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A NOTE ON A BIJECTIVE PROOF OF A THEOREM OF G. E. ANDREWS

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Abstract: In this paper we provide a bijective proof for the theorem 3.1 of Andrews [6]. This theorem involves the number of semi-Fibonacci partitions and the number of binary partitions in which each part appears an odd number of times.

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1. Introduction

In Andrews [6], the author gave analytic proofs for two results involving the semi-Fibonacci partitions and a class of binary partitions. In the final section of [6], the author wrote: "It would be nice to have combinatorial proofs of both theorems." In this sense we deal with a bijective proof for the first theorem of this paper. As shown in [6], we give the mathematical definitions for the mentioned two classes of integer partitions, as follows.

Definition 1.1. *Let $\mathscr{SF}(n)$ denote the set of semi-Fibonacci partitions of n.*

- *if n is even, all parts of a semi-Fibonacci partition is parts of a semi-Fibonacci partition of $n/2$, wherein each part has been multiplied by 2;*

- *if n is odd, $\mathscr{SF}(n)$ arises from two sources: first a 1 is inserted in each semi-Fibonacci partition of $n-1$ and second a 2 is added*[1] *to the single odd part of $\mathscr{SF}(n-2)$.*

The number of semi-Fibonacci partitions of n is denoted by $sf(n)$. For instance, $sf(9) = 6$ ($\mathscr{SF}(9) = \{8+1, 4+3+2, 6+3, 5+4, 7+2, 9\}$).

Definition 1.2. *An odd binary partition of n is a partition of n in which the parts are powers of 2 and each distinct part appears an odd number of times.*

We denote the set of odd binary partition of n by $\mathscr{OB}(n)$, and the cardinality of this set by $ob(n)$. For example, $\mathscr{OB}(9) = \{8+1, 4+2+1+1+1, 4+1+1+1+1+1, 2+2+2+1+1+1, 2+1+1+1+1+1+1+1, 1+1+1+1+1+1+1+1+1\}$.

Our goal in this short note is to find bijective proof for the next theorem.

Theorem 1.3. *For all positive integer n, we have $sf(n) = ob(n)$.*

Bijective proofs involving partition identities are widely used to establish simple results as found in Bressoud [7], Pak [8], Andrews [1, 2, 3, 4, 5] and Subbarao [9]. The importance of this type of proof is due to the fact of making a bijective association between one class of partition to another, using simple invertible operations, without the need to establish a generating function for both.

We claim that there exists a bijective function $f_n : \mathscr{SF}(n) \to \mathscr{OB}(n)$, for all positive integer n, as we explain in the next section. For example, we will show that the partition $4+3+2$ in $\mathscr{SF}(9)$ is unically associated to $4+2+1+1+1$ in $\mathscr{OB}(9)$.

2. Bijective proof of theorem 1.3

In order to obtain a bijection between $\mathscr{SF}(n)$ and $\mathscr{OB}(n)$ we will make use of a lemma. In order to proof this lemma, we will use the recursive nature of a semi-Fibonacci partition and an inductive argument.

Lemma 2.1. *In a semi-Fibonacci partition λ of n, there are no two parts of type $2^r m$ and $2^r s$ in λ, for m and s odd integers and a positive integer r.*

Proof. Suppose that $2^r m$ and $2^r s$ are parts of λ. First consider $r = 1$. If n is even, we have that m and n are parts of $\lambda/2$, where $\lambda/2$ is understood as a semi-Fibonacci partition of $n/2$, where each part is divided in half. Since m and s are odd numbers, so $2m$ and $2s$ cannot both be parts of λ.

Let n be an odd integer. If there is a part equal to 1 in λ, we have it comes from a semi-Fibonacci partition of $n-1$, and as $n-1$ is even, we have m and s , odd numbers are both parts of $\lambda/2$, which is a contradiction.

[1] As explained in [6], there are only one odd part in this kind of partition.

If there is no part equal to 1 in this partition, then it comes from a semi-Fibonacci partition of $n-2$, adding 2 in its single odd part. Being an odd part of this partition as $2j+1 \geq 3$, using the opposite recursion j times, we arrive at a partition with an odd part equal to 1. Considering the same argument in the previous paragraph, we also obtain a contradiction here. This complete a proof for the case $r = 1$.

Assume that $2^r m$ and $2^r s$ are part of a semi-Fibonacci partition. If 1 is part of this partition, we have $2^{r-1}m$ and $2^{r-1}s$ are parts of $(\lambda - 1)/2$. The partition $(\lambda - 1)/2$ is understood as a semi-Fibonacci partition of $(n-1)/2$, coming from the λ partition by removing the part 1 and dividing the remaining parts in half. Using the inductive hypothesis, we have another contradiction here.

If 1 is not part of λ, this partition comes from a $n-2$ semi-Fibonacci partition, adding 2 in the odd part. In this way, using the definition of semi-Fibonacci partitions, we have $2^{r-a}m$, $2^{r-a}s$ are parts of a semi-Fibonacci partition with 1 as part. Again, using the inductive hypothesis, we conclude the proof.

Let n a positive integer, and a function $f_n : \mathscr{SF}(n) \to \mathscr{OB}(n)$ acting in the parts of a semi-Fibonacci partition λ as follows.

- if $2^r m$, for an odd integer m, is a part of λ, then this part is associated to $\underbrace{2^r + 2^r + \cdots + 2^r}_{m \text{ times}}$;
- If s is a odd part of λ, then this part is associated to $\underbrace{1 + 1 + \cdots + 1}_{s \text{ times}}$.

