South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 3 (2020), pp. 311-328

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

EQUILIBRIUMS AND STABILITY OF AN SVIR EPIDEMIC MODEL WITH NON LINEAR SATURATED INCIDENCE

Monika Badole, S. K. Tiwari and Aayush Sharma*

School of Studies in Mathematics,
Vikram University, Ujjain, Madhya Pradesh - 456010, INDIA
E-mail : monikabadole@gmail.com

*Department of Mechanical,
Maulana Azad National Institute of Technology,
Bhopal, Madhya Pradesh - 462003, INDIA

(**Received:** *Sep. 04, 2020* **Accepted:** *Oct. 19, 2020* **Published:** *Dec. 30, 2020)*

Abstract: Vaccination is most essential for the elimination of infectious diseases. This paper investigates a susceptible-vaccinated-infectious-recovered epidemic model with a non-linear saturated incidence rate for infectious disease dynamics, including the role of preventive vaccine is proposed and analyzed. Incoming immigrants are proposed in this model. A model for the transmission dynamics of infectious disease has been studied and presented this model's equilibrium points. The model shows the disease-free and endemic equilibrium.

Keywords and Phrases: SVIR epidemic model, Reproduction number, Endemic Equilibrium, Diseases Free Equilibrium, Stability analysis.
2010 Mathematics Subject Classification: 92D25, 49J15, 93D20, 92D30, 34D23, 93A30.

1. Introduction

Vaccination is mainly considered as one of the outstanding medical achievements of modern civilization. Vaccination gained rising popularity and success after eradicating smallpox that was responsible for centuries of the outbreak of 1976. Because of vaccines, childhood diseases were commonplace less than a generation ago is now increasingly rare. However, now vaccination is an ordinarily used method to control illnesses such as polio, tuberculosis, and measles. Customarily,

there are diverse schedules of dosage for various diseases and vaccines. For some infections, vaccinated individuals should take doses several times, and there must be some fixed time interval between a couple of doses. For example, Gabbuti et al. [2] suggested that vaccination of hepatitis B can confer long-term immunity and that immunological memory can remain the loss of antibody. Hence, routine booster doses of the vaccine do not appear necessary to maintain long-lasting protection in successfully vaccinated immunocompetent individuals. For a given set of population, the ratio of susceptive who goes on to vaccination depends on myriad factors, one of which is the availability of the required resources.

Vaccination is administering a vaccine to spur an individual's immune system to improve adaptive immunity to an epidemic. Vaccines can limit or enhance morbidity from infection. Vaccination is the most effective tool for checking infectious diseases Gabbuti et al. [2]. Much thought has been paid to creating and investigating epidemic models with vaccination to gain insights into vaccination's role in the last decades. For example, by including a continuous vaccination strategy into a classical susceptible-infectious-recovered model, Liu et al. [6] presented a vaccination model characterized by a system of four ordinary differential equations (ODEs). It is ascertained that vaccination has an effect of reducing the basic reproduction number. It is also confirmed that the basic reproduction number governs the global dynamics of the model. In the presented model, the populations are divided into susceptible, vaccinated, infective, and recovered.

On the other hand, vaccinated individuals are considered to have partial immunity, and therefore the dynamic relationships with infectious individuals may moderate compared with those of susceptible. In this sense, the ongoing vaccination strategy can be assessed by the basic reproduction number because of the two infection paths: the vaccinated infection path and the susceptible infection path. Presently, it has been perceived that the transmission dynamics of several diseases could not be correctly interpreted by the traditional compartmental epidemic models with no age structure. Models with (continuous) age structures are represented by a hybrid system of ODEs and partial differential equations (PDEs) (Webb [11]). Following is the vitality of the research works [3, 4, 6, 7, 8], Wang et al. [9] formed an SVEIR model with the composition of infection age. Under the hypothesis that the infectious class's removal rate is constant rather than a function of the infection age, the model can be recreated as a differential equation with an infinite delay.

Duan et al. [1] studied an SVIR model's global stability with the vaccination age. Correlated with the models in [6, 9]; in this, the model did not consider that vaccinated peoples still have the possibility of being infected by contact with infected peoples. Following the line of Liu et al.[6] and Wang et al.[9] and believing

that before receiving immunity, the vaccinees still has the possibility of being infected by contact with infected peoples, Wang et al. [10] reviewed the dynamics of a hybrid system of the SVIR model with the infection age.

2. Model Formulation

In Islam [3], the Susceptible-Vaccinated-Infected-Recovered model with Bilinear incidence rate was considered as:

$$\left.\begin{aligned} \tfrac{dS}{dt} &= (1-p)A + \Lambda - \alpha SI - (\mu+\phi)S \\ \tfrac{dV}{dt} &= \phi S - \sigma\alpha VI - \mu V \\ \tfrac{dI}{dt} &= pA + \alpha SI + \sigma\alpha VI - (\mu+\gamma+\beta)I \\ \tfrac{dR}{dt} &= \gamma I - \mu R \end{aligned}\right\}$$

In this paper, we are considering an SVIR epidemic model with the saturated incidence rate $\frac{\alpha SI}{1+\alpha I}$, based on the above motivations. The model described under the framework of the following nonlinear ordinary differential equations:

$$\left.\begin{aligned} \tfrac{dS}{dt} &= (1-p)\Lambda + \beta - \tfrac{\alpha SI}{1+\alpha I} - (\mu+\varphi)S \\ \tfrac{dV}{dt} &= \varphi S - \sigma\alpha VI - \mu V \\ \tfrac{dI}{dt} &= p\Lambda + \tfrac{\alpha SI}{1+\alpha I} + \sigma\alpha VI - (\mu+\delta+\lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\} \tag{2.1}$$

where $S(t) > 0, V(t) > 0, I(t) > 0$ and $R(t) > 0$ denoted the divisions of the population that are susceptible, vaccinees, infectious, and recovered at time t, respectively. We model new infections using the saturated incidence rate $\frac{\alpha SI}{1+\alpha I}$, where α is the contact rate that is sufficient to transmit the disease. We also considered a constant recovery rate $\lambda \geq 0$. With the help of a factor σ, the vaccine has the effect of reducing the susceptibility to infection, so that $\sigma = 0$ indicates that the vaccine is thoroughly efficient in preventing infection, while $\sigma = 1$, implies that the vaccine is thoroughly inefficient. φ is the rate at which the susceptible population is vaccinated. $x' = f(x)$ is disease-related death rate, μ is natural death rate which is not related to the disease. The population is renewed in two type, birth and immigration. We believe that all newborns enter the susceptible class at the constant rate of β and there is a constant incoming flow of immigrants.

