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ON BOCHNER CURVATURE TENSOR ON KAEHLER-NORDEN MANIFOLDS

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Abstract: In this paper we prove that a Bochner flat Kaehler Norden manifold is holomorphically projectively flat provided the $*$-scalar curvature tensor $S(e_i, e_i)$ vanish. We also show that a Kaehler-Norden manifold is Bochner symmetric if and only if it is locally symmetric and a Kaehler-Norden manifold is Bochner semi-symmetric if and only if it is semi-symmetric.

Keywords and Phrases: Kaehler-Norden manifold, Bochner curvature tensor, holomorphic projective curvature tensor, semi-symmetric manifold.

2010 Mathematics Subject Classification: 53C26, 53C55.

1. Introduction

An even dimensional differentiable manifold M^{2n} is said to be an anti-Kaehler manifold (Kaehler-Norden manifold) [11] if a complex structure J of type (1, 1) and a pseudo-Riemannian metric g of the manifold satisfies the following conditions:

$$J^2 = -I, \tag{1.1}$$

$$g(JX, JY) = -g(X, Y), \tag{1.2}$$

and

$$\nabla J = 0, \tag{1.3}$$

for any $X, Y \in \chi(M)$, where $\chi(M)$ is Lie algebra of vector fields on M^{2n} and ∇ is Levi-Civita connection of g. The metric g necessary have neutral signature (n, n). We know that such type of two dimensional manifold is flat, so through out this paper we have considered the manifold of dimension ≥ 4. Arif Salimov and Sibel Turanli [13] studied curvature properties of anti-Kaehler-Codazzi manifolds in 2013. Other differential Geometers [14, 9, 10] also studied Kaehler-Norden manifold by different approaches. In 1997, F. Defever, R. Deszcz and L. Verstraelen [8] studied pseudosymmetric para-Kaehler manifold and proved that every semi-Riemannian Ricci-pseudosymmetric para-Kaehler manifold (M^{2n}, J, g) of dimension ≥ 4 is Ricci-semisymmetric. They also shown that the Weyl pseudosymmetric para-Kaehler manifold (M^{2n}, J, g) of dimension ≥ 4 is Weyl semi-symmetric. In 2000, K. Sluka [14] proved that every pseudosymmetric, Ricci-pseudosymmetric and Weyl pseudosymmetric Kaehler-Norden manifold (M^{2n}, J, g) are semi-symmetric, Ricci-semisymmetric and semi-symmetric respectively. She also constructed an example of holomorphically projectively flat as well as semi-symmetric and locally symmetric Kaehler-Norden manifolds. After then in 2014, De and Majhi [9] studied the properties of the quasi-conformal curvature tensor of Kaehler-Norden manifolds. They proved that a Kaehler-Norden manifold (M^{2n}, J, g) is quasi-conformally semi-symmetric if and only if it be semi-symmetric. We have gone through the above developments and then planed to study the Bochner semi-symmetric Kaehler-Norden manifold.
Before equation (1.4), Let R(X, Y) and R be curvature operator and Riemannian Christoffel curvature tensor respectively then

$$R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \tag{1.4}$$

$$R(X,Y,Z,W) = g(R(X,Y)Z,W). \tag{1.5}$$

We know that the Ricci tensor S is defined by

$$S(X,Y) = trace\,\{Z \to R(Z,X)Y\}\,. \tag{1.6}$$

According to [3] the tensors defined in (1.4), (1.5) and (1.6) have the following properties

$$\begin{aligned} &R(JX,JY) = -R(X,Y), \quad R(JX,Y) = R(X,JY), \\ &S(JY,Z) = trace\,\{X \to R(JX,Y)Z\}\,, \;\; S(JX,Y) = S(JY,X), \\ &S(JX,JY) = -S(X,Y). \end{aligned} \tag{1.7}$$

If we take Q as the Ricci operator then the Ricci tensor of type (0,2) in terms of Q is defined as

$$S(X,Y) = g(QX,Y), \tag{1.8}$$

where

$$QY = -\sum_i \epsilon_i R(e_i,Y)e_i,$$

and $\{e_1, e_2, e_3......., e_n\}$ is an orthonormal basis and ϵ_i are the indicators of e_i. The Reimannian metric g in terms of e_i and ϵ_i are given by

$$\begin{aligned} &(a) \quad \epsilon_i = g(e_i\,, e_i) = \pm 1, \\ &(b) \quad g(Je_i, e_i) = 0. \end{aligned} \tag{1.9}$$

