} = \alpha_{r-1}$ are the initial conditions. For such sequences, known in the literature as *r-generalized Fibonacci sequences*, the analytic formula for u_n is given by

$$u_n = \sum_{i=1}^{l} \left(\sum_{j=0}^{s_i-1} \beta_{i,j} n^j \right) \lambda_i^n, \qquad \text{for every integer } n \geq 0, \tag{5}$$

where $\lambda_1, \ldots \lambda_l$ are the roots of the characteristic polynomial $Q(z) = z^r - b_0 z^{r-1} - \cdots - b_{r-1}$, of multiplicities $s_1, \ldots, s_l$, respectively. Generally, the scalars $\beta_{i,j}$ $(1 \leq i \leq l,\ 0 \leq j \leq s_i - 1)$ are obtained by solving the generalized Vandermonde system of equations $\sum_{i=1}^{l} \left(\sum_{j=0}^{s_i-1} \beta_{i,j} n^j \right) \lambda_i^n = \alpha_n$, $n = 0, 1, \ldots, r-1$ (see, for example, [2, 18]). In addition, we will resort to the combinatorial formula of the general term u_n given by,

$$u_n = \sum_{i=0}^{r-1} A_i \rho(n-i, r), \text{ for every } n \geq r, \tag{6}$$

where $A_i = a_{r-1} u_i + \ldots + a_i u_{r-1}$, for $0 \leq i \leq r-1$, and

$$\rho(n, r) = \sum_{k_0 + 2k_1 + \cdots + r k_{r-1} = n-r} \frac{(k_0 + \cdots + k_{r-1})!}{k_0! k_1! \ldots k_{r-1}!} a_0^{k_0} \ldots a_{r-1}^{k_{r-1}}, \tag{7}$$

with $\rho(r,r)=1$ and $\rho(n,r)=0$ for $0\leq n\leq r-1$ (see, for instance, [[3], references therein], [18]). Expressions of type (7) where considered by Philippou *et al.* (see, for example, [15] and references therein).

The purpose of this study is to establish that the Bernoulli numbers B_n and $B_n^{(k)}$ are generated by some linear recursive relations of order infinity of type (3). We show that this method represents a natural way for obtaining linear recursive relation generating B_n and $B_n^{(k)}$, which is an answer to our precedent question. Moreover, we establish some compact combinatorial formulas for B_n and $B_n^{(k)}$. On the other side, we develop an analytic approximation process of B_n and $B_n^{(k)}$, based on the approximation of the sequence (3) by a family of linear recursive sequences of finite order (4). In the same way, we improve a combinatorial approximation process of B_n and $B_n^{(k)}$. The main idea behind our study, is the closed relation between the exponential generating function (1) and the *characteristic functions of the sequence (3)*. Connections with the partial Bell polynomials and Stirling numbers of the second kind are discussed. Moreover, applications to Euler numbers, Genocchi numbers and zeta functions are also addressed.

The content of this paper is organized as follows. In Section 2 we recall some basic view of sequences (3) and results on the Bernoulli numbers B_n and $B_n^{(k)}$, using properties of sequences (3), are established. Section 3 is devoted to explore a connection between partial Bell Polynomials, sequences (3), Bernoulli numbers and Stirling numbers of second order. In Section 4 an approximation processes of the Bernoulli numbers are provided from the approximation of the sequence (3), by a specific family of linear recursive sequences of finite order. In section 5 related applications to Genocchi numbers, Euler numbers and zeta function are also considered. Finally, concluding remarks are given.

2. Bernoulli numbers and truncated sequences (3)

2.1. Truncated sequences (3). In this subsection we recall some basic elements of the linear recursive sequences (3), which are important in sequel of this study. Let $\{w_n\}_{n\in\mathbb{Z}}$ be a sequence (3) of coefficients and initial data $\{a_i\}_{i\geq 0}$ and $\{\alpha_i\}_{i\geq 0}$, respectively. As said before, the general term w_n $(n\geq 1)$ is a numerical series. Therefore, for the existence of w_n, when $n\geq 1$, a necessary and sufficient condition, labeled (C_∞), was formulated in [3, Proposition 2.1] as follows:

(C_∞): *The series* $\sum_{i=0}^{\infty} a_{i+n-1}\alpha_i$ *converges for all* $n\geq 1$.

When $\alpha_j=0$, for all $j\geq k+1$, the condition (C_∞) is verified and we have $w_{n+1}=\sum_{j=0}^{n+k} a_j w_{n-j}$, for all $n\geq 0$. In such case, the sequence $\{w_n\}_{n\in\mathbb{Z}}$ is called a *k-truncated sequence (3)*. Especially, when $\alpha_j=\delta_{j,k}$ for all $j\geq 0$, where $\delta_{j,k}$ is the Kronecker symbol, the associated sequence (3), denoted by $\{\omega_n^{(k)}\}_{n\in\mathbb{Z}}$, is called an *elementary k-truncated sequence (3)*. Let $\mathcal{F}_\infty = \mathcal{F}_\infty(\{a_i\}_{i\geq 0})$ be the vector space (over $\mathbb{C}$) of sequences (3) of (fixed) coefficients $\{a_i\}_{i\geq 0}$ such that their initial sequences $\{\alpha_i\}_{i\geq 0}$ satisfy the condition (C_∞). A straightforward long verification allows us to establish that the set $\mathcal{S} = \{\{\omega_n^{(k)}\}_{n\in\mathbb{Z}}; k \in \mathbb{N}\}$ of elementary k-truncated sequences (3), is a basis of the vector space $\mathcal{F}_\infty$. More precisely, for every $\{w_n\}_{n\in\mathbb{Z}} \in \mathcal{F}_\infty$, we have $w_n = \sum_{k=0}^{+\infty} \alpha_k \omega_n^{(k)}$ for all $n \in \mathbb{Z}$. Moreover, it was proved in [13] that every $\{w_n\}_{n\in\mathbb{Z}} \in \mathcal{F}_\infty$ takes the combinatorial form $w_n = \sum_{s=1}^{n} A_s \rho(n-s, 0)$ with $A_s = \sum_{m=0}^{+\infty} a_{s+m-1}\alpha_m$ and

$$\rho(n,0) = \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} a_0^{k_0}\dots a_{n-1}^{k_{n-1}}, \tag{8}$$

such that $\rho(0,0) = 1$ and $\rho(-k,0) = 0$ for every $k \geq 1$. Especially, we have the following useful property.

Proposition 2.1. *(see [13]) The combinatorial formula of $\{\omega_n^{(0)}\}_{n\in\mathbb{Z}}$ is*

$$\omega_n^{(0)} = \rho(n,0), \quad \textit{for every} \quad n \geq 1,$$

where $\rho(n,0)$ is defined as in (8).

