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BILINEAR CONCOMITANT AND GREEN'S FORMULA ASSOCIATED WITH A MATRIX DIFFERENTIAL OPERATOR

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Abstract: In this paper, we have considered a matrix differential operator and the corresponding eigenvalue problem. The bilinear concomitant for the problem has been obtained. After this, the Lagrange's Identity and the Green's Formula has been derived.

Keywords and Phrases: Matrix differential operator, eigenvalue, bilinear concomitant, Lagrange's Identity, Green's Formula.

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1. Introduction

The differential equation, which is considered in the problem, is given below,

$$-\frac{d}{dx}\left(P_0\frac{du}{dx}\right)+pu+rv=\lambda(F_{11}u+F_{12}v)$$

$$i\frac{dv}{dx}+qv+ru=-\lambda(F_{21}u+F_{22}v) \tag{1}$$

where,

(i) P_0 is a real valued function of u, having continuous derivatives of the first order in $a \le x \le b$.

(ii) p, q, r are all real valued function of u continuous in $a \le x \le s$.

(iii) $P_0(x) > 0$ for $a \leq x \leq b$.

(iv) F_{11}, F_{12}, F_{21},F_{22} are real valued continuous functions of x such that the matrix $F = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}$ is symmetric and positive definite for $a \leq x \leq b$, and is a parameter, real or complex.

(v) For a given vector

$$\phi = \begin{pmatrix} u \\ v \end{pmatrix} \tag{2}$$

the operator

$$L = \begin{pmatrix} -\frac{d}{dx}\left(P_0\frac{d}{dx}\right) + p & r \\ r & i\frac{d}{dx} + q \end{pmatrix} \tag{3}$$

Therefore, the equation (1) reduces into

$$L\phi = -\lambda F\phi \tag{4}$$

Equation (4) is a matrix differential equation and L is a matrix differential operator. We impose the following boundary condition on $\phi = \begin{pmatrix} u \\ v \end{pmatrix}$.

$$u'(a) = 0 \tag{5}$$

$$v(a) = v(b) \tag{6}$$

and

$$u'(b) = 0 \tag{7}$$

Then the equation (4) together with equations (5), (6), (7) becomes a boundary value problem.

$$\phi(\alpha/x, \lambda) = \begin{pmatrix} u(\alpha/x, \lambda) \\ v(\alpha/x, \lambda) \end{pmatrix} \quad \text{where} \quad a \leq x \leq b, \quad a \leq \alpha \leq b. \tag{8}$$

Where $\phi(\alpha/x, \lambda)$ is the solution of equation 4.

2. Methodology

In this part, we discuss and prove the existence and Uniqueness theorem. After all, we also find the expression for Bi-linear concomitant and Green's Formula for the equation (1).

2.1. Existence and Uniquness Theorem

Let $P_0(x)$, $p(x)$, $q(x)$, $r(x)$ and F satisfy the condition of the problem, which is considered in introduction part. Here, we also let λ, B, C be the three constants not all vanishing. Simultaneously we get a unique solution of equation (4) which is discussed below.

$$\phi(x) = \begin{pmatrix} u(x) \\ v(x) \end{pmatrix} \tag{9}$$

. Here, equation (9) satisfies

$$u(\alpha) = \lambda_1 u'(\alpha) = \beta_1 v(\alpha) = C \tag{10}$$

Where,$a \leq \alpha \leq b$, be the accent denoting w.r.t. x.
Also for each x in the closed interval, $a(x)$, $a, b, u(x)$, $U'(x)$, $V'(x)$ and $u''(x)$ all are the integral functions of the complex variable λ.