The notion of Bochner curvature tensor B on a Kaehler manifold was given by S. Bochner in 1994. The Bochner curvature tensor B is defined by [2]

$$\begin{aligned} B(Y,Z,U,V) &= R(Y,Z,U,V) - \frac{1}{2(n+2)}\Big\{S(Y,V)g(Z,U) - S(Y,U)g(Z,V) \\ &+ g(Y,V)S(Z,U) - g(Y,U)S(Z,V) + S(JY,V)g(JZ,U) \\ &- S(JY,U)g(JZ,V) + S(JZ,U)g(JY,V) - g(JY,U)S(JZ,V) \\ &- 2S(JY,Z)g(JU,V) - 2g(JY,Z)S(JU,V)\Big\} \\ &+ \frac{r}{(2n+2)(2n+4)}\Big\{g(Z,U)g(Y,V) - g(Y,U)g(Z,V) \\ &+ g(JZ,U)g(JY,V) - g(JY,U)g(JZ,V) - 2g(JY,Z)g(JU,V)\Big\}, \end{aligned} \tag{1.10}$$

where r is a scalar curvature of the manifold.
Putting $Y = Je_i$, $Z = JZ$ and $U = e_i$ and using equation (1.9) in above equation we have

$$\begin{aligned} \sum_i \epsilon_i g(B(Je_i, JZ)e_i, V) &= \left\{1 + \frac{(\epsilon_i + 4)}{2n+4}\right\} S(Z,V) \\ &+ \frac{1}{2n+4}[r^* g(JZ,V) + rg(Z,V)] - \frac{r(\epsilon_i + 2)}{(2n+2)(2n+4)} g(Z,V), \end{aligned} \tag{1.11}$$

where r^* denote $*$-scalar curvature , which is defined as the trace of JQ.
The holomorphic projective curvature tensor is defined by [16]

$$\begin{aligned} P(Y,Z,U,V) = R(Y,Z,U,V) - \frac{1}{n-2}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) \\ - S(JZ,U)g(JY,V) + S(JY,U)g(JZ,V)]. \end{aligned} \tag{1.12}$$

From equation (1.12) by straight forward calculation we have

$$\begin{aligned} &P(Y,Z,U,V) = -P(Z,Y,U,V), \\ &P(JY,JZ,U,V) = -P(Y,Z,U,V), \\ &\sum_i \epsilon_i P(e_i,Z,U,Je_i) = 0, \quad \sum_i \epsilon_i P(Y,Z,e_i,e_i) = 0 \end{aligned} \tag{1.13}$$

2. On a Bochner Flat Kaehler-Norden Manifold

A Kaehler-Norden manifold (M^{2n}, J, g) is said to be Bochner flat Kaehler-Norden manifold if and only if the Bochner curvature tensor vanishes identically i.e.

$$B(Y,Z,U,V) = 0. \tag{2.1}$$

Therefore from equation (1.11), we get

$$\begin{aligned} &\left\{1 + \frac{(\epsilon_i + 4)}{2n+4}\right\} S(Z,V) + \frac{1}{2n+4}[r^* g(JZ,V) + rg(Z,V)] \\ &- \frac{r(\epsilon_i + 2)}{(2n+2)(2n+4)} g(Z,V) = 0, \end{aligned} \tag{2.2}$$

after equation (2.2), Putting $Z = V = e_i$, we get

$$\left\{1 + \frac{(2\epsilon_i + 4)}{2n+4} - \frac{\epsilon_i(\epsilon_i + 2)}{(2n+2)(2n+4)}\right\} r = 0, \tag{2.3}$$

this implies

$$r = 0, \tag{2.4}$$

Now from equations (2.2) and (2.4), we have

$$S(Z,V) = -\frac{r^*}{(2n + \epsilon_i + 8)} g(JZ,V). \tag{2.5}$$

Using (2.1), (2.4) and (2.5) in equation (1.10), we have

$$R(Y,Z,U,V) = -\frac{r^*}{(2n+\epsilon_i+8)(n+2)}[g(Y,Z)g(JU,V) + g(U,V)g(JY,Z)]. \tag{2.6}$$

A Kaehler-Norden manifold (M^{2n}, J, g) is said to be holomorphically flat if and only if the holomorphic projective curvature tensor vanishes identically i.e.

$$P(Y,Z,U,V) = 0. \tag{2.7}$$

Therefore from equation (1.12) and (2.7), we get

$$\begin{aligned} R(Y,Z,U,V) =& \frac{1}{(n-2)}[S(Z,U)g(Y,V) - S(Y,U)g(Z,V) \\ & - S(JZ,U)g(JY,V) + S(JY,U)g(JZ,V)]. \end{aligned} \tag{2.8}$$

From equation (2.5) and (2.8), we have

$$\begin{aligned} R(Y,Z,U,V) =& \frac{r^*}{(n-2)(2n+\epsilon_i+8)}[g(JY,U)g(Z,V) - g(JZ,U)g(Y,V) \\ & - g(Z,U)g(JY,V) + g(Y,U)g(JZ,V)], \end{aligned} \tag{2.9}$$

from equation (2.6) and (2.9), we get

$$\begin{aligned} & - \frac{r^*}{(2n+\epsilon_i+8)(n+2)}[g(Y,Z)g(JU,V) + g(U,V)g(JY,Z)] \\ & = \frac{r^*}{(n-2)(2n+\epsilon_i+8)}[g(JY,U)g(Z,V) - g(JZ,U)g(Y,V) \\ & - g(Z,U)g(JY,V) + g(Y,U)g(JZ,V)]. \end{aligned} \tag{2.10}$$