Throughout the rest of this work the sequence $\{\omega_n^{(0)}\}_{n\in\mathbb{Z}}$ will play a central role. Therefore, for reasons of necessity of the clarity of the rest of this text, the sequence $\{\omega_n^{(0)}\}_{n\in\mathbb{Z}}$ is simply denoted by $\{v_n\}_{n\in\mathbb{Z}}$.

On the other hand, a direct computation shows that the generating function of $\{v_n\}_{n\in\mathbb{Z}}$ is,

$$f(t) = \sum_{n=0}^{\infty} v_n t^n = \frac{1}{Q(t)}, \tag{9}$$

where $Q(t) = 1 - \sum_{j=0}^{\infty} a_j t^{j+1}$ is the so-called *the characteristic function of the sequence (3)* (for more details see [3, 13]). Conversely, let $Q(t)$ be a complex function which is analytic in open disk $D(0;R)$ centered at 0, with $R > 0$. Suppose that Q takes the form $Q(t) = 1 - \sum_{j=0}^{\infty} a_j t^{j+1}$ in $D(0;R)$. Since $Q(0) = 1 \neq 0$ then $f(t) = \dfrac{1}{Q(t)}$ has a Taylor expansion in the $D(0;R)$ given by,

$$f(t) = \frac{1}{1-\sum_{j=0}^{\infty} a_j t^{j+1}} = \sum_{n=0}^{\infty} w_n t^n. \tag{10}$$

And the identity $Q(t)f(t) = 1$ implies that we have $w_{n+1} = \sum_{j=0}^{n} a_j w_{n-j}$, for all $n \geq 0$, where $w_0 = 1$ and $w_{-j} = 0$, for all $j \geq 1$. Hence, $\{w_n\}_{n\in\mathbb{Z}}$ is nothing else but the linear sequence of infinite order $\{v_n\}_{n\in\mathbb{Z}}$.

2.2. Bernoulli numbers and sequences (3)

Let $\{B_n\}_{n\geq 0}$ be the sequence of Bernoulli numbers defined by their associated exponential generating function (1). Observing that we can write $\frac{t}{e^t-1} = \dfrac{1}{Q(t)}$, where $Q(t) = 1 - \sum_{n=0}^{+\infty} a_n t^{n+1}$ with $a_n = -\dfrac{1}{(n+2)!}$. Comparing the right sides of (1) and (10) where $Q(t) = 1 - \sum_{n=0}^{+\infty} a_n t^{n+1}$, with $a_n = -\dfrac{1}{(n+2)!}$, we get the result.

Theorem 2.2. *The Bernoulli numbers are expressed in terms of a specific linear recursive sequence (3) as follows,*

$$B_n = n! \times v_n^{[1]}, \tag{11}$$

where $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ is the sequence (3), whose coefficients and initial values are $a_n = -\dfrac{1}{(n+2)!}$ $(n \geq 0)$, $v_0^{[1]} = 1$ and $v_{-j}^{[1]} = 0$ for all $j \geq 1$, respectively. In addition, the combinatorial formula of B_n is given by,

$$B_n = n! \sum_{k_0+2k_1+\cdots+nk_{n-1}=n} (-1)^{k_0+\cdots+k_{n-1}} \frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!} \prod_{j=0}^{n-1} \left[\frac{1}{(j+2)!}\right]^{k_j}. \tag{12}$$

Moreover, the sequence $\left\{\frac{B_n}{n!}\right\}_{n\geq 0}$ satisfies the following linear recursive relation,

$$\frac{B_{n+1}}{(n+1)!} = a_0 \frac{B_n}{n!} + a_1 \frac{B_{n-1}}{(n-1)!} + \cdots + a_n \frac{B_0}{0!}. \tag{13}$$

Proof. Indeed, Expression (11) is obtained from (1) and (9) by identification of the terms of the two series. In addition, using Proposition 2.1 and Expression (8), we obtain the combinatorial formula (12) of the Bernoulli numbers. Second, Expression (11) shows that $v_n = \dfrac{B_n}{n!}$, and by substitution in (3) we derive the linear recursive formula (13). □

In fact, Expression (13) shows that the sequence $\left\{\frac{B_n}{n!}\right\}_{n\geq 0}$ satisfies the recursive relation of type (3). This assertion represents a linear recursive process for generating the Bernoulli numbers. Since $a_n = -\dfrac{1}{(n+2)!}$, Expression (13) takes the following form,

$$\frac{B_{n+1}}{(n+1)!} = -\sum_{k=0}^{n} \frac{B_{n-k}}{(k+2)!(n-k)!} = -\sum_{k=0}^{n} \frac{B_k}{k!(n-k+2)!}. \tag{14}$$

Moreover, the recursive formulas (13)-(14) allows us to recover the following classical expression,

Corollary 2.3. *The Bernoulli numbers B_n satisfy the identity*

$$\sum_{k=0}^{n+1} \frac{(n+2)!}{(n-k+2)!} \times \frac{B_k}{k!} = \sum_{k=0}^{n+1} \binom{n+2}{k} B_k = 0. \tag{15}$$

Proof. Expression (13) is equivalent to $\dfrac{B_{n+1}}{(n+1)!} + \displaystyle\sum_{k=0}^{n} \frac{B_k}{k!} \times \frac{1}{(n-k+2)!} = 0$, and multiplying both sides by $(n+2)!$ we get the requested identity, namely, formula (15). $\square$

Furthermore, since the function $f(t) = \frac{t}{e^t-1}$ satisfies $f(-t) = t + f(t)$, we show that the equality $f(-t) = \sum_{n=0}^{\infty}(-1)^n v_n t^n = t + \sum_{n=0}^{\infty} v_n t^n$ implies that $v_{2n+1} = 0$, for every $n \geq 1$. Consequently, Expression (11) attests that $B_{2n+1} = 0$, for every $n \geq 1$. Since all the odd Bernoulli numbers vanish except $B_1 = -\frac{1}{2}$ and $B_0 = 1$, Expression (14) gives,

B_n	Value	Decimal Value (5 digits)
B_0	1	1.0000
B_1	$\frac{-1}{2}$	−0.50000
B_2	$\frac{1}{6}$	0.16667
B_4	$\frac{-1}{30}$	−0.03333
B_6	$\frac{1}{42}$	0.02381
B_8	$\frac{-1}{30}$	−0.03333
B_{10}	$\frac{6}{66}$	0.07576
B_{12}	$\frac{-691}{2730}$	−0.25311
B_{14}	$\frac{7}{6}$	1.16667
B_{16}	$\frac{-3617}{510}$	−7.09216
B_{18}	$\frac{43867}{798}$	54.97118
B_{20}	$\frac{-174611}{330}$	−529.12424
B_{22}	$\frac{854513}{138}$	6192.12319

Table 1: Terms of Bernoulli numbers sequence

$$B_{2n+2} = -\frac{1}{2}\left[\frac{2n+1}{(2n+3)(2n+2)} + \sum_{s=0}^{n-1} \frac{1}{(s+1)} \binom{2n+2}{2s+2} B_{2n-2s}\right], \text{ for every } n \geq 0.$$

Let the first terms of the sequence in Table 1.