Table 1: Biological meanings of parameters in equation (2.1)

Parameter	**Meaning**
Λ	Constant transition of new members into the population per unit time, where fraction ρ of emigrants is infective $(0 \leq \rho \leq 1)$
α	Disease contact rate
φ	Rate at which the vaccination of the susceptible population
μ	Constant per capita natural death rate
λ	Fraction of infective recovers in unit time
β	Constant natural birth rate, with all newborns getting into the susceptible class
δ	Disease induced death rate
σ	Reflects the effect of vaccine reducing the infection rate
αN	Infectious contact rate per person in unit time.

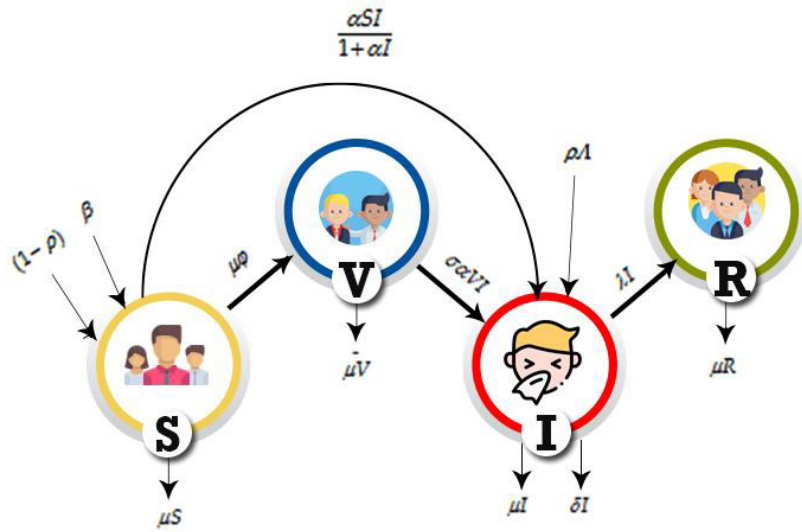

Diagram of SVIR Model

2.1. Boundedness of the system

We examine the boundness property of the total population in the system:

Theorem 2.1. *The solution of the system (2.1) is uniformly bounded.*

Proof. The sum of the susceptible, infective, vaccinated, and recovered class is the entire population, that is

$N(t) = S(t) + V(t) + I(t) + R(t)$, therefore

$$\frac{dN(t)}{dt} = \frac{dS(t)}{dt} + \frac{dV(t)}{dt} + \frac{dI(t)}{dt} + \frac{dR(t)}{dt} \tag{2.2}$$

Using equation (2.1) and (2.2)

$$\frac{dN(t)}{dt} = \Lambda + \beta - \mu N - \delta I$$

Then we obtained

$$\frac{dN}{dt} = \Lambda + \beta - \mu N - \delta I \quad \text{i.e} \quad \frac{dN}{dt} = \mu N \leq \Lambda + \beta$$

Now using the theory of the differential inequality and integrating both sides of the above inequality, we get

$$0 < N(S, V, I, R) \leq \frac{\Lambda + \beta}{\mu}(1 - e^{-\mu t}) + N\big(S(0), V(0), I(0), R(0)\big)e^{-\mu t}$$

Now taking limit, $t \to \infty$, we get $0 < N \leq \frac{\Lambda+\beta}{\mu}$

Consequently all the general solution of equation (2.1) that initiating at $\{R_+^4/0\}$ are confined in the region.

$$\mathfrak{R} = \left\{ (S, V, I, R) \in R_+^4 : S, V, I, R \geq 0, N = \frac{\Lambda + \beta}{\mu} + \xi \right\}$$

For any $\xi > 0$ and for $t \to \infty$.Hence the theorem proved.

We can obtain an substitute model but equivalent model by substituting S with $N - V - I - R$. Reformulate model can be expressed by following

$$\left.\begin{aligned} \tfrac{dV}{dt} &= \varphi(N - I - R) - \sigma\alpha VI - (\mu + \varphi)V \\ \tfrac{dI}{dt} &= \rho\Lambda + \tfrac{\alpha I}{1+\alpha I}(N - I - R) - \alpha VI\Big(\tfrac{1}{1+\alpha I} - \sigma\Big) - (\mu + \delta + \lambda)I \\ \tfrac{dR}{dt} &= \lambda I + \mu R \\ \tfrac{dN}{dt} &= \Lambda + \beta - \mu N - \delta I \end{aligned}\right\} \tag{2.3}$$

Jacobean matrix of above system is given by

$$J = \begin{bmatrix} -\sigma\alpha I - \mu - \varphi & -\varphi - \sigma\alpha V & -\varphi & \varphi \\ -\alpha I\Big(\frac{1}{1+\alpha I} - \sigma\Big) - \frac{\alpha I}{1+\alpha I} + (N - V - I - R) & \frac{\alpha}{(1+\alpha I)^2} + \sigma\alpha V - (\mu + \delta + \lambda) & \frac{-\alpha I}{1+\alpha I} & \frac{\alpha I}{1+\alpha I} \\ 0 & \lambda & -\mu & 0 \\ 0 & -\delta & 0 & -\mu \end{bmatrix}$$

3. Equilibrium Conditions

For equilibrium points of the system (2.3), we have

$$\varphi(N - I - R) - \sigma\alpha V I - (\mu + \varphi)V = 0 \tag{3.1}$$

$$\rho\Lambda + \frac{\alpha I}{1 + \alpha I}(N - I - R) - \alpha V I\Big(\frac{1}{1 + \alpha I} - \sigma\Big) - (\mu + \delta + \lambda)I = 0 \tag{3.2}$$

$$\lambda I - \mu R = 0 \tag{3.3}$$

$$\Lambda + \beta - \mu N - \delta I = 0 \tag{3.4}$$

From equation (3.1), (3.3) and (3.4) we get $N = \dfrac{\Lambda + \beta - \delta I}{\mu}, R = \dfrac{\lambda I}{\mu}, V = \dfrac{\varphi\{\Lambda + \beta - (\mu + \delta + \lambda)I\}}{\mu(\sigma\alpha I + \mu + \varphi)}$ and by substituting these value in equation (3.2), we get

