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A MODERN APPROACH BASED ON BERNSTEIN POLYNOMIAL MULTIWAVELETS TO SOLVE FREDHOLM INTEGRAL AND SYSTEM OF FREDHOLM INTEGRAL EQUATIONS

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Abstract: The objective of this paper is to obtain the approximate solution of integral and system of integral equations using Bernstein Polynomial Multiwavelets (BPMW). BPMW are used to obtain the approximate solution of Fredholm integral and system of Fredholm integral equations. These BPMW reduces the given equations into a system of linear (or nonlinear) algebraic equations, which are solved by appropriate methods. Error estimate of the proposed method is given. To illustrate our numerical findings a number of computational experiments are carried out.

Keywords and Phrases: Berstein polynomials, Berstein polynomial multiwavelets, Linear Fredholm integral equations, Nonlinear Fredholm integral equations, System of Fredholm integral equations.

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1. Introduction

Integral equations have been one of the essential tools in different areas of applied mathematics. Integral equations are extensively involved in several problems in science and technology [1, 9, 22]. In several physical models and fields of engineering, such as spectroscopy, radiography, image processing, and cosmic radiation Fredholm integral exist.

Many specific orthonormal basis functions have been used in recent years, such as Fourier functions, wavelets etc., to obtain approximate the solution of these integral equations. The most attractive, however, can be the wavelet bases, in which the kernel can be represented as a sparse matrix, particularly for large-scale problems. It is largely due to its outstanding locality and high-order moment-disappearing properties. In traditional wavelet-based methods for solving integral equations, the inner products of wavelets or the related scaling functions with normal square-integrable functions must be determined sooner or later. Since many forms of wavelets are not necessarily smooth, specific rules on quadrature are essential. This can be difficult and time consuming at times, particularly for those problems involving singular wavelet integrals and partial support [3, 5, 11]. Wavelets have been used in the last two decades to solve integral equations and they have made a lot of good approximations [10, 12, 13, 14, 15, 16] due to the MRA property (for detailed study of MRA see [21]).

We are concerned in this article with the application of Berstein polynomial multiwavelets (BPMW) [21] to find the approximate solution of linear, nonlinear, and system of linear Fredholm integral equations. We find many applications of Berstein polynomial multiwavelets. Some of them are found in [24, 17, 19, 20].

We consider the following linear, nonlinear, and system of linear Fredholm integral equation:

$$g(x) = u(x) + \int_a^b k_1(x,t)g(t)dt, \tag{1}$$

$$g(x) = u(x) + \int_a^b k_1(x,t)[g(t)]^p dt,\ p > 0, \tag{2}$$

and

$$G(x) = U(x) + \int_0^1 K(x,t)g(t)dt, \tag{3}$$

where, $G(x) = [g_1(x), g_2(x), ..., g_n(x)]$, $U(x) = [u_1(x), u_2(x), ..., u_n(x)]$, and $K(x,t) = [k_{i,j}(x,t)]$, $i, j = 1, 2, ..., n$.

This article is structured as follows: Bernstein Polynomial Multiwavelets and its function approximation are studied in section 2. In section 3 method of solution is given. Error estimate is studied in section 4. In section 5, to explain the efficiency of the proposed method, numerical examples are given. Finally, the conclusion is drawn in section 6.

2. Bernstein Polynomial Multiwavelets and Function Approximation

2.1. Bernstein Polynomial Multiwavelets(BPMW)

BPMW $\psi_{n,m}(x) = \psi(k, n, m, x)$ have four arguments: $n = 0, 1, ..., 2^k - 1$, k is assumed to be any positive integer, m is the order of Bernstein polynomials and x

is the normalized time. BPMW [21] are defined on the interval $[0,1)$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^{\frac{k}{2}} WB_m(2^k x - n), & \frac{n}{2^k} \leq x < \frac{n+1}{2^k}, \\ 0, & \text{otherwise.} \end{cases} \tag{4}$$

where, $m = 0, 1, ..., M$. The Berstein polynomials $B_m(x)$ of degree m are defined on the interval $[0,1)$ as

$$B_{i,m}(x) = \binom{m}{i} x^i (1-x)^{m-i}, \ \ i = 0, 1, ..., m. \tag{5}$$

Berstein polynomials are also recursively defined on the interval $[0,1)$ as,

$$B_{i,m}(x) = (1-x) B_{i,m-1}(x) + x B_{i-1,m-1}(x). \tag{6}$$

In equation (4), WB_m is the orthonormal form of Berstein polynomials of order m. These orthonormal form of Berstein polynomials are obtained by using Gram-Schmidt orthonormalization process on Berstein polynomials [21] $B_{i,m}(x)$. For instance, for $M = 3$, orthonormal polynomials are given by,

$$\begin{aligned} WB_0(x) &= \sqrt{7}\left[(1-x)^3\right], \\ WB_1(x) &= 2\sqrt{5}\left[3x(1-x)^2 - \frac{1}{2}(1-x)^3\right], \\ WB_2(x) &= \frac{10\sqrt{3}}{3}\left[3x^2(1-x) - 3x(1-x)^2 + \frac{3}{10}(1-x)^3\right], \end{aligned}$$

and

$$WB_3(x) = 4\left[x^3 - \frac{9}{2}x^2(1-x) + 3x(1-x)^2 - \frac{1}{4}(1-x)^3\right].$$