2.3. Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$ and sequences (3)

Recall that the Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$ are defined by the generating function (2). By considering (10) with $Q(t) = 1 + \sum_{j=0}^{+\infty} \frac{1}{(j+2)!} t^{j+1}$ we get,

$$\left(\frac{t}{e^t - 1}\right)^k = \left(\frac{1}{Q(t)}\right)^k = \left(\sum_{n=0}^{+\infty} v_n^{[1]} t^n\right)^k = \sum_{n=0}^{+\infty} \left[\sum_{p_1+\cdots+p_k=n} v_{p_1}^{[1]} \cdots v_{p_k}^{[1]}\right] t^n,$$

where $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ is the sequence (3), whose coefficients and initial values are $a_n = -\frac{1}{(n+2)!}$ $(n \geq 0)$, $v_0^{[1]} = 1$ and $v_{-j}^{[1]} = 0$, for all $j \geq 1$. Therefore, by applying (11)-(12), we succeed the result.

Theorem 2.4. *Bernoulli numbers $B_n^{(k)}$ $(n \geq 0)$ of order $k \geq 2$, satisfy the identity,*

$$B_n^{(k)} = n! \sum_{p_1+\cdots+p_k=n} v_{p_1}^{[1]} \cdots v_{p_k}^{[1]} = \sum_{p_1+\cdots+p_k=n} \binom{n}{p_1,\, \ldots,\, p_k} B_{p_1} \cdots B_{p_k}, \tag{16}$$

where the $B_{p_1}, \cdots, B_{p_k}$ are the Bernoulli numbers. Moreover, the combinatorial formula of the $B_n^{(k)}$ $(n \geq 1)$ is

$$B_n^{(k)} = n! \sum_{p_1+\cdots+p_k=n} \rho_v(p_1, 0) \cdots \rho_v(p_k, 0),$$

where the $\rho_v(n, 0)$ are as in (8) with $a_j = -\frac{1}{(j+2)!}$.

The simple decomposition $\left(\frac{t}{e^t - 1}\right)^{k+1} = \left(\frac{t}{e^t - 1}\right) . \left(\frac{t}{e^t - 1}\right)^k$ allows us to achieve a recursive process to construct the $B_n^{(k+1)}$ with the aid of $B_s^{(k)}$. Indeed, a direct computation, using (2) and (10), shows that $\frac{B_n^{(k+1)}}{n!} = \sum_{s=0}^{n} v_{n-s}^{[1]} \frac{B_s^{(k)}}{s!}$. Hence, we can state the following corollary.

Corollary 2.5. *For every $n \geq 1$, we have*

$$B_n^{(k+1)} = \sum_{s=0}^{n} \binom{n}{s} B_{n-s} B_s^{(k)} \quad \textit{and} \quad B_{n+1}^{(k+1)} = B_{n+1}^{(k)} - \sum_{s=0}^{n} \frac{1}{j+2} \binom{n+1}{s+1} B_{n-j}^{(k)}.$$

More generally, the forgoing process linking B_n and sequences (3), can also be considered for Bernoulli numbers $B_n^{(k)}$. Indeed, the $B_n^{(k)}$ can be expressed in terms of a specific sequence (3). More precisely, if we set $H(t) = 1 + \sum_{j=0}^{+\infty} \beta_j t^j$, where $\beta_j = \dfrac{1}{(j+1)!}$, a straightforward computation shows that $H(t)^k = 1 - \sum_{m=0}^{+\infty} b_{m,k} t^{m+1}$, where

$$b_{m,k} = - \sum_{j_1+\cdots+j_k=m+1} \prod_{s=1}^{k} \frac{1}{(j_s+1)!}. \tag{17}$$

Therefore, using (10), we obtain $\left(\dfrac{t}{e^t-1}\right)^k = \dfrac{1}{H(t)^k} = \sum_{n=0}^{+\infty} W_{n,k} t^n$, where $\{W_{n,k}\}_{n\in\mathbb{Z}}$ is a sequence (3) of initial data $W_{0,k} = 1$, $W_{-j,k} = 0$, for $j \geq 1$, and coefficients $b_{n,k}$ ($n \geq 0$) are as in (17). Thus, following (2) we derive the property.

Theorem 2.6. *Under the preceding data, the Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$ take the following form,*

$$B_n^{(k)} = n! W_{n,k}, \text{ for every } n \geq 0, \tag{18}$$

where $\{W_{n,k}\}_{n\in\mathbb{Z}}$ is a sequence of type (3), of initial data $W_{0,k} = 1$, $W_{-j,k} = 0$ for $j \geq 1$ and coefficients $b_{n,k}$ are as in (17). Moreover, the combinatorial formula of the $B_n^{(k)}$ in terms of the coefficients $b_{n,k}$ ($n \geq 0$) is given by,

$$B_n^{(k)} = n! \rho_k(n,0) = n! \sum_{s_0+2s_1+\cdots+ns_{n-1}=n} \frac{(s_0+s_1+\cdots+s_{n-1})!}{s_0! s_1! \ldots s_{n-1}!} b_{0,k}^{s_0} b_{1,k}^{s_1} \cdots b_{n-1,k}^{s_{n-1}}, \tag{19}$$

where the $b_{n,k}$ are as in (17).

Results of Theorem 2.4, namely, expression (11), implies that $\dfrac{B_n^{(k)}}{n!} = W_{n,k}$. Hence, it ensue that the sequence $\left\{\dfrac{B_n^{(k)}}{n!}\right\}_{n\geq 0}$ satisfies the recursive relation of type (3), whose coefficients $b_{n,k}$ are given by (17). Therefore, we have a linear recursive process for generating the Bernoulli numbers $B_n^{(k)}$. As a consequence of Theorems 2.4 and 2.6, we have the corollary.

Corollary 2.7. *Under the preceding data we have the following identity,*

$$\sum_{p_1+\cdots+p_k=n} \binom{n}{p_1, \ldots, p_k} B_{p_1} \cdots B_{p_k} =$$

$$n! \sum_{s_0+2s_1+\cdots+ns_{n-1}=n} \frac{(s_0+\cdots+s_{n-1})!}{s_0!\cdots s_{n-1}!} b_{0,k}^{s_0}\cdots b_{n-1,k}^{s_{n-1}},$$

where B_n are the Bernoulli numbers and the $b_{n,k}$ are as in (17).

It seems for us that the identity of the Corollary 2.7 is not current in the literature.