$$A_1 I^3 + A_2 I^2 + A_3 I + A_4 = 0 \tag{3.5}$$

Where, $A_1 = \alpha^2\sigma\big\{(\mu + \delta + \lambda)(1 + \mu + \varphi)\big\}$

$$\begin{aligned} A_2 &= (\Lambda + \beta)\alpha^2\sigma(1 - \varphi) + (\mu + \delta + \lambda)\{\mu + \alpha\sigma\varphi - \alpha\varphi - \alpha\mu^2 - \varphi\mu\} \\ &\quad + \alpha(\mu + \varphi)(\delta + \lambda) - \rho\Lambda\alpha^2\mu\sigma \\ A_3 &= \mu(\mu + \varphi)(\alpha - \delta - \lambda - \mu - \rho\alpha\Lambda)\rho\Lambda\alpha\mu\sigma - \alpha(\Lambda + \beta)(\mu + \sigma\varphi) \\ A_4 &= -\rho\Lambda\mu(\mu + \varphi) \end{aligned}$$

4. Vaccine Reproductive Number

Suppose that the case when there are no infective immigrants that is. $\rho = 0$, therefore from equation (2.3) we get

$$\left.\begin{aligned} \tfrac{dV}{dt} &= \varphi(N - I - R) - \sigma\alpha V I - (\mu + \varphi)V \\ \tfrac{dI}{dt} &= \tfrac{\alpha I}{1+\alpha I}(N - V - I - R) + \alpha\sigma V I - (\mu + \delta + \lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \\ \tfrac{dN}{dt} &= \Lambda + \beta - \mu N - \delta I \end{aligned}\right\} \tag{4.1}$$

For equilibrium points of the system (4.1), we have

$$\varphi(N-I-R)-\sigma\alpha VI-(\mu+\varphi)V=0 \tag{4.2}$$

$$\frac{\alpha I}{1+\alpha I}(N-V-I-R)+\alpha\sigma VI-(\mu+\delta+\lambda)I=0 \tag{4.3}$$

$$\lambda I-\mu R=0 \tag{4.4}$$

$$\Lambda+\beta-\mu N-\delta I=0 \tag{4.5}$$

From equation (4.4) and (4.5), we have $N=\dfrac{\Lambda+\beta-\delta I}{\mu}$ and $R=\dfrac{\lambda I}{\mu}$
From equation (4.3) we can easily see that $I=0$ that is diseases dies out.
By using equation (4.2), we can evaluate the other equilibrium values at disease-free equilibrium (DFE) of $R=0, N=\dfrac{\Lambda+\beta}{\mu}$, and $V=\frac{\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)}$.
So the diseases free equilibrium $E^0(V,I,R,N)=\left(\dfrac{\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)},0,0,\dfrac{(\Lambda+\beta)}{\mu}\right)$
Now Jacobean matrix at E^0 is

$$J=\begin{bmatrix} -(\mu+\varphi) & -\varphi-\sigma\alpha\frac{\phi(\Lambda+\beta)}{\mu(\mu+\varphi)} & -\varphi & \varphi \\ 0 & \alpha\Big(\frac{\Lambda+\beta}{\mu}-(1-\sigma)\frac{\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)}\Big)-(\mu+\delta+\lambda) & 0 & 0 \\ 0 & \lambda & -\mu & 0 \\ 0 & -\delta & 0 & -\mu \end{bmatrix}$$

The eigenvalues of the J_0 are given by

$$\lambda_1=\lambda_2=-\mu,\ \lambda_3=-(\mu+\varphi),\ \lambda_4=\alpha\Big(\frac{\Lambda+\beta}{\mu}-(1-\sigma)\frac{\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)}\Big)-(\mu+\delta+\lambda)$$

Since for the positive parameter λ_1,λ_2 and λ_3 are negative, only condition for stability of disease free equilibrium is $\lambda_4<0$ i.e.

$$\alpha\Big(\frac{\Lambda+\beta}{\mu}-(1-\sigma)\frac{\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)}\Big)<(\mu+\delta+\lambda)\Rightarrow\frac{(\alpha\varphi\sigma+\alpha\mu)(\Lambda+\beta)}{\mu(\mu+\varphi)(\mu+\delta+\lambda)}<1$$

Therefore we can represent a vaccine reproductive number $R(\varphi)=\frac{(\alpha\varphi\sigma+\alpha\mu)(\Lambda+\beta)}{\mu(\mu+\varphi)(\mu+\delta+\lambda)}<1$.

The vaccine reproduction number is less than one if and only if a disease-free equilibrium is locally asymptotically stable, in the lack of vaccine ($\varphi=0$). We represent the basic reproduction number as $R_0=\dfrac{\alpha(\Lambda+\beta)}{\mu(\mu+\delta+\lambda)}$

5. When There Are No Infective Immigrants

Case (I) for the case $\sigma = 1$ and $\rho = 0$

Investigate two extreme cases of endemic equilibrium for the model (2.1), case is when the vaccine is useless, and there is no infective immigrant that is $\sigma = 1$ and $\rho = 0$ then our system (2.1) becomes

$$\left.\begin{aligned} \tfrac{dS}{dt} &= \Lambda + \beta - \tfrac{\alpha SI}{1+\alpha I} - (\mu+\varphi)S \\ \tfrac{dV}{dt} &= \varphi S - \alpha VI - \mu V \\ \tfrac{dI}{dt} &= \tfrac{\alpha SI}{1+\alpha I} + \alpha VI - (\mu+\delta+\lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\} \tag{5.1}$$

Where, $N(t), S(t) + V(t) + I(t) + R(t)$, note that $R(\varphi)$ diminishes to R_0 when $\sigma = 1$. By using equilibrium conditions and substitution, we have the following relations are mentioned below:

$$\begin{aligned} &R^* = \frac{\lambda I}{\mu}, \quad V^* = \frac{\varphi(\Lambda+\beta)(1+\alpha I)}{(\mu+\alpha I)\{\alpha I(1-\mu-\varphi) - (\mu+\varphi)\}}, \text{ and} \\ &B_1 I^2 + B_2 I + B_3 = 0, \quad \text{where} \\ &B_1 = \alpha^2(1-\mu-\varphi) \\ &B_2 = -\alpha\left\{\mu^2 + \mu\varphi + \varphi + \frac{(\Lambda+\beta)(1+\varphi)\alpha}{(\mu+\delta+\lambda)}\right\}, \\ &B_3 = -(\mu+\varphi)\left\{\mu + \frac{\alpha(\Lambda+\beta)}{(\mu+\delta+\lambda)}\right\} \end{aligned} \tag{5.2}$$