2.2. Function Approximation

Suppose $u(x) \in [0,1)$ is expanded in terms of the BPMW as

$$u(x) = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} g_{n,m} \psi_{n,m}(x) = G^T \psi(x). \tag{7}$$

Truncating the above infinite series, we get

$$u(x) = \sum_{m=0}^{M-1} \sum_{n=1}^{2^{k-1}} g_{n,m} \psi_{n,m}(x) = G^T \psi(x), \tag{8}$$

where G and $\psi(x)$ are $\hat{m} \times 1$ ($\hat{m} = (2^k - 1)(2M + 2)$) vectors given by:

$$G = \left[g_{0,0}, g_{0,1}, .., g_{0,M}, g_{1,0}, .., g_{1,M}, .., g_{2^k-1,0}, .., g_{2^k-1,M}\right]^T, \tag{9}$$

and

$$\psi(x) = \left[\psi_{0,0}, \psi_{0,1}, .., \psi_{0,M}, \psi_{1,0}, .., \psi_{1,M}, .., \psi_{2^k-1,0}, .., \psi_{2^k-1,M}\right]^T. \tag{10}$$

Using the collocation point $x_j = \frac{j-0.5}{\hat{m}}$, equation (6) reduces to $\hat{m} \times \hat{m}$ BPMW coefficient matrix. For instance, for $k = 1$ and $M = 2$, we get

$$\psi(x) = \begin{bmatrix} \psi_{0,0}(x) \\ \psi_{0,1}(x) \\ \psi_{0,2}(x) \\ \psi_{1,0}(x) \\ \psi_{1,1}(x) \\ \psi_{1,2}(x) \end{bmatrix} = \begin{bmatrix} 2.1653 & 0.4677 & 0.0173 & 0 & 0 & 0 \\ 0.3660 & 1.9764 & 0.4246 & 0 & 0 & 0 \\ -0.8505 & 0.3062 & 2.2794 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2.1653 & 0.4677 & 0.0173 \\ 0 & 0 & 0 & 0.3660 & 1.9764 & 0.4246 \\ 0 & 0 & 0 & -0.8505 & 0.3062 & 2.2794 \end{bmatrix}.$$

3. Method of solution

3.1. Linear Fredholm Integral Equations

Let us consider the following linear Fredholm integral equation:

$$g(x) = u(x) + \int_a^b k_1(x,t)g(t)dt. \tag{11}$$

For the sake of simplicity, without loss of generality, we assume that $(a, b) = (0, 1)$. Approximating $g(x)$, $u(x)$ and $k_1(x, t)$, with respect to BPMW as follows:

$$g(x) \simeq G^T\psi(x) = G\psi^T(x), \tag{12}$$

where G is given in equation (9) and is the unknown vector to be determined.

$$u(x) \simeq U^T\psi(x) = U\psi^T(x), \tag{13}$$

and

$$k_1(x,t) \simeq \psi^T(x)K_1\psi(t) = \psi^T(t)K_1^T\psi(x), \tag{14}$$

where G and U are BPMW coefficient vectors and K_1 is the BPMW matrix. Substituting (12), (13) and (14) in (11), we get

$$G^T\psi(x) = U^T\psi(x) + G^T\left(\int_0^1 \psi(t)\psi^T(t)dt\right)K_1\psi(x).$$

Using the relation $\int_0^1 \psi(t)\psi^T(t)dt = 1$, we get

$$G^T\psi(x) = U^T\psi(x) + G^T K_1\psi(x),$$

hence

$$G^T - G^T K_1 = U^T.$$

That is:

$$G^T = (I - K_1)^{-1}U^T, \tag{15}$$

where I is a identity matrix of order $\hat{m} \times \hat{m}$. Solving this linear system of equations, we get the unknown vector G. Substituting this unknown vector in equation (12), we get the BPMW solution of the linear Fredholm integral equation given in equation (11).

3.2. Nonlinear Fredholm Integral Equations

Let us consider the following nonlinear Fredholm integral equation:

$$g(x) = u(x) + \int_a^b k_1(x,t)[g(t)]^p dt,\ p > 0. \tag{16}$$

For the sake of simplicity, without loss of generality, we assume that $(a, b) = (0, 1)$. Approximating $g(x)$, $u(x)$ and $k_1(x,t)$, and $[g(x)]^p$ with respect to BPMW as follows:

$$g(x) \simeq G^T\psi(x) = G\psi^T(x), \tag{17}$$

where G is given in equation (9) and is the unknown vector to be determined.