3. Bernoulli numbers, sequences (3) and partial Bell polynomials

It seems for us that the combinatorial formula (12) for the Bernoulli numbers is not known in the literature under this form. Nonetheless, this formula allows us to connect the Bernoulli numbers with some special case of the alternate sum of partial Bell polynomials (see [5]), defined by their generating function as follows,

$$\sum_{n=k}^{\infty} B_{n,k}\left(x_1, x_2, \ldots\right) \frac{t^n}{n!} = \frac{1}{k!}\left(\sum_{m=1}^{\infty} x_m \frac{t^m}{m!}\right)^k. \tag{20}$$

The explicit formula of the $B_{n,k}\left(x_1, x_2, \ldots\right)$ is

$$B_{n,k}\left(x_1, x_2, \ldots\right) = \sum_{\substack{k_0+2k_1+\cdots=n \\ k_0+k_1+\cdots=k}} \frac{n!}{k_0!k_1!\cdots(1!)^{k_0}(2!)^{k_1}\cdots} x_1^{k_0} x_2^{k_1}\cdots. \tag{21}$$

Note that the Formula (21) admits a finite number of terms according to $k_0+2k_1+\cdots=n$, and at most the last term of the product $x_1^{k_0}x_2^{k_1}\cdots$ is x_n. In the sequel we can use one of the two notations $B_{n,k}\left(x_1, x_2, \ldots\right)$ or $B_{n,k}\left(x_1, x_2, \ldots, x_n\right)$, for more details we can refer to [5], where Comtet gave an important impulsion for the development of Bell polynomials. The (exponential) partial Bell polynomials $B_{n,k}\left(x_1, x_2, \ldots\right)$ make it possible to generate several family of Stirling numbers of the second kind $S(n,k)$ and the m-associate Stirling numbers of the second kind $S_m(n,k)$, where $m \geq 1$, namely,

$$S(n,k) = B_{n,k}\left(1,1,1,\ldots\right) \text{ and } S_m(n,k) = B_{n,k}\left(0,\ldots,0,1,1,\ldots\right). \tag{22}$$

where there is m consecutive zeros in the expression of $B_{n,k}\left(0,\ldots,0,1,1,\ldots\right)$.

Lemma 3.1. *(see [5]) For every $k, m \geq 1$ and $n \geq k$, we have,*

$$B_{n,k}\left(\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right) = \frac{n!}{(n+k)!} B_{n+k,k}\left(0,1,1,\ldots\right) \tag{23}$$

$$B_{n,k}\left(\binom{m+1}{1}^{-1},\binom{m+2}{2}^{-1},\ldots\right)=\frac{n!}{(n+mk)!}B_{n+mk,k}\left(0,\ldots,0,1,1,\ldots\right) \quad (24)$$

The following theorem shows that the partial Bell polynomials are related to sequences (3).

Theorem 3.2. *The linear recursive sequence of infinite order* $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ *satisfies the following identity,*

$$v_n^{[1]} = \sum_{k\geq 0}(-1)^k B_{n,k}\left(\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n+1}\right).$$

Proof. We develop the summation formula according to the expression of partial Bell polynomials,

$$\begin{aligned}
v_n^{[1]} &= \sum_{k_0+2k_1+\cdots+nk_{n-1}=n}(-1)^{k_0+\cdots+k_{n-1}}\frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!}\prod_{j=0}^{n-1}\left[\frac{1}{((j+2)!)^{k_j}}\right]\\
&= \sum_{k\geq 0}(-1)^k\sum_{\substack{k_0+2k_1+\cdots+nk_{n-1}=n\\ k_0+k_1+\cdots+k_{n-1}=k}}\frac{k!}{k_0!k_1!\cdots k_{n-1}!}\left[\frac{1}{2!}\right]^{k_0}\left[\frac{1}{3!}\right]^{k_1}\cdots\left[\frac{1}{(n+1)!}\right]^{k_{n-1}}\\
&= \sum_{k\geq 0}(-1)^k\sum_{\substack{k_0+2k_1+\cdots+nk_{n-1}=n\\ k_0+k_1+\cdots+k_{n-1}=k}}\frac{k!}{k_0!\cdots k_{n-1}!\,(1!)^{k_0}\cdots(n!)^{k_{n-1}}}\times\\
&\times\left(\frac{1}{2}\right)^{k_0}\cdots\left(\frac{1}{n+1}\right)^{k_{n-1}}\\
&= \sum_{k\geq 0}(-1)^k B_{n,k}\left(\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n+1}\right). \qquad \square
\end{aligned}$$

As a consequence of the proof of Theorem 3.2 and Lemma 3.1, it ensues that the Bernoulli numbers B_n are linked to the partial Bell polynomials as follows.

Corollary 3.3. *Bernoulli numbers* B_n *satisfy the property,*

$$B_n = n!\sum_{k\geq 0}(-1)^k B_{n,k}\left(\frac{1}{2},\frac{1}{3},\ldots,\frac{1}{n+1}\right),\ \textit{for every}\quad n\geq 0.$$

Furthermore, Expressions (22) and (23) imply that the following identity holds.

Corollary 3.4. *The linear recursive sequence of infinite order $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ satisfy the identity,*

$$v_n^{[1]} = n!\sum_{k\geq 0}\frac{(-1)^k}{(n+k)!}S(n+k,k).$$

Involving (22)-(23) and Corollary 3.4 we get the following corollary, which allows us to formulate the Bernoulli numbers in terms 1-associate Stirling numbers of the second kind.

Corollary 3.5. *Bernoulli numbers B_n satisfy the equality,*

$$B_n = (n!)^2\sum_{k\geq 0}\frac{(-1)^k}{(n+k)!}S(n+k,k) \text{ for every } \quad n\geq 0.$$

Moreover, in a similar way to that of the association between Bernoulli numbers B_n and sequences (3), we can also study the closed relation between the sequence $\{W_{n,k}\}_{n\in\mathbb{Z}}$ and partial Bell polynomials. Indeed, we have the following results.

Theorem 3.6. *The sequence $\{W_{n,k}\}_{n\in\mathbb{Z}}$ satisfy the following property*

$$W_{n,k} = \sum_{m\geq 0}\left(\frac{-1}{k!}\right)^m B_{n,m}\left(\binom{k+1}{1}^{-1},\binom{k+2}{2}^{-1},\ldots,\binom{k+n}{n}^{-1}\right).$$

Proof. We develop the summation formula to get the expression of partial Bell polynomials. A direct computation implies that the $b_{m,k}$ given by (17) can be written under the form,

$$b_{m,k} = -\sum_{j_1+\cdots+j_k=m+1}\prod_{s=1}^{k}\frac{1}{(j_s+1)!} = -\frac{1}{(m+k+1)!}k^{m+k+1}. \tag{25}$$