Putting $U = V = e_i$ in equation (2.10), we get

$$r^* \, \epsilon_i \, g(JY,Z) = 0, \tag{2.11}$$

which implies

$$r^* = 0. \tag{2.12}$$

Thus we conclude:

Theorem 2.1. *If a Bochner flat Kaehler-Norden manifold (M^{2n}, J, g) be holomorphically projectively flat then $*$-scalar curvature tensor will vanish.*

3. Bochner Semisymmetric Kaehler-Norden Manifolds

Let (M,g) be a Riemannian manifold and ∇ be the Levi-Civita connection of (M,g) then a Riemannian manifold is said to be locally symmetric if $\nabla R = 0$, where R is the Riemannian curvature tensor of (M,g). After [1], The locally symmetric manifold have been extended by different differential Geometer such as semi-symmetric manifold by Szabo [15] and B. B. Chaturvedi and B. K. Gupta [4, 5, 6, 7, 12]. According to Z. I. $Szab^{'}o$ [15], a manifold M is said to be semi-symmetric manifold if

$$(R(X,Y).R)(U,V)W = 0, \quad X,Y,U,V,W \in \chi(M) \tag{3.1}$$

where X and Y are vector fields.
A Bochner curvature tensor is said to be Bochner parallel if the covariant derivative of Bochner curvature tensor vanish i.e. $\nabla B = 0$, and this type of manifold is called Bochner symmetric manifold.
Taking covariant derivative of equation (1.11) and using $\nabla B = 0$, we get

$$\begin{aligned}&(1+\frac{(\epsilon_i+4)}{2n+4})(\nabla_X S)(Z,V))\\&+\frac{1}{2n+4}[dr^*(X)g(JZ,V)+dr(X)g(Z,V)]-\frac{dr(X)(\epsilon_i+2)}{(2n+2)(2n+4)}g(Z,V)=0.\end{aligned} \tag{3.2}$$

Now putting $Z = V = e_i$ in above equation we have

$$[1+\frac{(2\epsilon_i+4)}{2n+4}-\frac{\epsilon_i(\epsilon_i+2)}{(2n+2)(2n+4)}]\,dr(X)=0, \tag{3.3}$$

which implies

$$dr(X)=0. \tag{3.4}$$

Again putting above equation in (3.2), we get

$$(\nabla_X S)(Y,V)=-\frac{1}{(2n+\epsilon_i+8)}dr^*(X)g(JY,V). \tag{3.5}$$

Putting $Y = JY$ in above equation we get

$$(\nabla_X S)(JY,V)=\frac{1}{(2n+\epsilon_i+8)}dr^*(X)g(Y,V), \tag{3.6}$$

again replacing Y and V in equation (3.6) by e_i, we get

$$(1-\frac{\epsilon_i}{(2n+\epsilon_i+8)})dr^*(X)=0, \tag{3.7}$$

this implies

$$dr^*(X)=0, \tag{3.8}$$

putting above value in equation (3.5), we have

$$(\nabla_X S)(Y,V)=0. \tag{3.9}$$

Now taking covariant derivative of equation (1.10) and using equation (3.4) and (3.9), we get

$$(\nabla_X B)(Y,Z,U,V)=(\nabla_X R)(Y,Z,U,V). \tag{3.10}$$

Thus we conclude:

Theorem 3.1. *A Kaehler-Norden manifold* (M^{2n}, J, g) *is Bochner symmetric if and only if it is locally symmetric.*

A Kaehler-Norden manifold is said to be Bochner semi-symmetric Kaehler-Norden manifold if Bochner curvature tensor of the manifold satisfies

$$(R(X,Y).B)(U,V)W = 0, \quad X,Y,U,V,W \in \chi(M) \tag{3.11}$$

for all vector fields X and Y.
Now we propose:

Theorem 3.2. *A Kaehler Norden manifold* (M^{2n}, J, g) *is Bochner semi-symmetric if and only if it is semi-symmetric.*

From equation (1.11) we have

$$\sum_i \epsilon_i B(Je_i, JZ)e_i, = \left\{1 + \frac{(\epsilon_i + 4)}{2n+4}\right\} QZ + \frac{1}{2n+4}[r^* JZ + rZ] - \frac{r(\epsilon_i + 2)}{(2n+2)(2n+4)} Z, \tag{3.12}$$

where r^* is the $*$-scalar curvature which is defined by the trace of JQ.
If Bochner curvature tensor in Kaehler-Norden manifold satisfies $R.B = 0$ then from equation (3.12) we have $R.Q = 0$ and hence $R.S = 0$. Since we know that the Ricci tensors are defined by $S(X,Y) = g(QX,Y)$ and $S(JX,Y) = g(QJX,Y)$ then from equation (1.10) if $R.B = 0$ and $R.S = 0$ then we get $R.R = 0$. Conversely if

$$R.R = 0 \Rightarrow R.S = 0 \Rightarrow R.Q = 0, \tag{3.13}$$

then from (3.12), we have R.B=0.

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