$$u(x) \simeq U^T\psi(x) = U\psi^T(x), \tag{18}$$

$$k_1(x,t) \simeq \psi^T(x)K_1\psi(t) = \psi^T(t)K_1^T\psi(x), \tag{19}$$

and

$$[g(t)]^p \simeq (G^*)^T\psi(x) = (G^*)\psi^T(x), \tag{20}$$

where G and U are BPMW coefficient vectors, K_1 is the BPMW matrices and G^* is a column vector function of the elements of vector G. Substituting (17), (18), (19) and (20) in (16), we get

$$G^T\psi(t) = U^T\psi(x) + (G^*)^T\left(\int_0^1 \psi(t)\psi^T(t)dt\right)K_1\psi(x).$$

Using the relation $\int_0^1 \psi(t)\psi^T(t)dt = 1$, we get

$$G^T\psi(x) = U^T\psi(x) + (G^*)^T K_1\psi(x),$$

hence

$$G^T - (G^*)^T K_1 = U^T. \tag{21}$$

Solving this nonlinear system of equations, we get the unknown vector G. Substituting this unknown vector in equation (17), we get the BPMW solution of the nonlinear Fredholm integral equation given in equation (16).

3.3. System of Fredholm Integral Equations

Let us consider the following system of linear integral equation:

$$G(x) = U(x) + \int_0^1 K(x,t)g(t)dt, \tag{22}$$

where,

$$G(x) = [g_1(x), g_2(x), ..., g_n(x)]\,, \tag{23}$$

$$U(x) = [u_1(x), u_2(x), ..., u_n(x)]\,, \tag{24}$$

and

$$K(x,t) = [k_{i,j}(x,t)]\,,\;\; i,j = 1,2,...,n. \tag{25}$$

Substituting (23), (24), and (25) in (22), we get

$$g_i(x) = u_i(x) + \int_0^1 \sum_{j=1}^{n} k_{i,j}(x,t)g_j(t)dt,\;\; i,j = 1,2,...,n. \tag{26}$$

where, $u_i \in L^2[0,1)$, $k_{i,j} \in L^2([0,1) \times [0,1))$, and g_i is the unknown function. Now, we approximate u_i, $k_{i,j}$, and g_i with respect to BPMW as follows:

$$g_i(x) \simeq G_i^T\psi(x) = G_i\psi^T(x), \tag{27}$$

$$u_i(x) \simeq U_i^T\psi(x) = U_i\psi^T(x), \tag{28}$$

and

$$k_{i,j}(x,t) \simeq \psi^T(x)K_{i,j}\psi(t) = \psi^T(t)K_{i,j}^T\psi(x). \tag{29}$$

Substituting (27), (28), and (29) in (26), we get

$$\begin{aligned} G_i\psi^T(x) &= U_i\psi^T(x) + \int_0^1 \sum_{j=1}^{n} \psi^T(x)K_{i,j}\psi(t)G_j\psi^T(t)dt \\ &= U_i\psi^T(x) + \left[\sum_{j=1}^{n} K_{i,j}\left(\int_0^1 \psi(t)\psi^T(t)dt\right)G_j\right]\psi^T(x) \\ &= U_i\psi^T(x) + \left(\sum_{j=1}^{n} K_{i,j}G_j\right)\psi^T(x). \end{aligned} \tag{30}$$

And this gives the linear system of equations,

$$G_i = U_i + \sum_{j=1}^{n} K_{i,j}G_j. \tag{31}$$

Solving this linear system of equations, we get the unknown vector G_i. Substituting this unknown vector in equation (27), we get the BPMW solution of the system of linear Fredholm integral equation given in equation (22).

4. Error Estimate

We compare the approximate solution and exact solution of the given equation at the some selected points via the definition of absolute error defined as:

$$e(x) = |g(x) - g^*(x)|, \ \ x \in [0, 1),$$

where, $g(x)$ and $g^*(x)$ denote the exact and approximate solution of the given equation.

5. Numerical Experiments

Test Problem 1. Let us consider the linear Fredholm integral equation [6]

$$g(x) = e^{(2x+\frac{1}{3})} - \frac{1}{3}\int_0^1 e^{(2x-\frac{5}{3}t)}g(t)dt, \tag{32}$$

with exact solution

$$g(x) = e^{2x}. \tag{33}$$

By implying the method described in section 3.1, we obtain the approximate solution of test problem 1. Absolute errors for different values of $\hat{m}$ are shown and compared with those of absolute errors of Jacobi wavelets (JW) solution defined in

[4] in table 1 and figure 1 shows the exact and approximate solution of test problem 1 for $\hat{m} = 8$.

Table 1: Absolute errors of test problem 1 for different values of $\hat{m}$.