Therefore, according to (17), (25) and (19), we have

$$\begin{aligned}W_{n,k} &= \sum_{k_0+2k_1+\cdots+nk_{n-1}=n}(-1)^{k_0+\cdots+k_{n-1}}\frac{(k_0+\cdots+k_{n-1})!}{k_0!\cdots k_{n-1}!}\prod_{j=0}^{n-1}\left[\frac{1}{((j+k+1)!)^{k_j}}\right]\\ &= \sum_{m\geq 0}(-1)^m\sum_{\substack{k_0+2k_1+\cdots+nk_{n-1}=n\\ k_0+k_1+\cdots+k_{n-1}=m}}\frac{m!}{k_0!\cdots k_{n-1}!}\left[\frac{1}{(k+1)!}\right]^{k_0}\cdots\left[\frac{1}{(k+n)!}\right]^{k_{n-1}}\end{aligned}$$

$$= \sum_{m\geq 0}(-1)^m \sum_{\substack{k_0+2k_1+\cdots+nk_{n-1}=n\\k_0+k_1+\cdots+k_{n-1}=m}} \frac{m!}{k_0!\cdots k_{n-1}!}\left[\frac{1!}{(k+1)!}\right]^{k_0}\cdots\left[\frac{n!}{(k+n)!}\right]^{k_{n-1}}$$

$$= \sum_{m\geq 0}\left(\frac{-1}{k!}\right)^m \sum_{\substack{k_0+2k_1+\cdots+nk_{n-1}=n\\k_0+k_1+\cdots+k_{n-1}=m}} \frac{m!}{k_0!\cdots k_{n-1}!}\left[\frac{1!k!}{(k+1)!}\right]^{k_0}\cdots\left[\frac{n!k!}{(k+n)!}\right]^{k_{n-1}}$$

$$= \sum_{m\geq 0}\left(\frac{-1}{k!}\right)^m B_{n,m}\left(\binom{k+1}{1}^{-1},\binom{k+2}{2}^{-1},\ldots,\binom{k+n}{n}^{-1}\right). \qquad \square$$

As a consequence of Theorem 3.6, the Bernoulli numbers $B_n^{(k)}$ are related to the partial Bell polynomials as follows.

Corollary 3.7. *The Bernoulli numbers $B_n^{(k)}$ satisfy the following property,*

$$B_n^{(k)} = n!\sum_{m\geq 0}\left(\frac{-1}{k!}\right)^m B_{n,m}\left(\binom{k+1}{1}^{-1},\binom{k+2}{2}^{-1},\ldots,\binom{k+n}{n}^{-1}\right),$$

for every $n \geq 0$.

In addition, Expressions (22) and (24) imply that the following identity holds.

Corollary 3.8. *The sequence $\{W_{n,k}\}_{n\in\mathbb{Z}}$ satisfies the following identity*

$$W_{n,k} = n!\sum_{m\geq 0}\left(\frac{-1}{k!}\right)^m \frac{1}{(n+mk)!}S_k(n+mk,m),$$

for every $n \geq 0$ and $k \geq 1$.

Here also, the Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$, can be expressed in terms of the m-associate Stirling numbers of the second kind. Using relations (22) and (24), we derive the following corollary.

Corollary 3.9. *The Bernoulli numbers $B_n^{(k)}$ satisfy the following identity,*

$$B_n^{(k)} = (n!)^2\sum_{m\geq 0}\left(\frac{-1}{k!}\right)^m \frac{1}{(n+mk)!}S_k(n+mk,m),$$

for every $n \geq 0$.

4. Approximation of Bernoulli numbers and sequences of type (3)

Many results and algorithms have been provided in the literature for approximating the Bernoulli numbers (see for example [1] and references therein). In this section, we are concerned in a new type of approximation of Bernoulli numbers B_n and $B_n^{(k)}$, founded on the approximation of sequences (3) by a specific family of generalized Fibonacci sequences of finite order.

4.1. Linear and combinatorial approximation of Bernoulli numbers

Let $\{w_n\}_{n\in\mathbb{Z}}$ be a sequence (3). It was established in [3] that $\{w_n\}_{n\in\mathbb{Z}}$ can be approximated by a family of generalized Fibonacci sequence of order $r \geq 2$. That is, let $\{w_n^{(r)}\}_{n\geq -r+1}$, where $r \geq 2$, be the sequence defined by $w_j^{(r)} = \alpha_j$ for $-r+1 \leq j \leq 0$ and

$$w_{n+1}^{(r)} = \sum_{i=0}^{r-1} a_{i,r} w_{n-i+1}^{(r)} \qquad \text{for} \qquad n \geq 0, \tag{26}$$

where $a_{i,r} = a_i$ $(0 \leq i \leq r-1)$. The approximation [3, Theorem 3.1] asserts that the general term w_n given by (3) exists if and only if the sequence $\{w_n^{(r)}\}_{r\geq 1}$ converges, for every fixed $n \geq 1$. Furthermore, in this case we have $\lim_{r\to+\infty} w_n^{(r)} = w_n$, for all $n \geq 1$.

Now, we apply the preceding linear approximation of sequences (3) to the sequence $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ defining the Bernoulli numbers (1), where the coefficients are $a_j = -\dfrac{1}{(j+2)!}$ and initial data $v_0^{[1]} = 1$ and $v_{-j}^{[1]} = 0$, for $j \geq 1$. Since $v_{n+1}^{[1]} = \sum_{j=0}^{n} a_j v_{n-j}^{[1]}$, for all $n \geq 0$, it is clear that $v_n^{[1]}$ exists, for every $n \geq 1$. Therefore, for each fixed n, the general term $v_n^{[1]}$ is the limit of the sequence $\{v_n^{(r)}\}_{r\geq 1}$ (see [3, Theorem 3.1]. On the other hand, it is derived from (6)- (7) that the combinatorial formula of the linear recursive sequence $\{v_n^{(r)}\}_{n\geq -r+1}$ is given by,

$$v_n^{(r)} = \sum_{s_0+2s_1+\cdots+rs_{r-1}=n} \frac{(s_0+s_1+\cdots+s_{r-1})!}{s_0!s_1!\cdots s_{r-1}!} a_0^{s_0} a_1^{s_1} \cdots a_{r-1}^{s_{r-1}}, \text{ for every } n \geq 0,$$

where $a_j = -\dfrac{1}{(j+2)!}$ (see [3] and references therein). Combining this discussion with Theorem 2.2 we show that the following linear and combinatorial approximations properties of the Bernoulli numbers, with the aid of a specific family of linear recursive sequences of finite order (26), are given as follows.

Proposition 4.1. *Under the preceding data, the Bernoulli numbers B_n are approximated in terms of sequences (26) under the form,*

$$B_n = n! \times \lim_{r\to+\infty} v_n^{(r)},$$

where $\{v_n^{(r)}\}_{n\geq -r+1}$ *is of type (26), with coefficients* $a_j = -\dfrac{1}{(j+2)!}$ *for* $j = 0,\ 1,$ $\dots,\ r-1$, *and initial data* $v_0^{(r)} = 1$, $v_{-j}^{(r)} = 0$ *for* $j = 1, \dots, r-1$. *Moreover, a combinatorial approximation of the* B_n *is given by,*

$$B_n = n! \lim_{r\to+\infty}\left[\sum_{s_0+2s_1+\cdots+rs_{r-1}=n} (-1)^{\sum_{j=0}^{r-1} s_j} \binom{\sum_{j=0}^{r-1} s_j}{s_0, \dots, s_{r-1}} \left[\prod_{j=0}^{r-1} \frac{1}{((j+2)!)}\right]^{s_j}\right]. \tag{27}$$

Formula (27) shows that for a large r we can write,

$$B_n \approx n! v_n^{(r)}$$

and

$$B_n \approx n! \sum_{s_0+2s_1+\cdots+rs_{r-1}=n} (-1)^{\sum_{j=0}^{r-1} s_j} \binom{\sum_{j=0}^{r-1} s_j}{s_0, \dots, s_{r-1}} \left[\prod_{j=0}^{r-1} \frac{1}{((j+2)!)}\right]^{s_j}.$$

Similarly, Theorem 2.6, namely identities (18)- (19), shows that the approximation of the $B_n^{(k)}$ in terms of family of linear recursive sequences (26), can also be provided as follows.

