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NEW RESULTS IN CONE PENTAGONAL METRIC SPACES

Shishir Jain and Pooja Chaubey

Department of Mathematics,
Shri Vaishnav Vidyapeeth Vishwavidyalaya,
Sanwer Road, Indore - 453331, Madhya Pradesh, INDIA

E-mail : jainshishir11@rediffmail.com, chaubey.pooja8991@gmail.com

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Abstract: In the present paper, we establish the proof of a fixed point result for ordered Reich type contraction in cone pentagonal metric spaces with the cone which is not necessarily normal. The obtained results are new and generalize the results of Reich [17], Auwalu A. [1] and Garg et. al. [7] in ordered cone pentagonal metric spaces.

Keywords and Phrases: Cone pentagonal metric space, fixed point, ordered Reich type contraction.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In literature, the concept of substituting the set of real numbers by ordered "set" was first given by Kurepa [14]. Zabreiko [18] in 1997 worked on the idea of Kurepa [14] and introduced K-metric and K-normed spaces in which the set of real numbers were replaced by an ordered Banach space. Following the work of Zabreiko [18], in 2007 Huang and Zhang [8] introduced cone metric spaces by replacing Banach space in usual metric with ordered Banach space and defined convergence criteria in these new settings with fixed point theorem. After that several authors([1], [5], [6], [9], [10], [16], [15]) worked on the results of Huang and Zhang [8]. Branciari [4] introduced rectangular metric spaces in which triangular inequality is substituted by rectangular inequality and proved Banach contraction principle in rectangular metric spaces. Azam et. al. [3] followed the concept given by Huang and Zhang [8] on rectangular metric spaces and defined cone rectangular metric space. After that Garg et. al. [7] defined cone pentagonal metric spaces and established the proof of famous contraction principle named as Banach.

In this sequel, Auwalu A. [2] proved Kannan fixed point theorem in cone pentagonal metric spaces. But Reich type mappings and fixed point theorem particularly for ordered contractions is not proven yet in cone pentagonal metric spaces . In the present work we extend and generalize the results of Auwalu A. [1]and Garg et. al. [7] in ordered cone pentagonal metric spaces by proving a fixed point result for the Reich type contractions in this new setting. The outcomes of the present work are the generalization and extension of Kannan [12, 13] and Reich [17].
Now, we recall some definitions from Huang and Zhang [8] which are useful in our work.

Definition 1.1. [8] *Let E be a real Banach space and $P \subset E$ is a cone if and only if:*

(i) $P \neq \emptyset$ *, closed and* $P \neq \{\theta\}$.

(ii) $c, d \in \mathbb{R}, c, d \geq 0$ *and* $y, z \in P$ *implies that* $cy + dz \in P$.

(iii) $y \in P$ *and* $-y \in P$ *implies that* $y = \theta$.

For a given cone P which is a subset of E, we defined a partial ordering $\preceq$ with respect to P by $y \preceq z \Longleftrightarrow z - y \in P$. We could write $y \prec z$ which implies $y \preceq z$ but $y \neq z$ while $y \ll z$ will imply $z - y \in int(P)$ where the $int(P)$ denotes the interior of P. If in a cone P there exists a number $k > 0$ such that for all $y, z \in E$

$$\theta \preceq y \preceq z \Rightarrow \| y \| \leq k \| z \|$$

then P is called normal cone and normal constant of P is the least positive number k which satisfy the above.

Definition 1.2. [8] *Let X be a non-void set and let $d : X \times X \to E$ be a mapping such that for all $r, s, t \in X$, fascinate the below settings:*

(i) $\theta \preceq d(r, s)$ *and* $d(r, s) = \theta$ *if and only if* $r = s$;

(ii) $d(r, s) = d(s, r)$;

(iii) $d(r, s) \preceq d(r, t) + d(t, s)$.

Then, the pair (X, d) is said to be cone metric space over Banach space E. In our paper we consistently suppose that E is a real Banach space, P be a solid cone and $\preceq$ be a partial order with respect to P.

Remark 1.1. [11] *Let $P \subset E$ is a cone where $d, e, f \in P$,then*

(i) $d \preceq e$ *and* $e \ll f$*, then* $d \ll f$.

(ii) $d \ll e$ *and* $e \ll f$*, then* $d \ll f$.

(iii) $\theta \preceq v \ll f$ *for each* $f \in P^0$*, then* $v = \theta$.

(iv) $f \in P^0$ *and* $d_n \to \theta$*, then there exists* $n_0 \in \mathbb{N}$ *such that for all* $n > n_0$*, implies* $d_n \ll f$.