	BPMW solution			JW solution		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	4.4883e-01	4.5601e-02	2.7549e-05	5.8273e-03	1.1743e-02	1.1581e-02
0.1	4.7757e-01	4.7592e-02	2.8891e-05	6.2006e-03	1.2255e-02	1.2145e-02
0.2	5.7925e-01	5.8459e-02	3.5155e-05	7.5211e-03	1.5054e-02	1.4778e-02
0.3	7.0506e-01	7.1475e-02	4.2922e-05	9.1547e-03	1.8406e-02	1.8043e-02
0.4	8.5129e-01	8.6641e-02	5.2619e-05	1.1053e-02	2.2312e-02	2.2120e-02
0.5	1.0587e-00	1.0681e-01	6.4363e-05	1.3747e-02	2.7505e-02	2.7057e-02
0.6	1.0587e-00	1.0681e-01	6.4363e-05	1.3747e-02	2.7505e-02	2.7057e-02
0.7	1.5745e-00	1.5891e-01	9.5559e-05	2.0445e-02	4.0922e-02	4.0170e-02
0.8	1.9166e-00	1.9429e-01	1.1667e-04	2.4885e-02	5.0033e-02	4.9046e-02
0.9	2.2642e-00	2.3552e-01	1.4303e-04	2.9399e-02	6.0649e-02	6.0127e-02

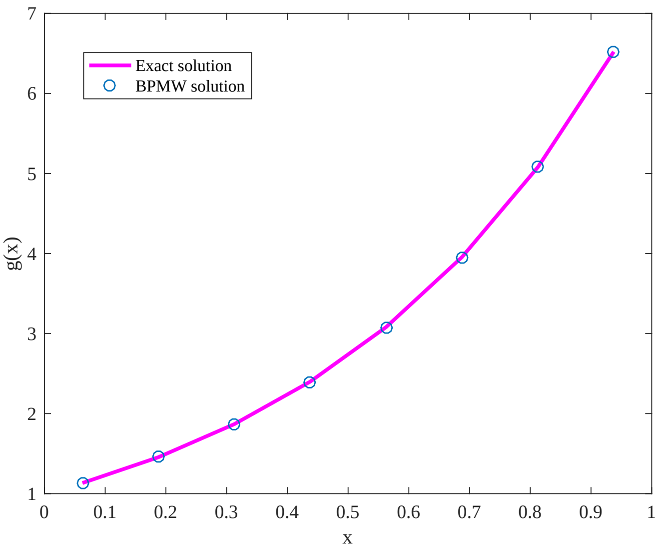

Figure 1: Exact and approximate solution of test problem 1 for $\hat{m} = 8$.

Test Problem 2. Let us consider the linear Fredholm integral equation [7]

$$g(x) = e^x + 2\int_0^1 e^{x+t} g(t)dt, \tag{34}$$

with exact solution

$$g(x) = \frac{e^x}{2 - e^2}. \tag{35}$$

By implying the method described in section 3.1, we obtain the approximate solution of test problem 2. Absolute errors for different values of $\hat{m}$ are shown and compared with those of absolute errors of Jacobi wavelets (JW) solution defined in [4] in table 2 and figure 2 shows the exact and approximate solution of test problem 2 for $\hat{m} = 8$.

Table 2: Absolute errors of test problem 2 for different values of $\hat{m}$.

	BPMW solution			JW solution		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	1.5139e-01	2.0003e-02	4.5014e-06	3.4913e-03	2.3081e-03	2.2994e-03
0.1	1.6529e-01	2.1830e-02	4.9166e-06	3.8119e-03	2.5189e-03	2.5115e-03
0.2	1.8336e-01	2.4163e-02	5.4285e-06	4.2286e-03	2.7881e-03	2.7729e-03
0.3	2.0255e-01	2.6708e-02	5.9992e-06	4.6713e-03	3.0817e-03	3.0645e-03
0.4	2.2311e-01	2.9465e-02	6.6363e-06	5.1454e-03	3.3997e-03	3.3899e-03
0.5	2.4776e-01	3.2638e-02	7.3367e-06	5.7139e-03	3.7658e-03	3.7477e-03
0.6	2.4776e-01	3.2638e-02	7.3367e-06	5.7139e-03	3.7658e-03	3.7477e-03
0.7	3.0230e-01	3.9839e-02	8.9501e-06	6.9718e-03	4.5968e-03	4.5719e-03
0.8	3.3394e-01	4.4034e-02	9.8913e-06	7.7017e-03	5.0809e-03	5.0524e-03
0.8	3.6618e-01	4.8578e-02	1.0941e-05	8.4451e-03	5.6052e-03	5.5890e-03

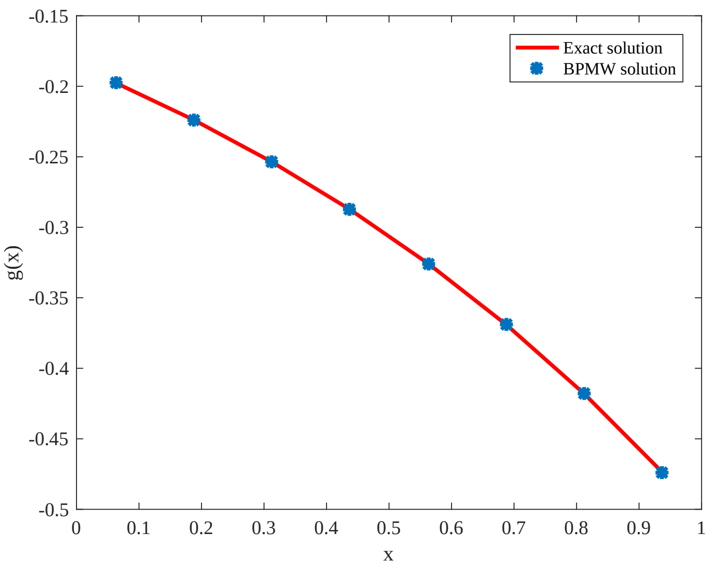

Figure 2: Exact and approximate solution of test problem 2 for $\hat{m} = 8$.