Proposition 4.2. *The approximations of the* $B_n^{(k)}$ *in terms of sequences (26) and their combinatorial form are given by,*

$$B_n^{(k)} = n! \lim_{r\to+\infty} W_{n,k}^{(r)} =$$

$$= n! \lim_{r\to+\infty}\left[\sum_{s_0+2s_1+\cdots+rs_{r-1}=n} (-1)^{\sum_{j=0}^{r-1} s_j} \binom{\sum_{j=0}^{r-1} s_j}{s_0, \dots, s_{r-1}} \prod_{j=0}^{r-1} (b_{j,k}^{(r)})^{s_j}\right],$$

where $\{W_{n,k}^{(r)}\}_{n\geq -r+1}$ *is of type (26), the coefficients* $b_{n,k}^{(r)}$ *are as in (17) and initial data are* $W_{0,k}^{(r)} = 1$, $W_{-j,k}^{(r)} = 0$ *for* $j = 1, \dots, r-1$.

Similarly, formula of Proposition 4.2 shows that, for a large r, we can write,

$$B_n^{(k)} \approx n! W_n^{(r)}$$

and

$$B_n^{(k)} \approx n! \sum_{s_0+2s_1+\cdots+rs_{r-1}=n} (-1)^{\sum_{j=0}^{r-1} s_j} \binom{\sum_{j=0}^{r-1} s_j}{s_0, \dots, s_{r-1}} \prod_{j=0}^{r-1} (b_{j,k}^{(r)})^{s_j}.$$

4.2. Analytic approximation of Bernoulli numbers

Let $\{w_n\}_{n\in\mathbb{Z}}$ be a sequence (3). For every fixed $r \geq 2$, consider the polynomial $Q_r(z) = 1 - \sum_{j=0}^{r-1} a_j z^{j+1}$. It is clear that the characteristic polynomial of sequences (26) is $P_r(z) = z^r Q_r(\frac{1}{z})$. Let $\lambda_{1,r}, \ldots, \lambda_{s_r,r}$ be the roots of $P_r(z)$, of multiplicities $d_{1,r}, \ldots, d_{s_r,1}$, respectively. Formula (5) implies that the analytic formula of the sequence $\{w_n^{(r)}\}_{n\geq -r+1}$ is $w_n^{(r)} = \sum_{i=1}^{s_r}\left[\sum_{j=0}^{d_{i,r}-1} \beta_{i,j}^{(r)} n^j\right]\lambda_{i,r}^n$ for $n \geq -r+1$ (see for example [2, 18]).

Lemma 4.3. *Let $P_r(z) = z^r + \sum_{j=0}^{r-1}\frac{1}{(j+2)!}z^{r-j+1}$ be the characteristic polynomial of the linear recursive sequence (26). Then, every (characteristic) root λ of $P_r(z)$ is simple.*

Proof. Let $S_{r+1}(z) = \sum_{j=0}^{r}\frac{1}{j!}z^j = e^z - \sum_{j=r+2}^{+\infty}\frac{1}{j!}z^j$. We show easily that $\frac{dS_{r+1}}{dz}(z) = S_r(z)$ and $S_{r+1}(z) = 1 + zH_r(z)$, where $H_r(z) = 1 + \sum_{j=0}^{r-1}\frac{1}{(j+2)!}z^{j+1}$ and we verify that $P_r(z) = z^r H_r(\frac{1}{z})$. If λ is a root of $P_r(z)$, then $\lambda \neq 0$ and $H_r(\frac{1}{\lambda}) = 0$, $S_{r+1}(\frac{1}{\lambda}) = 1$. Suppose now that λ is of order ≥ 2, then $\frac{dP_r}{dz}(\lambda) = 0$. Since $\frac{dP_r}{dz}(z) = rz^{r-1}H_r(\frac{1}{z}) - z^{r-2}\frac{dH_r}{dz}(\frac{1}{z})$, we derive that $\frac{dH_r}{dz}(\frac{1}{\lambda}) = 0$. Moreover, the relation $\frac{dS_{r+1}}{dz}(z) = H_r(z) + z\frac{dH_r}{dz}(z)$ implies that $\frac{dS_{r+1}}{dz}(\frac{1}{\lambda}) = 0$ and thus $S_r(\frac{1}{\lambda}) = 0$. On the other hand, $S_{r+1}(z) = S_r(z) + \frac{1}{(r+1)!}z^{r+1}$ therefore

$$S_{r+1}(\frac{1}{\lambda}) = S_r(\frac{1}{\lambda}) + \frac{1}{(r+1)!}\frac{1}{\lambda^{r+1}} = \frac{1}{(r+1)!}\frac{1}{\lambda^{r+1}} = 1.$$

Hence, we have $\lambda^{r+1} = \frac{1}{(r+1)!}$ and then $\lambda = \frac{1}{\sqrt[n+1]{(r+1)!}} > 0$ is a root of $P_r(z)$, which is impossible since $P_r(\frac{1}{\sqrt[n+1]{(r+1)!}}) > 0$. Therefore, every root λ of $P_r(z)$ is simple. □

For reason of clarity, let recall the following result of [2].

Lemma 4.4. *(see [2, Theorem 2.2]) Suppose that the roots $\lambda_1, \ldots, \lambda_r$ of the characteristic polynomial $Q(z) = z^r - b_0 z_{r-1} - \cdots - b_{r-2}z - b_{r-1}$ of the sequence (26), are simple. Then, the analytic formula of the sequence $\{u_n\}_{n\geq 0}$ is given by*

$$u_n = \sum_{i=1}^{r}\frac{1}{Q'(\lambda_i)}\left(\sum_{p=1}^{r}\frac{A_p}{{\lambda_i}^{p+1}}\right){\lambda_i}^n,$$

for $n \geq r$, where $A_m = b_{r-1}u_m + \cdots + b_m u_{r-1}$.