The preceding equation will have a positive root if $\Delta > 0$ and $R_0 > 1$, and I^* is given by

$$I^* = \frac{\alpha\left\{\mu^2 + \mu\varphi + \varphi + \frac{(\Lambda+\beta)(1+\varphi)\alpha}{(\mu+\delta+\lambda)}\right\} \pm \sqrt{\Delta}}{2\alpha^2(1-\mu-\varphi)}$$

where

$$\Delta = \sqrt{\alpha^2\left\{\mu^2 + \mu\varphi + \varphi + \frac{(\Lambda+\beta)(1+\varphi)\alpha}{(\mu+\delta+\lambda)}\right\}^2 + 4\alpha^2(1-\mu-\varphi)(\mu+\varphi)\left\{\mu + \frac{\alpha(\Lambda+\beta)}{(\mu+\delta+\lambda)}\right\}}$$

Presently an endemic equilibrium allows to a real positive solution of equation (5.2). Observe that $B_1 > 0$ and that $B_3 < 0$ precisely when $R_0 > 1$, It can

also be inferred that $B_2^2 - 4B_1B_3$ when $B_3 < 0$. Simply conclude that there is precisely endemic equilibrium when $R_0 > 1$ as there are two real roots, and the multiplication of those two roots are negative. On the contrariety, we can observe that $B_3 > 0$, if $R_0 < 1$. Perceive that there are precisely two changes in the sign of coefficients of the equation (5.2) if coefficient $B_2 < 0$ and none when $B_2 > 0$. By Descartes' rule of signs, one can conclude that the maximum number of endemic equilibrium is two when basic reproduction number less than one, and $B_2 < 0$, and if $B_2 > 0$ and basic reproduction number is smaller than one, then there is no endemic equilibrium. However, it is explained that if basic reproduction number less than one then it is always the case that the system does not have any endemic equilibrium.

Case (II) for the case $\sigma = 0$ and $\rho = 0$

If we assume that there are no infective emigrants and vaccine is completely effective that is $\rho = 0$ and $\sigma = 0$ then model can be represented by

$$\left.\begin{aligned} \tfrac{dS}{dt} &= \Lambda + \beta - \tfrac{\alpha SI}{1+\alpha I} - (\mu + \varphi)S \\ \tfrac{dV}{dt} &= \varphi S - \mu V \\ \tfrac{dI}{dt} &= \tfrac{\alpha SI}{1+\alpha I} - (\mu + \delta + \lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\} \tag{5.3}$$

with $N(t) = S(t) + V(t) + I(t) + R(t)$, By using equilibrium conditions $R^* = \dfrac{\lambda I}{\mu}, V^* = \dfrac{\varphi S}{\mu}$ and $S^* = \dfrac{(\mu + \delta + \lambda)(1 + \alpha I)}{\alpha}$ and substitution, we have the following relation are mentioned below:

$$I^* = \frac{\{\alpha(\Lambda + \beta) - (\mu + \varphi)(\mu + \delta + \lambda)\}}{\alpha(1 + \mu + \varphi)(\mu + \delta + \lambda)} \text{ which exist only when } R_0 > 1$$

Case (III) for the case $0 < \sigma < 1$ and $\rho = 0$

We need to assume the more general case when the vaccine is partly infective and when there are no infective emigrants. Symbolic denoted by $0 < \sigma < 1$, and

$\rho = 0$. The model given by

$$\left.\begin{aligned} \tfrac{dS}{dt} &= \Lambda + \beta - \tfrac{\alpha SI}{1+\alpha I} - (\mu+\varphi)S \\ \tfrac{dV}{dt} &= \varphi S - \sigma\alpha VI - \mu V \\ \tfrac{dI}{dt} &= \tfrac{\alpha SI}{1+\alpha I} + \sigma\alpha VI - (\mu+\delta+\lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\} \tag{5.4}$$

With relation $N(t) = S(t) + V(t) + I(t) + R(t)$, By using equilibrium conditions $R^* = \frac{\lambda I}{\mu}$,

$$S^* = \frac{(1+\alpha I)(\Lambda+\beta)}{\{\alpha I + (\mu+\varphi)(1+\alpha I)\}}, \quad V^* = \frac{\varphi(1+\alpha I)(\Lambda+\beta)}{(\mu+\sigma\alpha I)\{\alpha I + (\mu+\varphi)(1+\alpha I)\}}$$

After complicated calculation, we get the quadratic equation of I as

$$C_1 I^2 + C_1 I + C_3 = 0$$

Where, $C_1 = (1+\mu+\varphi)(\mu+\delta+\lambda)\sigma\alpha^2$

$$\begin{aligned} C_2 &= \alpha\mu(1+\mu+\varphi)(\mu+\delta+\lambda) + (\mu+\delta+\lambda)(\mu+\varphi)\alpha\sigma - \alpha^2\sigma(\Lambda+\beta) \\ &\quad - \alpha^2\sigma\varphi(\Lambda+\beta) \\ C_3 &= \mu(\mu+\varphi)(\mu+\delta+\lambda) - \alpha(\Lambda+\beta)(\mu+\varphi\sigma) = \mu(\mu+\varphi)(\mu+\delta+\lambda)[1-R(\varphi)] \end{aligned}$$

Here, endemic equilibrium allows for a real positive solution of the equation (5.5). Observe that $C_1 > 0$ and that $C_3 < 0$ accurately when vaccine reproduction number is less than one. Also seen that $C_2^2 - 4C_1C_3 > 0$, when $C_3 < 0$. It can be easily concluded that there is precisely one endemic equilibrium when vaccine reproduction number is greater than one, as there are two real roots, and the multiplication of those two roots are negative. On the other hand, we can see that $C_3 > 0$ if vaccine reproduction number is less than one. See that there are precisely two changes in the sign of coefficients of the equation (5.5) if coefficient $C_2 < 0$ and none when $C_2 > 0$. By Descartes' rule of signs one can conclude that the maximum number of endemic equilibrium is two when $C_2 < 0$ and vaccine reproduction number $R(\varphi) < 1$, and if $R(\varphi) < 1$ and $C_2 > 0$ then there is no endemic equilibrium. However, it is explained that if vaccine reproduction number less than one then it is always the case that the system does not have any endemic equilibrium.