Test Problem 3. Let us consider the nonlinear Fredholm integral equation [23]

$$g(x) = \frac{5}{6}x^2 - \frac{8}{105}x - 1 + \int_0^1 (x^2t + xt^2)[g(t)]^2 dt, \tag{36}$$

with exact solution

$$g(x) = x^2 - 1. \tag{37}$$

By implying the method described in section 3.2, we obtain the approximate solution of test problem 3. Absolute errors for different values of $\hat{m}$ are shown and compared with those of absolute errors of Jacobi wavelets (JW) solution defined in [4] in table 3 and figure 3 shows the exact and approximate solution of test problem 3 for $\hat{m} = 8$.

Table 3: Absolute errors of test problem 3 for different values of $\hat{m}$.

	BPMW solution			JW solution		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	1.3279e-05	7.9331e-06	7.6488e-11	2.7491e-01	4.5237e-01	6.0593e-01
0.1	2.7325e-04	1.6749e-03	3.3285e-06	3.7554e-02	4.2845e-01	5.7897e-01
0.2	6.1726e-03	3.7312e-03	7.1657e-06	2.5997e-01	6.1619e-01	4.8478e-01
0.3	1.3236e-02	6.0901e-03	1.1753e-05	5.9833e-01	7.5002e-01	7.9734e-01
0.4	2.2419e-02	8.7514e-03	1.7089e-05	7.9612e-01	8.2995e-01	8.2560e-01
0.5	3.7956e-02	1.1919e-02	2.3068e-05	5.7221e-01	4.5864e-01	4.4533e-01
0.6	3.7956e-02	1.1919e-02	2.3068e-05	5.7221e-01	4.5864e-01	4.4533e-01
0.7	7.5385e-02	1.9180e-02	3.6954e-05	2.9733e-01	3.8741e-01	4.7960e-01
0.8	9.9394e-02	2.3374e-02	4.4968e-05	3.0400e-01	4.2931e-01	4.5677e-01
0.9	1.2457e-01	2.7871e-02	5.3732e-05	1.4912e-01	2.4448e-01	2.4433e-01

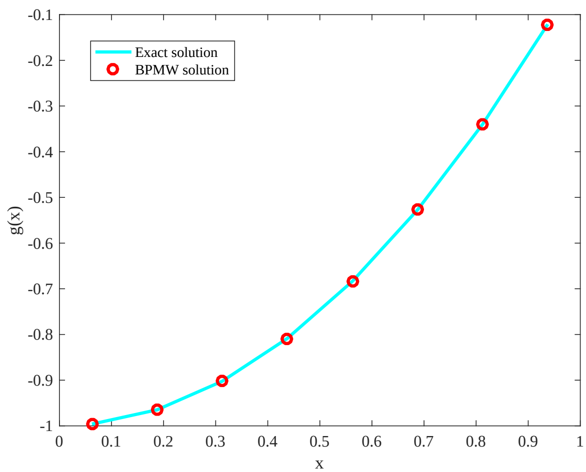

Figure 3: Exact and approximate solution of test problem 3 for $\hat{m} = 8$.

Test Problem 4. Let us consider the nonlinear Fredholm integral equation [2]

$$g(x) = -x^2 - \frac{x}{3}(2\sqrt{2} - 1) + 2 + \int_0^1 xt\sqrt{g(t)}dt, \tag{38}$$

with exact solution

$$g(x) = -x^2 + 2. \tag{39}$$

By implying the method described in section 3.2, we obtain the approximate solution of test problem 4. Absolute errors for different values of $\hat{m}$ are shown and compared with those of absolute errors of Jacobi wavelets (JW) solution defined in [4] in table 4 and figure 4 shows the exact and approximate solution of test problem 4 for $\hat{m} = 8$.

Table 4: Absolute errors of test problem 4 for different values of $\hat{m}$.