2.2. Bilinear Concomitant and Green's Formula

In this part of paper, we shall deduce the Bilinear concomitant and Green's formula for the equation (1).

$$\phi_j = \begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} u_j(x,\lambda) \\ v_j(x,\lambda) \end{pmatrix} = \phi_j(x,\lambda) \tag{11}$$

. and

$$\phi_k = \begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} u_k(x,\lambda) \\ v_k(x,\lambda) \end{pmatrix} = \phi_k(x,\lambda) \tag{12}$$

. Equation (11) and equation (12) be the two solutions of equation (4) where ϕ_j & ϕ_k are complex valued vectors and u_j, v_j, u_k, v_k are complex function of x. Also u_j and u_k have derivatives of second order where as v_j and v_k have derivatives of first order only w.r.t x. Now from equation (4) we get,

$$\phi_k^{-T} L\phi_j = (\bar{u_k}\bar{v_k}) = \begin{pmatrix} -\frac{d}{dx}\left(P_0\frac{d}{dx}\right) + p & r \\ r & i\frac{d}{dx} + q \end{pmatrix} \begin{pmatrix} u_j \\ v_j \end{pmatrix} \tag{13}$$

$$= (\bar{u_k}\bar{v_k}) \begin{pmatrix} (-P_0 u_j)' + Pu_j + rv_j \\ ru_j + iv_j + qv_j \end{pmatrix}$$

$$= -(P_0 u_j)'\bar{u_k} + Pu_j\bar{u_k} + rv_j\bar{v_k} + ru_j\bar{v_k} + iv_j\bar{v_k} + qv_j\bar{u_k}$$

$$\phi_k^{-T} L\phi_j = -P_0\bar{u_k}u_k - P_0\bar{u_j}u_k + Pu_j\bar{u_k} + rv_j\bar{u_k}v_k + iv_j\bar{v_k} + qv_j\bar{v_k} \tag{14}$$

and now the similar expression for $\phi_j^T\overline{L\phi_k}$

$$\phi_j^T\overline{L\phi_k} = -P_0\bar{u_k'}u_j - P_0\overline{u_k''}u_j + Pu_j\bar{u_k} + rv_j\bar{u_k}v_k + ru_j\bar{v_k} - iv_j\bar{v_k} + qv_j\bar{v_k} \tag{15}$$

Therefore, we get the expression by subtracting equation (15) from equation (14)

$$\phi_k^{-T}L\phi_j - \phi_j^T\overline{L\phi_k} = \left(P_0\bar{u_k'}\right)' u_j - (P_0\bar{u_j})'\bar{u_k} + i(v_j\bar{v_k} + v_j\bar{v_k'})$$

$$\phi_k^{-T}L\phi_j - \phi_j^T\overline{L\phi_k} = [P_0(u_j\bar{u_k'} - u_j\bar{u_k})]' + i(v_j\bar{v_k})'$$

$$\phi_k^{-T}L\phi_j - \phi_j^T\overline{L\phi_k} = \frac{d}{dx}[P_0(u_j\bar{u_k} - u_j\bar{u_k}) + iv_j\bar{v_k}] \tag{16}$$

Which is in the form of

$$\phi_k^{-T}L\phi_j - \phi_j^T\overline{L\phi_k} = \frac{d}{dx}\phi_j\phi_k \tag{17}$$

Where,

$$[\phi_j\phi_k] = P_0(u_j\bar{u_k'} - u_j\bar{u_k}) + iv_jv_k \tag{18}$$

Equation (18) is bilinear concomitant for the equation (4). Equation (17) is the Lagrange's identity for the problem.
Now, we integrating equation (17), then we get

$$\int_a^b (\phi_k^{-T}L\phi_j - \phi_j^T\overline{L\phi_k})dx = [\phi_j\phi_k](b) - \phi_j\phi_k(a) \tag{19}$$

The outcomes in equation (19) is Green's formula for our problem, which is given in equation (3), is the most suitable formula for our boundary value problem.

3. Conclusions

In this paper, our main aim is to find the bilinear concomitant and Green's formula for the differential equation problem, which is considered. Here, after analysis, we got the solution in which equation (18) is the bilinear concomitant and equation (19) is the Green's formula. By solving these equations, we construct the Green's matrix, which is discussed, in my paper "Construction of Green Matrix for the solution of a Matrix Differential Equation" which is given in the reference. After finding these values and use of boundary value conditions, we find the Green matrix $G_1 \& G_2$ where (2×2) determinants are evaluated at (y, λ). After this reduction of Green matrix in suitable form is performed and then we find a singular solution of the equation.

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