Proposition 5.1. *If $R(\varphi) < 1$ then for the model (5.4) with $R(\varphi) = \frac{(\alpha\varphi\sigma+\alpha\mu)(\Lambda+\beta)}{\mu(\mu+\varphi)(\mu+\delta+\lambda)}$, there is no endemic equilibrium.*

Proof. Firstly we consider $R(\varphi) < 1$ and $C_2 < 0$

$$R(\varphi) < 1 \Leftrightarrow (\alpha\varphi\sigma + \alpha\mu)(\Lambda + \beta) < \mu(\mu + \varphi)(\mu + \delta + \lambda) \tag{5.6}$$

$$\begin{aligned} C_2 < 0 \Leftrightarrow\ & \alpha\mu(1 + \mu + \varphi)(\mu + \delta + \lambda) + (\mu + \delta + \lambda)(\mu + \varphi)\alpha\sigma - \alpha^2\sigma(\Lambda + \beta) \\ & - \alpha^2\sigma\varphi(\Lambda + \beta) < 0 \end{aligned} \tag{5.7}$$

Combining equation (5.6) and (5.7), we have

$$(\mu + \delta + \lambda)(\mu^2 + \mu + \mu\varphi + \mu\sigma + \varphi\sigma) < \frac{\sigma\mu(\mu + \varphi)(\mu + \delta + \lambda)}{(\mu + \sigma\varphi)}$$

After some expansion and calculation we have

$$\sigma\varphi\mu + \sigma\varphi\mu^2 + \sigma\varphi^2\mu + \sigma^2\varphi\mu + \sigma^2\varphi^2 + \mu^2 + \mu^3 + \mu^2\varphi < 0$$

This is a contradiction for all non negative parameters. Therefore $R(\varphi) < 1$ and $C_2 < 0$ is impossible. So when $R(\varphi) < 1$ and $C_2 > 0$. Observed that $R(\varphi) < 1$ corresponds to $C_3 > 0$. Also clearly $C_1 > 0$, so by Descartes rule of signs there is no endemic equilibrium for $R(\varphi) < 1$.

6. When There Are No Disease Fatalities But Infective Immigrants

In this section, we assume that there is a constant flow of infective immigrants into host population and no disease fatality i.e. $\delta = 0$ and $\rho \neq 0$. Based on this supposition equation (2.1) becomes:

$$\left.\begin{aligned} \tfrac{dS}{dt} &= (1-\rho)\Lambda + \beta - \tfrac{\alpha SI}{1+\alpha I} - (\mu+\varphi)S \\ \tfrac{dV}{dt} &= \varphi S - \sigma\alpha VI - \mu V \\ \tfrac{dI}{dt} &= \rho\Lambda + \tfrac{\alpha SI}{1+\alpha I} + \sigma\alpha VI - (\mu+\lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\} \tag{6.1}$$

With relation $N(t) = S(t) + V(t) + I(t) + R(t) \Rightarrow \frac{dN}{dt} = \Lambda + \beta - \mu N$
By using theory of autonomous system $\lim_{t\to\infty} \frac{dN}{dt} = 0$ so $\lim_{t\to\infty} N(t) = \frac{\Lambda+\beta}{\mu} = \xi$ (say)

So replacing S by $\xi - I - V - R$, we reduced system of equations

$$\left.\begin{aligned} \tfrac{dV}{dt} &= \varphi(\xi - I - V - R) - \sigma\alpha VI - \mu V \\ \tfrac{dI}{dt} &= \rho\Lambda + \tfrac{\alpha I}{1+\alpha I}(\xi - V - I - R) + \alpha\sigma VI - (\mu+\lambda)I \\ \tfrac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\} \tag{6.2}$$

6.1. Equilibrium conditions

Endemic equilibrium conditions are

$$\varphi(\xi - I - V - R) = \sigma\alpha VI + \mu V \Rightarrow \xi - I - V - R = \frac{\sigma\alpha VI + \mu V}{\varphi}$$

$$\frac{\rho\Lambda}{I} = (\mu+\lambda) - \frac{\alpha}{1+\alpha I}(\xi - V - I - R) - \alpha\sigma V,\ \xi = \frac{\Lambda+\beta}{\mu}$$

By using above conditions, we get
$R^* = \frac{\lambda I}{\mu}$, $V^* = \frac{\varphi(\Lambda+\beta)-\varphi I(\mu+\lambda)}{\mu+\varphi+\sigma\alpha I}$ and I is given by following relation
$f(I) = D_1I^3 + D_2I^2 + D_3I + D_4 = 0$, where

$$\begin{aligned} D_1 &= 2\alpha^2\sigma\varphi(\mu+\lambda) \\ D_2 &= \alpha\varphi\big[(\mu+\lambda)(\mu+\varphi) + (\mu+\lambda)\sigma + (\mu+\alpha\sigma\varphi)(\mu+\lambda) - \rho\Lambda\alpha\sigma - \alpha\sigma(\Lambda+\beta)\big] \\ D_3 &= \varphi(\mu+\lambda)(\mu+\varphi) - \rho\Lambda\alpha\varphi\sigma - (\mu+\varphi)\rho\Lambda\alpha\varphi - \alpha\varphi(\mu+\sigma\alpha\varphi)(\Lambda+\beta) \\ &\quad - \rho\Lambda\varphi(\mu+\varphi) \\ D_4 &= -\rho\Lambda\varphi(\mu+\varphi) \end{aligned}$$

Since $D_4 < 0$ for positive parameter, then $f(0) < 0$ and $\lim_{t\to\infty} f(I) = \infty$ so that $\exists$ one or three positive roots I^*
We have $\frac{df(I)}{dt} = 3D_1I^2 + 2D_2I + D_3$.
If $f(I) = 0$ has three different positive roots then $\frac{df(I)}{dt} = 0$ necessity have two different roots by using Rolle's Theorem. $D_2 < 0$ and $D_3 > 0$ is necessary condition to three positive endemic equilibrium.

7. Stability Analysis

7.1. Local Stability

In this section, the local stability analysis of disease-free and endemic equilibrium is discussed.