	BPMW solution			JW solution		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	2.0000e-00	2.5387e-05	2.0104e-10	7.2277e-02	2.5387e-05	1.3110e-00
0.1	1.9900e-00	7.6300e-03	3.8137e-06	3.5645e-01	7.6300e-03	1.2293e-00
0.2	1.9469e-00	1.5296e-02	7.6275e-06	6.2355e-01	1.5296e-02	8.8446e-01
0.3	1.8969e-00	2.2952e-02	1.1441e-05	1.3013e-00	2.2952e-02	1.6043e-00
0.4	1.8344e-00	3.0597e-02	1.5255e-05	1.6448e-00	3.0597e-02	1.7791e-00
0.5	1.7344e-00	3.8247e-02	1.9069e-05	9.8507e-01	3.8247e-02	1.3417e-00
0.6	1.7344e-00	3.8247e-02	1.9069e-05	9.8507e-01	3.8247e-02	1.3417e-00
0.7	1.4969e-00	5.3555e-02	2.6696e-05	4.9208e-01	5.3555e-02	8.7184e-01
0.8	1.3469e-00	6.1205e-02	3.0509e-05	9.3426e-01	6.1205e-02	1.2609e-00
0.9	1.1900e-00	6.8849e-02	3.4324e-05	9.3335e-01	6.8849e-02	1.1969e-00

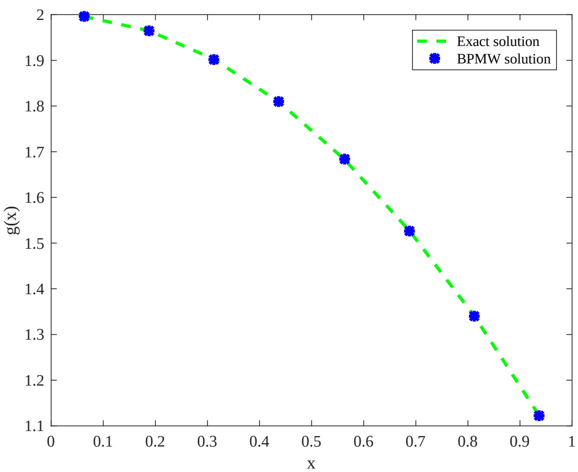

Figure 4: Exact and approximate solution of test problem 4 for $\hat{m} = 8$.

Test Problem 5. Let us consider the system of Fredholm integral equation [8]

$$g_1(x) = \frac{cosx}{3} + \frac{xsin^2 1}{2} + x - \int_0^1 t\,cosx\,g_1(t)dt - \int_0^1 x\,sint\,g_2(t)dt,$$
$$g_2(x) = \frac{e^x - 1}{2x} + cosx + (x+1)sin1 + cos1 - 1 - \int_0^1 e^{xt^2} g_1(t)dt -$$
$$- \int_0^1 (x+t)g_2(t)dt, \tag{40}$$

with exact solution

$$g_1(x) = x,$$
$$g_2(x) = cosx. \tag{41}$$

By implying the method described in section 3.3, we obtain the approximate solution of test problem 5. Absolute errors for different values of $\hat{m}$ are shown and compared with those of absolute errors of Jacobi wavelets (JW) solution defined in [4] in table 5a and table 5b and figure 5 shows the exact and approximate solution of test problem 5 for $\hat{m} = 8$.

Table 5a: Absolute errors of test problem 5 (BPMW solution) for different values of $\hat{m}$.

	$y_1(x)$				$y_2(x)$		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	8.5167e-01	1.8153e-02	3.5504e-06	0	1.6354e-02	3.8754e-02	4.5003e-05
0.1	8.0251e-01	1.8492e-02	5.0278e-06	0.1	6.6677e-02	4.2005e-02	2.4976e-05
0.2	7.4125e-01	1.8710e-02	6.4919e-06	0.2	1.3699e-01	4.5540e-02	2.5988e-05
0.3	6.7790e-01	1.8782e-02	7.9179e-06	0.3	2.1246e-01	4.9206e-02	2.5451e-05
0.4	6.0965e-01	1.8708e-02	9.3071e-06	0.4	2.9468e-01	5.3002e-02	2.2403e-05
0.5	5.2666e-01	1.8405e-02	1.0666e-05	0.5	3.9714e-01	5.7048e-02	1.6289e-05
0.6	5.2666e-01	1.8405e-02	1.0666e-05	0.6	3.9714e-01	5.7048e-02	1.6289e-05
0.7	3.4787e-01	1.7416e-02	1.3298e-05	0.7	6.2675e-01	6.5642e-02	8.8555e-06
0.8	2.4778e-01	1.6706e-02	1.4571e-05	0.8	7.6212e-01	7.0282e-02	3.1537e-05
0.9	1.4472e-01	1.5887e-02	1.5817e-05	0.9	9.0058e-01	7.5120e-02	6.4581e-05

Table 5b: Absolute errors of test problem 5 (JW solution) for different values of $\hat{m}$.