Let $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ be the sequence of type (3) defining the Bernoulli numbers B_n. Since $v_{-j}^{[1]} = 0$, for $j \geq 1$, the conditions (C_∞) is satisfied. The approximation process of [3, Theorem 3.1] shows that, for each fixed n, the general term $v_n^{[1]}$ can be expressed in terms of sequences of type (4) as in (26), namely, $v_n^{[1]} = \lim_{r\to+\infty} v_n^{(r)}$. Therefore, since Lemma 4.3 proclaim that the characteristic roots $\lambda_{1,r}, \ldots, \lambda_{r,r}$ of $P_r(z) = z^r + \frac{1}{2}z^{r-1} + \cdots + \frac{1}{r!}z + \frac{1}{(r+1)!}$ are simple, Lemma 4.4 implies that the analytic expression (5) of the sequence $\{v_n^{(r)}\}_{n\geq -r+1}$ takes the form,

$$v_n^{(r)} = \sum_{i=1}^{r} \frac{1}{P_r'(\lambda_{i,r})} \left(\sum_{p=1}^{r} \frac{A_{p,r}}{{\lambda_{i,r}}^{p+1}} \right) \lambda_{i,r}^n,$$

where $A_{m,r} = a_{r-1}v_m^{(r)} + \cdots + a_m v_{r-1}^{(r)}$. In summary, combining the approximation process of sequences (3) by linear recursive sequences of finite order of [3, Theorem 3.1], with Lemma 4.3, we derive the following result on the approximation of Bernoulli numbers.

Theorem 4.5. *Under the preceding data and notations of Propositions 4.1 and 4.2, the analytic approximation of Bernoulli numbers as follows,*

$$B_n = n! \times \lim_{r\to+\infty} \left[\sum_{i=1}^{r} \frac{1}{P_r'(\lambda_{i,r})} \left(\sum_{p=1}^{r} \frac{A_{p,r}}{{\lambda_{i,r}}^{p+1}} \right) \lambda_{i,r}^n \right],$$

where $\lambda_{1,r}, \ldots, \lambda_{r,r}$ are the roots of $P_r(z) = z^r + \frac{1}{2}z^{r-1} + \cdots + \frac{1}{r!}z + \frac{1}{(r+1)!}$, and $A_{m,r} = a_{r-1}v_m^{(r)} + \cdots + a_m v_{r-1}^{(r)}$.

The former result shows that, for a large r, we can write,

$$B_n \approx n! \times \left[\sum_{i=1}^{r} \frac{1}{P_r'(\lambda_{i,r})} \left(\sum_{p=1}^{r} \frac{A_{p,r}}{{\lambda_{i,r}}^{p+1}} \right) \lambda_{i,r}^n \right],$$

where $\lambda_{1,r}, \ldots, \lambda_{r,r}$ are the simple roots of $P_r(z) = z^r + \frac{1}{2}z^{r-1} + \cdots + \frac{1}{r!}z + \frac{1}{(r+1)!}$, and $A_{m,r} = a_{r-1}v_m^{(r)} + \cdots + a_m v_{r-1}^{(r)}$.

Results of Theorem 4.5 and formulas (16), may contribute to establish the analytic approximations of the Bernoulli numbers $B_n^{(k)}$ of order $k \geq 2$.

5. Applications to Genocchi numbers, Euler numbers and Zeta function

It is well known that the Bernoulli numbers B_n are related to other important classes of numbers, especially the Genocchi numbers G_n and Euler numbers E_n (see for example [1, 5, 6, 8, 18]). These two classes of numbers are defined by the two following generating functions,

$$\frac{2t}{e^t+1} = \sum_{n=0}^{+\infty} G_n \frac{t^n}{n!} \quad \text{and} \quad \frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{+\infty} E_n \frac{t^n}{n!}.$$

On the other hand, these two families of numbers are related to Bernoulli numbers B_n thorough the two identities,

$$G_{2n} = 2(2^{2n} - 1)B_{2n} \quad \text{and} \quad E_{2n+1} = (2^{2n} - 1)\frac{2^{2n+1}}{n+1}B_{2n}.$$

(see for example, [6, 8]). Therefore, results of Sections 2 and 4 on Bernoulli numbers can contribute to obtain some properties for the Genocchi and Euler numbers, with the aid of those of sequences (3). Particularly, Theorem 2.2 permits to get the proposition.

Proposition 5.1. *Under the preceding data, the Genocchi and Euler numbers G_{2n} and E_{2n+1} are expressed in terms of the linear recursive sequences of infinite order, as follows,*

$$G_{2n} = 2(2^{2n} - 1)(2n)! \times v_{2n}^{[1]} \quad \textit{and} \quad E_{2n+1} = (2^{2n} - 1)(2n)!\frac{2^{2n+1}}{n+1} \times v_{2n}^{[1]},$$

where $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ is a sequence (3), whose coefficients are $a_n = -\dfrac{1}{(n+2)!}$ $(n \geq 0)$ and initial values are $v_0^{[1]} = 1$ and $v_{-j}^{[1]} = 0$ for all $j \geq 1$. Moreover, the combinatorial formulas of G_{2n} and E_{2n+1} are,

$$G_{2n} = 2(2^{2n} - 1)(2n)!\ \Omega(n) \ \textit{and} \ E_{2n+1} = (2^{2n} - 1)(2n)!\ \frac{2^{2n+1}}{n+1}\ \Omega(n),$$

where

$$\Omega(n) = \sum_{\sum_{j=0}^{2n-1}(j+1)k_j = 2n} (-1)^{\sum_{j=0}^{2n-1} k_j} \frac{(\sum_{j=0}^{2n-1} k_j)!}{k_0! \cdots k_{2n-1}!} \prod_{j=0}^{2n-1} \left[\frac{1}{((j+2)!)^{k_j}}\right].$$

On the other side, Propositions 4.1, 5.1 and Theorem 4.5 show that the approximation process of [3, Theorem 3.1], can also be applied for approaching the Genocchi and Euler numbers G_{2n} and E_{2n+1}, in terms of the family of linear sequences $\{v_n^{(r)}\}_{n\geq -r+1}$ of finite order of type (26).

Proposition 5.2. Linear approximation. *Under the preceding data, the Genocchi and Euler numbers G_{2n} and E_{2n+1} are approximated as follows,*

$$G_{2n} = 2(2^{2n} - 1)(2n)! \lim_{r\to+\infty} v_{2n}^{(r)} \quad \textit{and} \quad E_{2n+1} = (2^{2n} - 1)(2n)!\frac{2^{2n+1}}{n+1} \lim_{r\to+\infty} v_{2n}^{(r)},$$

where $\{v_n^{(r)}\}_{n\geq -r+1}$ *is of type (26), with coefficients* $a_{j,r} = -\dfrac{1}{(j+2)!}$ *for* $n = 0, 1, \ldots, r-1$, *and initial data* $v_0^{(r)} = 1$, $v_{-j}^{(r)} = 0$ *for* $j = 1, \ldots, r-1$.