Theorem 7.1. *The disease-free equilibrium E^0 is locally asymptotically stable when the basic reproduction number $R_0 < 1$, and is unstable when $R_0 > 1$.*

Proof. The Jacobean matrix of the system (1) at $E^0(S,V,I,R) = E^0\left(\frac{\Lambda+\beta}{\mu+\varphi}, \frac{\Lambda+\beta}{\mu(\mu+\varphi)}, 0, 0\right)$ is given by

$$J(E^0) = \begin{bmatrix} -(\mu+\varphi) & 0 & -\frac{\alpha(\Lambda+\beta)}{(\mu+\varphi)} & 0 \\ \varphi & -\mu & -\frac{\alpha\sigma(\Lambda+\beta)}{\mu(\mu+\varphi)} & 0 \\ \frac{\alpha I}{1+\alpha I} & 0 & \frac{\alpha(\Lambda+\beta)}{(\mu+\varphi)} + \frac{\alpha\sigma\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)} - (\mu+\delta+\lambda) & 0 \\ 0 & 0 & \lambda & -\mu \end{bmatrix}$$

The characteristic equation of $J(E^0)$ is

$$\begin{vmatrix} -(\mu+\varphi)-\lambda_1 & 0 & -\frac{\alpha(\Lambda+\beta)}{(\mu+\varphi)} & 0 \\ \varphi & -\mu-\lambda_1 & -\frac{\alpha\sigma(\Lambda+\beta)}{\mu(\mu+\varphi)} & 0 \\ \frac{\alpha I}{1+\alpha I} & 0 & \frac{\alpha(\Lambda+\beta)}{(\mu+\varphi)} + \frac{\alpha\sigma\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)} - (\mu+\delta+\lambda) - \lambda_1 & 0 \\ 0 & 0 & \lambda & -\mu-\lambda_1 \end{vmatrix} = 0$$

$$\Rightarrow (\mu+\varphi+\lambda_1)(\mu+\lambda_1)^2\left\{\frac{\alpha(\Lambda+\beta)}{(\mu+\varphi)} + \frac{\alpha\sigma\varphi(\Lambda+\beta)}{\mu(\mu+\varphi)} - (\mu+\delta+\lambda) - \lambda_1\right\}$$

Clearly, the three eigenvalues have strictly negative real parts and fourth eigenvalue is given by $\lambda_1 = (\mu+\delta+\lambda)\left\{\frac{(\Lambda+\beta)(\alpha\mu+\alpha\sigma\varphi)}{\mu(\mu+\varphi)(\mu+\delta+\lambda)} - 1\right\}$. Fourth eigenvalue will negative if $R(\varphi) < 1$. Hence the system (2.1) is locally asymptotically stable at disease-free equilibrium E^0 if $R(\varphi) < 1$, and unstable if $R(\varphi) > 1$.

Theorem 7.2. *If the basic reproduction number $R(\varphi) > 1$, then the endemic equilibrium point $E^*(S^*, V^*, I^*, R^*)$ is locally asymptotically stable.*

Proof. The original system (2.1) can be reduced to the following system replacing S by $\xi - I - V - R$, by using equation (6.2).

$$\left.\begin{aligned} \frac{dV}{dt} &= \varphi(\xi - I - V - R) - \sigma\alpha VI - \mu V \\ \frac{dI}{dt} &= \rho\Lambda + \frac{\alpha I}{1+\alpha I}(\xi - V - I - R) + \alpha\sigma VI - (\mu+\lambda)I \\ \frac{dR}{dt} &= \lambda I - \mu R \end{aligned}\right\}$$

Jacobean matrix of the above system at E^* is given by

$$J = \begin{bmatrix} -(\alpha\sigma I^* + \mu + \varphi) & -\alpha\sigma V^* - \varphi & \varphi \\ \alpha\sigma I^* - \frac{\alpha I^*}{1+\alpha I^*} & \alpha\sigma V^* + \frac{\alpha(\xi - I^* - V^* - R^*)}{(1+\alpha I^*)} - \frac{\alpha^2 I^*}{(1+\alpha I^*)^2} - \frac{\alpha I^*}{1+\alpha I^*} - (\mu+\lambda) & -\frac{\alpha I^*}{1+\alpha I^*} \\ 0 & \lambda & -\mu \end{bmatrix}$$

Using equilibrium condition $\frac{\rho\Lambda}{I} = (\mu+\lambda) - \alpha\sigma V - \frac{\alpha(\xi-I-V-R)}{1+\alpha I}$, we have

$$J = \begin{bmatrix} -(\alpha\sigma I^* + \mu + \varphi) & -\alpha\sigma V^* - \varphi & -\varphi \\ \alpha\sigma I^* - \frac{\alpha I^*}{1+\alpha I^*} & -\frac{\rho\Lambda}{I^*} - \frac{\alpha^2 I^*}{(1+\alpha I^*)^2} - \frac{\alpha I^*}{1+\alpha I^*} & -\frac{\alpha I^*}{1+\alpha I^*} \\ 0 & \lambda & -\mu \end{bmatrix}$$

The characteristic equation is given by $\lambda_1^3 + E_1\lambda_1^2 + E_2\lambda_1 + E_3 = 0$
Where $E_1 = 2\mu + \varphi + \sigma\alpha I + \frac{\rho\Lambda}{I^*} + \frac{\alpha I^*}{1+\alpha I^*} + \frac{\alpha^2 I^*}{(1+\alpha I^*)^2}$

$$E_2 = (2\mu + \varphi + \sigma\alpha I)\left\{\frac{\rho\Lambda}{I^*} + \frac{\alpha I^*}{1+\alpha I^*} + \frac{\alpha^2 I^*}{(1+\alpha I^*)^2}\right\} + \mu(\alpha\sigma I^* + \mu + \varphi) + \frac{\lambda\alpha I^*}{1+\alpha I^*}$$
$$+ (\alpha\sigma V^* + \varphi)\left(\alpha\sigma I^* - \frac{\alpha I^*}{1+\alpha I^*}\right)$$

$$E_3 = (\alpha\sigma I^* + \mu + \varphi)\left[\left\{\frac{\rho\Lambda}{I^*} + \frac{\alpha I^*}{1+\alpha I^*} + \frac{\alpha^2 I^*}{(1+\alpha I^*)^2}\right\}\mu + \frac{\lambda\alpha I^*}{1+\alpha I^*}\right]$$
$$+ \left(\alpha\sigma I^* - \frac{\alpha I^*}{1+\alpha I^*}\right)\left\{\mu(\alpha\sigma V^* + \varphi) + \lambda\varphi\right\}$$

It is easy to seen $E_1 > 0$, and $E_2, E_3 > 0$ if $\frac{\alpha I^*}{1+\alpha I^*} < 0$, and $E_1E_2 - E_3 > 0$. Hence E^* is locally asymptotically stable (by Routh-Hurwitz Theorem).