	$y_1(x)$				$y_2(x)$		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	9.0418e-03	8.4363e-03	8.4296e-03	0	1.6139e-02	1.2638e-02	1.2504e-02
0.1	8.3240e-03	7.9680e-03	7.9615e-03	0.1	1.1364e-02	8.9416e-03	8.9100e-03
0.2	7.4776e-03	7.4433e-03	7.4611e-03	0.2	8.3700e-03	7.6120e-03	7.6101e-03
0.3	6.6091e-03	6.8508e-03	6.8704e-03	0.3	6.6942e-03	6.1781e-03	6.1868e-03
0.4	5.6885e-03	6.1906e-03	6.1925e-03	0.4	4.9398e-03	4.6398e-03	4.6240e-03
0.5	4.6115e-03	5.4238e-03	5.4427e-03	0.5	2.9495e-03	2.8944e-03	2.9285e-03
0.6	4.6115e-03	5.4238e-03	5.4427e-03	0.6	2.9495e-03	2.8944e-03	2.9285e-03
0.7	2.3215e-03	3.7116e-03	3.7398e-03	0.7	2.9217e-03	1.8681e-03	9.0719e-04
0.8	1.0629e-03	2.7558e-03	2.7847e-03	0.8	6.2016e-03	3.2847e-03	3.1052e-03
0.9	2.2721e-04	1.7489e-03	1.7664e-03	0.9	8.8450e-03	5.7213e-03	5.5320e-03

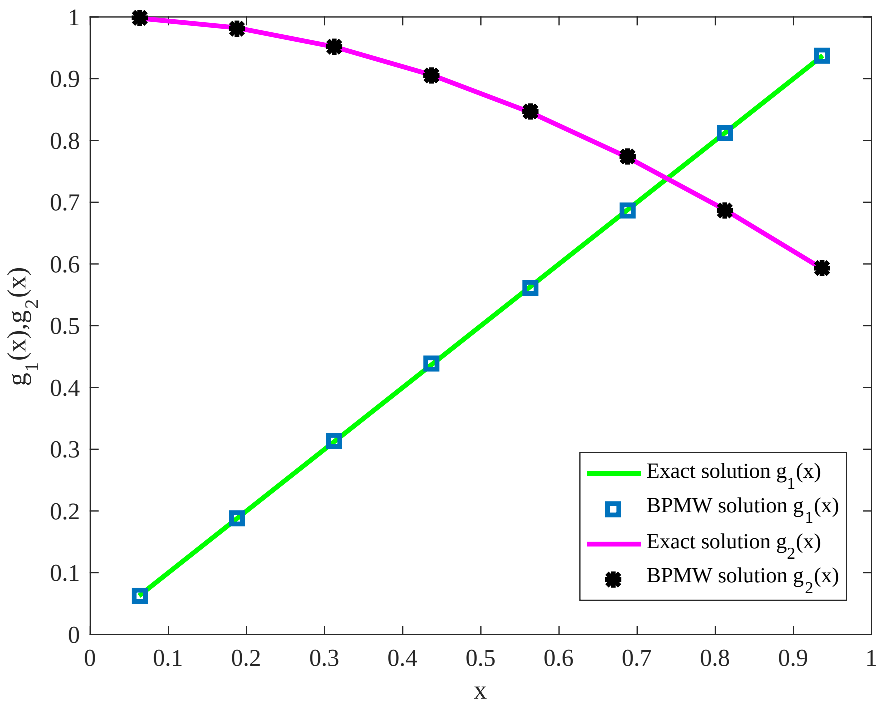

Figure 5: Exact and approximate solution of test problem 5 for $\hat{m}=8$.

Test Problem 6. Let us consider the system of Fredholm integral equation [18]

$$
\begin{aligned}
g_1(x) &= \frac{11}{6}x + \frac{11}{15} - \int_0^1 (x+t)\, g_1(t)dt - \int_0^1 (x+2t^2)\, g_2(t)dt, \\
g_2(x) &= \frac{5}{4}x^2 + \frac{1}{4}x - \int_0^1 xt^2 g_1(t)dt - \int_0^1 x^2 t g_2(t)dt,
\end{aligned} \tag{42}
$$

with exact solution

$$
\begin{aligned}
g_1(x) &= x, \\
g_2(x) &= x^2.
\end{aligned} \tag{43}
$$

By implying the method described in section 3.3, we obtain the approximate solution of test problem 6. Absolute errors for different values of $\hat{m}$ are shown and compared with those of absolute errors of Jacobi wavelets (JW) solution defined in [4] in table 6a and table 6b and figure 6 shows the exact and approximate solution of test problem 6 for $\hat{m} = 8$.

Table 6a: Absolute errors of test problem 6 (BPMW solution) for different values of $\hat{m}$.

	$y_1(x)$				$y_2(x)$		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	7.2912e-01	2.3522e-02	1.1072e-05	0	7.2000e-07	2.5837e-08	4.5685e-12
0.1	7.2032e-01	2.7250e-02	1.1813e-05	0.1	1.5000e-02	4.4862e-04	2.9142e-07
0.2	7.1152e-01	3.0978e-02	1.2555e-05	0.2	4.4000e-02	1.2325e-03	6.6586e-07
0.3	7.0271e-01	3.4706e-02	1.3295e-05	0.3	7.5000e-02	2.2847e-03	1.1626e-06
0.4	6.9391e-01	3.8434e-02	1.4036e-05	0.4	1.1000e-01	3.6050e-03	1.7818e-06
0.5	6.8511e-01	4.2162e-02	1.4777e-05	0.5	1.6000e-01	5.3725e-03	2.5057e-06
0.6	6.8511e-01	4.2162e-02	1.4777e-05	0.6	1.6000e-01	5.3725e-03	2.5057e-06
0.7	6.6751e-01	4.9617e-02	1.6259e-05	0.7	2.8000e-01	9.7123e-03	4.2683e-06
0.8	6.5870e-01	5.3345e-02	1.7000e-05	0.8	3.6000e-01	1.2374e-02	5.3244e-06
0.9	6.4990e-01	5.7073e-02	1.7741e-05	0.9	4.3000e-01	1.5304e-02	6.5028e-06

Table 6b: Absolute errors of test problem 6 (JW solution) for different values of $\hat{m}$.