The combinatorial approximation of G_{2n} and E_{2n+1} can be provided from (27) as follows,

$$G_{2n} = 2(2^{2n}-1)(2n)! \lim_{r\to+\infty} \Omega_n(r) \text{ and } E_{2n+1} = (2^{2n}-1)(2n)! \frac{2^{2n+1}}{n+1} \lim_{r\to+\infty} \Omega_n(r),$$

where

$$\Omega_n(r) = \sum_{s_0+2s_1+\cdots+rs_{r-1}=n} (-1)^{\sum_{j=0}^{r-1} s_j} \binom{\sum_{j=0}^{r-1} s_j}{s_0, \cdots, s_{r-1}} \left[\prod_{j=0}^{r-1} \frac{1}{((j+2)!)}\right]^{s_j}.$$

In addition, the analytic approximation of G_n and E_n are obtained using Lemma 4.3 and result of Proposition 5.2. Indeed, we have the proposition.

Proposition 5.3. Analytic approximation. *The analytic approximations of Genocchi and Euler numbers* G_{2n} *and* E_{2n+1} *are given by the formulas,*

$$G_{2n} = 2(2^{2n}-1)(2n)! \times \lim_{r\to+\infty} \Lambda_n(r) \text{ and } E_{2n+1} = (2^{2n}-1)(2n)! \frac{2^{2n+1}}{n+1} \times \lim_{r\to+\infty} \Lambda_n(r),$$

where

$$\Lambda_n(r) = \sum_{i=1}^{r} \frac{1}{P_r'(\lambda_{i,r})} \left(\sum_{p=1}^{r} \frac{A_{p,r}}{\lambda_{i,r}{}^{p+1}}\right) \lambda_{i,r}^{2n}, \tag{28}$$

such that $\lambda_{1,r}, \ldots, \lambda_{r,r}$ *are the simple roots of* $P_r(z) = z^r + \frac{1}{2}z^{r-1} + \cdots + \frac{1}{r!}z + \frac{1}{(r+1)!}$ *and* $A_{m,r} = a_{r-1}v_m^{(r)} + \cdots + a_m v_{r-1}^{(r)}$.

Consequently, we deduce from Propositions 5.2-5.3, that for a large r we can write,

$$G_{2n} \approx 2(2^{2n}-1)(2n)! v_{2n}^{(r)} \quad \text{and} \quad E_{2n+1} \approx (2^{2n}-1)(2n)! \frac{2^{2n+1}}{n+1} v_{2n}^{(r)},$$

and

$$G_{2n} \approx 2(2^{2n}-1)(2n)! \times \Lambda_n(r) \text{ and } E_{2n+1} \approx (2^{2n}-1)(2n)! \frac{2^{2n+1}}{n+1} \times \Lambda_n(r),$$

where $\Lambda_n(r)$ is as in (28).

Finally, Bernoulli numbers are also related to the well known zeta function defined by $\zeta(n) = 1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \cdots$. The well known result of Euler asserts that,

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2[(2n)!]} B_{2n},$$

for every $n \geq 1$ (see for example [6]). Therefore, by Theorem 2.2, we have the following properties of the zeta function.

Proposition 5.4. *Under the preceding data, the zeta function satisfies the identity,*

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2} v_{2n}^{[1]}, \quad \textit{for every} \ \ n \geq 1,$$

where $\{v_n^{[1]}\}_{n\in\mathbb{Z}}$ *is the sequence (3) defining the Bernoulli numbers. Moreover, the combinatorial formula of the zeta function is,*

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2} \sum_{\sum_{j=0}^{2n-1}(j+1)k_j=2n} (-1)^{\sum_{j=0}^{2n-1} k_j} \frac{(\sum_{j=0}^{2n-1} k_j)!}{k_0! \cdots k_{2n-1}!} \prod_{j=0}^{2n-1} \frac{1}{((j+2)!)^{k_j}}.$$

Similarly, as for Bernoulli numbers, Euler and Genocchi numbers, the approximation process of [3, Theorem 3.1], can also be applied for approaching the zeta function.

Proposition 5.5. *For every* $n \geq 1$*, the zeta function can be approached in terms of the linear sequence* $\{v_n^{(r)}\}_{n\geq -r+1}$ *of type (26) as follows,*

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2} \lim_{r\to+\infty} v_{2n}^{(r)},$$

Moreover, its analytic approximation is given,

$$\zeta(2n) = \frac{(-1)^{n+1}(2\pi)^{2n}}{2} \times \lim_{r\to+\infty} \left[\sum_{i=1}^{r} \frac{1}{P_r'(\lambda_{i,r})} \left(\sum_{p=1}^{r} \frac{A_{p,r}}{{\lambda_{i,r}}^{p+1}} \right) \lambda_{i,r}^{2n} \right],$$

where where $\lambda_{1,r}, \ldots, \lambda_{r,r}$ *are the roots of* $P_r(z) = z^r + \frac{1}{2}z^{r-1} + \cdots + \frac{1}{r!}z + \frac{1}{(r+1)!}$, *and* $A_{m,r} = a_{r-1}v_m^{(r)} + \cdots + a_m v_{r-1}^{(r)}$.

In the best of our knowledge, it seems for us that results of Propositions 5.1, 5.2 and 5.4 are not known in the literature, at least under these forms.

6. Concluding remarks and perspectives

In this study, we had emphases the closed connection between Bernoulli numbers B_n, Bernoulli numbers $B_n^{(k)}$ of order k ($k \geq 1$) and the generalized Fibonacci sequences of order infinity. This new relationship has allowed us to establish some linear and combinatorial compact formulas of Bernoulli numbers B_n and $B_n^{(k)}$ ($k \geq 1$). Moreover, new and known identities were founded. In addition, the approximations method of B_n and $B_n^{(k)}$ ($k \geq 1$) are also provided, starting from the approximation of sequences (3) by a specific family of linear recursive sequences of finite order (4). The link with the partial Bell polynomials has been considered. Moreover, the closed relationship between Bernoulli numbers and other king of numbers such that Genocchi numbers G_n and Euler numbers E_n, allows us to provide properties and explicit formulas for G_n and E_n, similar to those of B_n and $B_n^{(k)}$. Finally, the expression of the zeta function $\zeta(n)$ in terms of the Bernoulli numbers B_n, also permits to set analogous properties and explicit formulas for $\zeta(n)$.

It seems for us that our approach of Bernoulli numbers B_n and $B_n^{(k)}$ is not current in the literature. Furthermore, this approach can be also applied to the study of the Genocchi numbers $G_n^{(k)}$ and Euler numbers $E_n^{(k)}$ of order $k \geq 2$ can also be studied with the aid of properties of sequences (3) and their underlying techniques. Moreover, applications to Bernoulli polynomials, to Genocchi and Euler polynomials are provided. Some results in this direction have been already established.

Finally, the asymptotic behavior for a sequence (3) has been studied in [3], and the open question consists in how to apply result of [3, Theorem 5.2] for studying the asymptotic behavior of the Bernoulli numbers B_n and $B_n^{(k)}$. Some numerical tests show us that the treatment of this question is not an easy task.

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