7.2. Global Stability

In this section, the Lyapunov function analyzes the global stability of the disease-free and endemic equilibrium. Disease-free stability is presented in the following form.

Theorem 7.3. *The disease free equilibrium of the model (2.1) is globally asymptotically stable if $R(\varphi) < 1$.*

Proof. To prove this result, we construct the following Lyapunov function:

$$L = k_1(S - S^0) + k_2(V - V^0) + k_3 I + k_4 R \tag{7.1}$$

where k_1, k_2, k_3, k_4 are positive constant to be determined later. By differentiating equation (7.1) with respect to time, we get

$$L' = k_1\left[(1-\rho)\Lambda + \beta - \frac{\alpha SI}{1+\alpha I} - (\mu+\varphi)S\right] + k_2[\varphi S - \sigma\alpha VI - \mu V]$$
$$+ k_3\left[\rho\Lambda + \frac{\alpha SI}{1+\alpha I} + \sigma\alpha VI - (\mu+\delta+\lambda)I\right] + k_4[\lambda I - \mu R]$$

By some arrangements, we get

$$L^{'} = \frac{\alpha SI}{1+\alpha I}(k_3 - k_1) + \sigma\alpha VI(k_3 - k_2) + S\{\varphi k_2 - (\mu+\varphi)k_1\} - k_2\mu V + k_1\{(1-\rho)\Lambda + \beta\} + k_3\rho\Lambda - k_3(\mu+\delta+\lambda)I$$

Let us select the constants $k_1 = k_2 = k_3 = 1$, then we get

$$L^{'} = -\big[(\mu N - \Lambda - \beta) + \delta I\big]$$

$$L^{'} = \frac{\mu(\mu+\varphi)(\mu+\delta+\lambda)}{(\alpha\varphi\sigma+\alpha\mu)}\left[\frac{\mu N(\alpha\varphi\sigma+\alpha\mu)}{\mu(\mu+\varphi)(\mu+\delta+\lambda)} - R(\varphi)\right] - \delta I < 0$$

Since all parameters are positive and $R(\varphi) < 1$ then $L^{'} < 0$. Thus, by Lasalle's invariant principle (Lasalle [5]), the disease free equilibrium E^0 of the system (2.1) is globally asymptotically stable in $\mathfrak{R}$, if $R(\varphi) < 1$.

8. Numerical Solution

We observed that vaccine reproduction number $R(\varphi)$ plays an important role to control the disease. Our main results show that if $R(\varphi) < 1$, the disease-free equilibrium is globally stable and if $R(\varphi) > 1$ then the endemic equilibrium endures and is globally stable. Besides this analytical study, we provide some numerical solutions as under:

8.1. Case I: For diseases free equilibrium

If the parameters are chosen as follows:
$\alpha = 0.3, \Lambda = 0.009, \delta = 0.01, \beta = 0.0002, \sigma = 0.09\rho = 0.5, \varphi 0.009$ and $\lambda = 0.01$ then by computing, vaccine reproduction number $R(\varphi) = 0.2557 < 1$ and system (2.1) has a disease-free equilibrium. In this case, $S(t)$ and $V(t)$ tends to its steady-state value while $I(t)$ and $R(t)$ tends to zero $t \to \infty$ a. Hence the disease disappeared in population and dies out. The numerical simulation is shown in Figure 1(a) to 1(d). It follows that E^0 is globally asymptotically stable.

8.2. Case II: For endemic equilibrium

If the parameters are chosen as follows:

$\alpha = 0.3, \Lambda = 0.09, \mu = 0.09, \delta = 0.001, \beta = 0.09, \sigma = 0.09\rho = 0.5, \varphi 0.08$ and $\lambda = 0.01$ then by computing, Vaccine reproduction number $R(\varphi) = 3.396622 > 1$ and system (2.1). It is seen that all the component $S(t), V(t), I(t)$ and $R(t)$ tends to their steady-state values as $t \to \infty$, the diseases becomes endemic (See in Figure

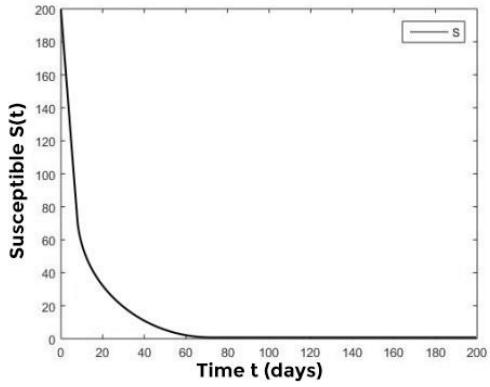

Figure: 1(a)

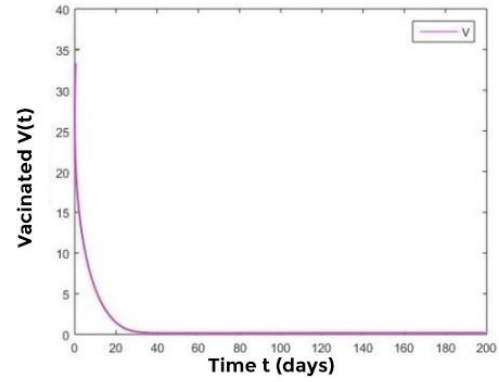

Figure: 1(b)

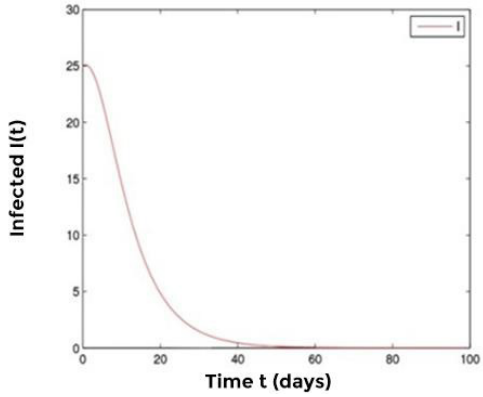

Figure: 1(c)

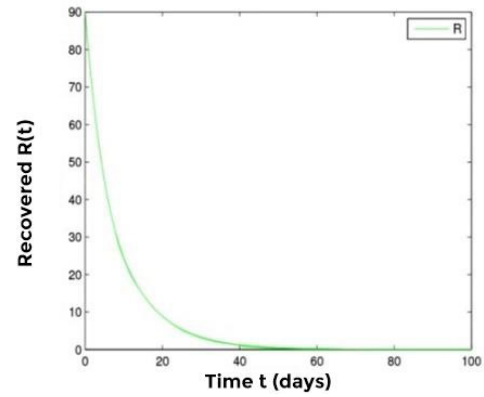

Figure: 1(d)

Figure 1: For diseases free equilibrium

2(a)-2(d)).