	$y_1(x)$				$y_2(x)$		
x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$	x	$\hat{m}=4$	$\hat{m}=6$	$\hat{m}=8$
0	2.9105e-02	2.5268e-02	2.5316e-02	0	1.0625e-07	1.2000e-07	5.0134e-08
0.1	2.7051e-02	2.3205e-02	2.3250e-02	0.1	9.1709e-04	7.8000e-04	7.7999e-04
0.2	2.4996e-02	2.1142e-02	2.1184e-02	0.2	2.1106e-03	1.7000e-03	1.6579e-03
0.3	2.2941e-02	1.9079e-02	1.9118e-02	0.3	3.3616e-03	2.7000e-03	2.6801e-03
0.4	2.0886e-02	1.7016e-02	1.7052e-02	0.4	4.7169e-03	3.8000e-03	3.8466e-03
0.5	1.8832e-02	1.4952e-02	1.4985e-02	0.5	6.3851e-03	5.2000e-03	5.1368e-03
0.6	1.8832e-02	1.4952e-02	1.4985e-02	0.6	6.3851e-03	5.2000e-03	5.1368e-03
0.7	1.4723e-02	1.0826e-02	1.0853e-02	0.7	1.0035e-02	8.1000e-03	8.0882e-03
0.8	1.2668e-02	8.7624e-03	8.7867e-03	0.8	1.2121e-02	9.8000e-03	9.7701e-03
0.9	1.0613e-02	6.6991e-03	6.7204e-03	0.9	1.4264e-02	1.2000e-02	1.1596e-02

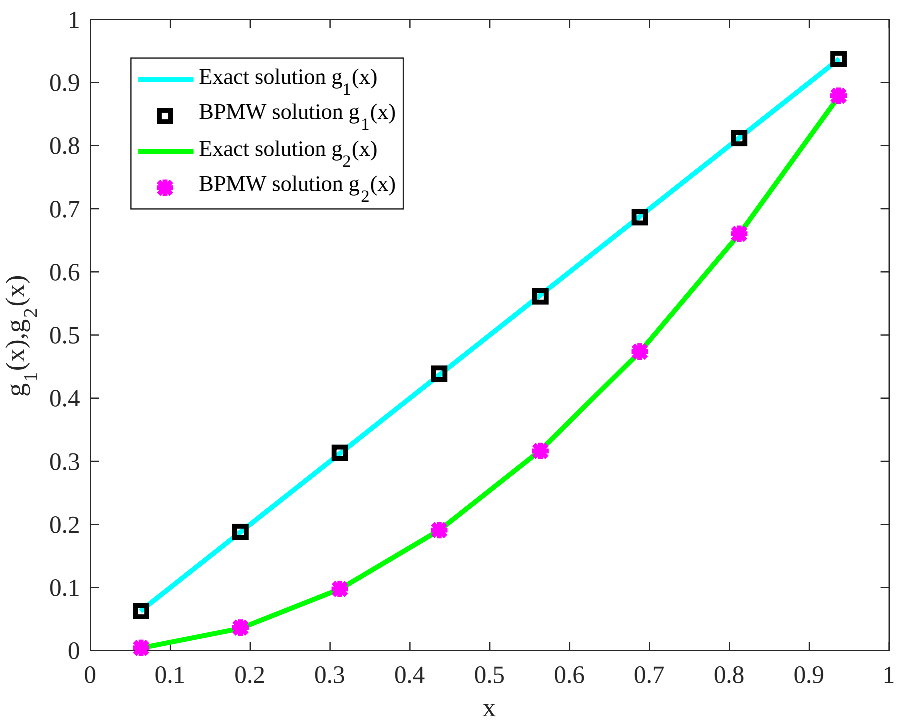

Figure 6: Exact and approximate solution of test problem 6 for $\hat{m}=8$.

6. Conclusions

In this paper, we approximate Fredholm integral and system of Fredholm integral equations using Bernstein Polynomial Multiwavelets. This method reduces the Fredholm integral equations and the system of Fredholm integral equations to system of linear (or nonlinear) algebraic equations, which are solved by appropriate methods. Error estimate of the proposed method is given. The numerical experiments show that the obtained results are in good agreement with that of exact solution. Also, we have compared the obtained solutions with those of solutions obtained using Jacobi wavelets. The results obtained by using the methods described in section 3 using BPMW are more accurate and precise compared to those of results obtained by using Jacobi wavelets. And hence, the proposed methods are efficient for solving Fredholm integral and system of Fredholm integral equations.

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