(v) $d \preceq \alpha d$ *where* $0 < \alpha < 1$*, then* $d = \theta$.

Definition 1.3. [3] *Let X be a non-void set and let $d_1 : X \times X \to E$ be a mapping such that for all $r, s \in X$, fascinate the below settings:*

(R1) $\theta \preceq d_1(r, s)$ *and* $d_1(r, s) = \theta$ *if and only if* $r = s$;

(R2) $d_1(r, s) = d_1(s, r)$;

(R3) $d_1(r, s) \preceq d_1(r, w) + d_1(w, t) + d_1(t, s)$ *for all* $r, s \in X$ *and for all distinct points* $w, t \in X - \{r, s\}$ *this is known as rectangular property.*

Then, the pair (X, d_1) is said to be cone rectangular metric space over Banach space E.

Definition 1.4. [7] *Let X be a non-void set and let $\eta : X \times X \to E$ be a mapping such that for all $r, s \in X$, fascinate the below settings:*

(i) $\theta \preceq \eta(r, s)$ *and* $\eta(r, s) = \theta$ *if and only if* $r = s$;

(ii) $\eta(r, s) = \eta(s, r)$;

(iii) $\eta(r, s) \preceq \eta(r, t) + \eta(t, w) + \eta(w, u) + \eta(u, s)$ *for all* $r, s, t, w, u \in X$ *and for all distinct points* $t, w, u \in X - \{r, s\}$ *this is known as pentagonal property.*

Then, the pair (X, η) is said to be cone pentagonal metric space over Banach space E.

2. Criteria of convergence [7]

Definition 2.1. *Let (X, η) be a cone pentagonal metric space and a sequence*

$\{u_m\} \in X$ is said to be convergent and converge to u if for every $c \in E$ $\exists$ a positive integer m_0 such that $\eta(u_m, u) \ll c$ $\forall m > m_0$. In this case u is said to be the limit of u_m, denoted by $\lim_{n\to\infty} u_m = u$.

Definition 2.2. *Let (X, η) be a cone pentagonal metric space and a sequence $\{u_m\} \in X$ is said to be Cauchy sequence if $\eta(u_m, u_n) \ll c$, $\forall m, n > m_0$ and (X, η) is called complete cone pentagonal metric space if every Cauchy sequence $\{u_m\}$ converges to $u \in X$.*

Example 2.1. Let us assume that $X = \{5, 6, 7, 8, 9\}$, $E = \mathbb{R}^2$, $P = \{(r, s) : r, s \geq 0\}$ (normal cone) in E, Define $\eta : X \times X \to E$ by the following :

$$\eta(5,7) = \eta(7,5) = \eta(7,8) = \eta(8,7) = \eta(6,7) = \eta(7,6) = \eta(6,8) = \eta(8,6) =$$
$$\eta(5,8) = \eta(8,5) = (1,2)$$
$$\eta(5,6) = \eta(6,5) = (4,8)$$
$$\eta(5,9) = \eta(9,5) = \eta(6,9) = \eta(9,6) = \eta(7,9) = \eta(9,7) = \eta(8,9) = \eta(9,8) = (3,6).$$

Then (X, η) is a cone pentagonal metric space. Since rectangular property absences here, therefore define metric space is not a cone rectangular metric space that is,

$$(4,8) = \eta(5,6) > \eta(5,7) + \eta(7,8) + \eta(8,6) = (1,2) + (1,2) + (1,2) = (3,6)$$

as $(4,8) - (3,6) = (1,2) \in P$.

Definition 2.3. *An ordered cone pentagonal metric space is denoted by $(X, \sqsubseteq, \eta)$ where (X, η) is a cone pentagonal metric space , $X \neq \emptyset$ equipped with partial order " $\sqsubseteq$ ". Let us assume that R be a self map on X with R is non-decreasing and " $\sqsubseteq$ " is partial order, if for each $u, v \in X$, $u \sqsubseteq v$ if and only if $Ru \sqsubseteq Rv$.*
R is ordered Banach type contraction on $(X, \sqsubseteq, \eta)$ if for all $u, v \in X$ with $u \sqsubseteq v$, there exists $\alpha \in [0, 1)$ such that

$$\eta(Ru, Rv) \preceq \alpha\eta(u, v) \tag{1}$$

R is ordered Kannan type contraction on $(X, \sqsubseteq, \eta)$ if for all $u, v \in X$ with $u \sqsubseteq v$, there exists $\alpha \in [0, \frac{1}{2})$ such that

$$\eta(Ru, Rv) \preceq \alpha[\eta(u, Ru) + \eta(v, Rv)] \tag{2}$$

R is ordered Reich type contraction on $(X, \sqsubseteq, \eta)$ if for all $u, v \in X$ with $u \sqsubseteq v$, there exists $\alpha, \beta, \gamma \in [0, 1)$ such that $\alpha + \beta + \gamma < 1$ and

$$\eta(Ru, Rv) \preceq \alpha\eta(u, v) + \beta\eta(u, Ru) + \gamma\eta(v, Rv) \tag{3}$$

One may notice here that on putting $\beta = \gamma = 0$ and $\alpha = 0$, $\beta = \gamma$ Reich type contraction can be turn into Banach and Kannan type contractions respectively.