9. Conclusions

The mathematical analysis of the SVIR epidemic model with non-linear incidence has been presented. We have constructed a compartmental model with vaccination and investigated their dynamical behaviors. Utilizing the Jacobean matrix, we obtained their vaccine reproductive number $R(\varphi)$ and basic reproduction number R_0, which play a significant role. Vaccination is helpful for disease control by decreasing the basic reproduction number. It has been observed that the DFE is locally asymptotically stable if and only if the vaccine reproductive number $R(\varphi) < 1$. When the vaccine is ineffective or completely useful, and there

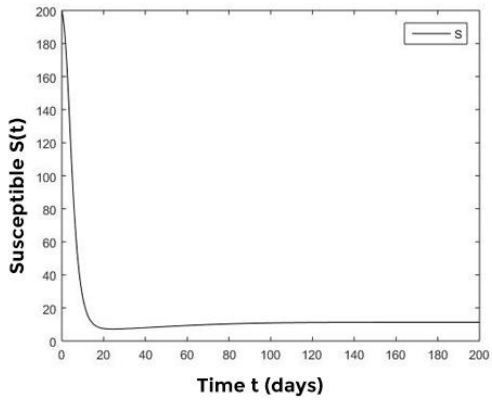

Figure: 2(a)

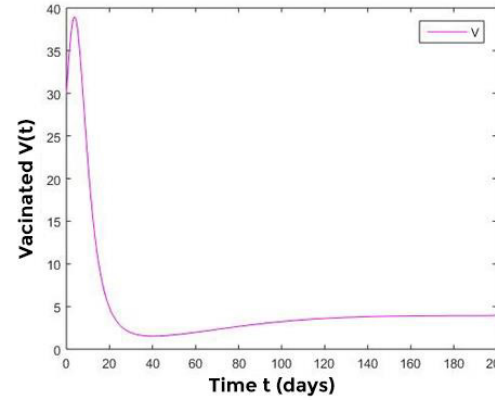

Figure: 2(b)

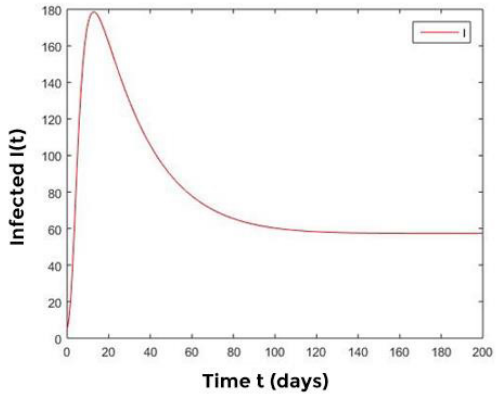

Figure: 2(c)

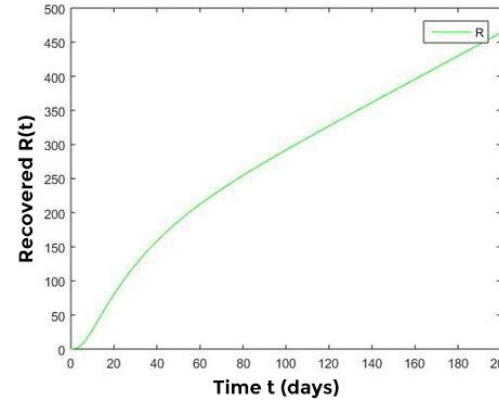

Figure: 2(d)

Figure 2: For endemic equilibrium

is no infective emigrant, the endemic equilibrium exists if and only if $R_0 > 1$. With infective immigrants, but when there are no disease fatalities, we give the condition for endemic equilibrium and their stability condition. This work also provides a considerable role in the correlation of mathematical modelling and dynamical aspects of some specific epidemic diseases.

References

[1] Duan X., Yuan S., and Li X., Global stability of an SVIR model with age of vaccination, Applied Mathematics and Computation, 226, (2014), 528–540.

[2] Gabbuti, L. Romanò, P. Blanc et al., Long-term immunogenicity of hepatitis

B vaccination in a cohort of Italian healthy adolescents, Vaccine, 25(16), (2007), 3129–3132.

[3] Islam M. S., Equilibriums and stability of an SVIR Epidemic Model, BEST: International Journal of Humanities, Arts, Medicine and Sciences, 3(1), (2015), 1-10.

[4] Khan M. A, Ali Z., Dennis C., Khan I, Islam S, Ullah M., Gul T., Stability Analysis of an SVIR Epidemic Model with Non-linear Saturated Incidence Rate, HIKARI Ltd, Applied Mathematical Sciences, 9(23), (2015), 1145-1158.

[5] Lasalle J. P., The stability of dynamical system, SIAM, Philadelphia, PA, (1976).

[6] Liu X.,. Takeuchi Y., and Iwami S., Theoret J., SVIR epidemic models with vaccination strategies. Biol. 253, (2008), 1–11.

[7] McCluskey C. C., Global stability for an SEIR epidemiological model with varying infectivity and infinite delay, Mathematical Bioscience Engineering, 6(3), (2009), 603–610.

[8] Röst G. and Wu J., SEIR epidemiological model with varying infectivity and infinite delay, Mathematical Bioscience Engineering, 5, (2008), 389–402.

[9] Wang J., Huang G., Takeuchi Y., and Liu S., SVEIR epidemiological model with varying infectivity and distributed delays, Mathematical Bioscience. Eng. 8, (2011), 875–888.

[10] Wang J., R. Zhang, and Kuniya T., The dynamics of an SVIR epidemiological model with infection age, IMA Journal of Applied Mathematics, 81(2), (2015), 321-343.

[11] Webb G. F., Theory of Nonlinear Age-Dependent Population Dynamics, Marcel Dekker, New York, (1985).