We claim that the function f_n is bijective. In order to proof this claim, we will divide the proof in two as follows.

From a semi-Fibonacci partition to a odd binary partition:

Let λ be a semi-Fibonacci partition of n. Clearly $f_n(\lambda)$ is a partition whose parts are powers of 2. By lemma 2.1, λ does not have multiple parts of the same power of 2, so the split process results in a binary partition $f_n(\lambda)$ in which all equal parts are repeated an odd number of times.

From a odd binary partition to a semi-Fibonacci partition:

For a partition $\lambda' \in \mathscr{OB}(n)$, we must perform the process of adding all the powers of 2 that are repeated an odd number of times. We claim that the resultant partition λ is a semi-Fibonacci partition. We will justify this claim next.

First, let n an odd number. Performing the previous operations on λ', we find $\lambda = m_1 2^{r_1} + m_2 2^{r_2} + \ldots + m_k 2^{r_k}$, with m_i odd, for $1 \leq i \leq k$, and $m_1 2^{r_1} > m_1 2^{r_1} > \ldots > m_k 2^{r_k}$.

We claim that there is only an odd part in λ. Indeed, if there are two, say m_i and m_j, with $m_i > m_j$, these come from sums of $m_i + m_j$ ones, but since $m_i + m_j$ is even, this is impossible . An odd number of odd and distinct parts it is also impossible. The same argument works to prove that there is no an even number of odd parts in λ.

From the previous arguments, if the odd part in λ is equal to 1, remove this part and divide the others in half, obtaining a partition $(\lambda - 1)/2$ de $(n-1)/2$. The same process is repeated until a part of this partition is odd. We designate this partition as $\mu = m_1 2^{r_1 - j} m_2 2^{r_2 - j} + \ldots + m_{l-1} 2^{r_{l-1} - j} + m_l$. We claim that μ is a semi-Fibonacci partition. Indeed, as the number of equal parts in λ' is odd, it is not the case that $m_i 2^{r_i - j} = m_s 2^{r_s - j}$, for two parts $m_i 2^{r_i - j}$ e $m_s 2^{r_s - j}$ of μ. If $m_l = 1$, this partition comes from a semi-Fibonacci partitions $\mu - 1$ adding 1 as part. If $m_l > 1$, this partition comes from a semi-Fibonacci partitions $\mu - 2$ adding 2 to m_l.

If the odd part of λ is $m_l = 2l+1 > 1$, removing $2l$ units from m_l, and repeating the same argument from the previous paragraph, one may justify that the resulting partition μ is a semi-Fibonacci partition.

Thus for n odd, retracing the steps, the partition λ is a semi-Fibonacci partition.

If n is an even number, we split all parts of λ in half, and repeating the process as done in the previous paragraph, we conclude that λ is a semi-Fibonacci partition.

Example 2.2. For instance, if $n = 14$, we have $f_{14}(8+4+2) = 8+4+2$, $f_{14}(8+6) = 8+2+2+2$, $f_{14}(12+2) = 4+4+4+2$, $f_{14}(10+4) = 4+2+2+2+2+2$, $f_{14}(14) = 2+2+2+2+2+2+2$.

If each part of the partition is a power of 2, f_n works as the identity function.

The inverse f_9^{-1} acts in $\mathscr{OB}(9)$ as follows:

$f_9^{-1}(8+1) = 8+1$, $f_9^{-1}(4+2+1+1+1) = 4+3+2$, $f_9^{-1}(2+2+2+1+1+1) = 6+3$, $f_9^{-1}(4+1+1+1+1+1) = 5+4$, $f_9^{-1}(2+1+1+1+1+1+1+1) = 7+2$, $f_9^{-1}(9) = 1+1+1+1+1+1+1+1+1$.

The next table shows completely the associations made by the function f_{15}.

$\mathscr{SF}(15)$		$\mathscr{OB}(15)$
$8+4+2+1$	$\longleftrightarrow$	$8+4+2+1$
$12+2+1$	$\longleftrightarrow$	$4+4+4+2+1$
$8+6+1$	$\longleftrightarrow$	$8+2+2+2+1$
$10+4+1$	$\longleftrightarrow$	$4+2+2+2+2+2+1$
$14+1$	$\longleftrightarrow$	$2+2+2+2+2+2+2+1$
$8+4+3$	$\longleftrightarrow$	$8+4+1+1+1$
$12+3$	$\longleftrightarrow$	$4+4+4+1+1+1$
$8+5+2$	$\longleftrightarrow$	$8+2+1+1+1+1+1$
$6+5+4$	$\longleftrightarrow$	$4+2+2+2+1+1+1+1+1$
$10+5$	$\longleftrightarrow$	$2+2+2+2+2+1+1+1+1+1$
$8+7$	$\longleftrightarrow$	$8+1+1+1+1+1+1+1$
$9+4+2$	$\longleftrightarrow$	$4+2+1+1+1+1+1+1+1+1+1$
$9+6$	$\longleftrightarrow$	$2+2+2+1+1+1+1+1+1+1+1+1$
$11+4$	$\longleftrightarrow$	$4+1+1+1+1+1+1+1+1+1+1+1$
$13+2$	$\longleftrightarrow$	$2+1+1+1+1+1+1+1+1+1+1+1+1+1$
15	$\longleftrightarrow$	$1+1+1+1+1+1+1+1+1+1+1+1+1+1+1$

Table 1: Example of bijective association partitions in $\mathscr{GF}(15)$ and in $\mathscr{OB}(15)$.

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