Definition 2.4. *Let $(X, \sqsubseteq, \eta)$ be an ordered cone pentagonal metric space and B be the non-empty subset of X is said to be well ordered if $\forall\ u, v \in B$ are comparable with respect to partial order $\sqsubseteq$.*

3. Main Results

Theorem 3.1. *Let us assume that $(X, \sqsubseteq, \eta)$ be an ordered cone pentagonal metric space and $R : X \to X$ be a self map on X which satisfies the given settings*

(i) *R is an ordered Reich type contraction.*

(ii) *there exists $u_0 \in X$ such that $u_0 \sqsubseteq Ru_0$;*

(iii) *R is nondecreasing with respect to $\sqsubseteq$;*

(iv) *If $\{u_n\}$ is a nondecreasing sequence in X, converging to some w,then $u_n \sqsubseteq w$.*

Then R has a unique fixed point. Moreover, the set of fixed points of R is well ordered if and only if fixed point of R is unique.

Proof. Let us take the iterative sequence $\{u_n\}$ for given u_0. Since by (ii) $u_0 \in X$ such that $u_0 \sqsubseteq Ru_0$. Now set $Ru_0 = u_1$, this implies that $u_0 \sqsubseteq u_1$. Again by (iii) we get $Ru_0 \sqsubseteq Ru_1$. Now assume $Ru_1 = u_2$. By proceeding like this, we get nondecreasing sequence $\{u_n\}$ such that

$$u_0 \sqsubseteq u_1 \sqsubseteq \ldots \sqsubseteq u_n \sqsubseteq u_{n+1} \sqsubseteq \ldots \text{ and } u_{n+1} = Ru_n \ \forall n \geq 0.$$

Here one may notice that if $u_{n+1} = u_n$ for any n implies that u_n become a fixed point of R so always suppose $u_{n+1} \neq u_n$ $\forall n \geq 0$.
Since $u_n \sqsubseteq u_{n+1}$, for any $n \geq 0$, from (i) we have

$$\begin{aligned}
\eta(u_n, u_{n+1}) &= \eta(Ru_{n-1}, Ru_n) \\
&\preceq \alpha\eta(u_{n-1}, u_n) + \beta\eta(u_{n-1}, Ru_{n-1}) + \gamma\eta(u_n, Ru_n) \\
&= \alpha\eta(u_{n-1}, u_n) + \beta\eta(u_{n-1}, u_n) + \gamma\eta(u_n, u_{n+1}) \\
\eta(u_n, u_{n+1}) &\preceq \frac{\alpha + \beta}{1 - \gamma}\eta(u_{n-1}, u_n),
\end{aligned}$$

i.e., $\eta_n \preceq \lambda\eta_{n-1}$, where $\lambda = \dfrac{\alpha + \beta}{1 - \gamma}$ and $\eta_n = \eta(u_n, u_{n+1})$.
Repeating above process, we get

$$\eta_n \preceq \lambda^n \eta_0 \quad \forall n \geq 1. \tag{4}$$

We now show that u_0 is not a periodic point of R. If $u_0 = u_n$ for any $n \geq 2$, then from (4), we obtained

$$\begin{aligned} \eta(u_0, Ru_0) &= \eta(u_n, Ru_n) = \eta(u_n, u_{n+1}) \\ &= \eta_n \\ &\preceq \lambda^n \eta_n. \end{aligned}$$

Since $\lambda = \dfrac{\alpha+\beta}{1-\gamma} < 1$ (since $\alpha+\beta+\gamma < 1$), then above inequality gives, $\eta_0 = \theta$, i.e.,$\eta(u_0, Ru_0) = \theta$. So that u_0 become the fixed point of R. Therefore we can assume that $u_n \neq u_m$ $\forall$ distinct $n, m \in \mathbb{N}$.
Now since $u_n \sqsubseteq u_{n+2}$ therefore by (i) and (4)

$$\begin{aligned} \eta(u_n, u_{n+2}) &= \eta(Ru_{n-1}, Ru_{n+1}) \\ &\preceq \alpha\eta(u_{n-1}, u_{n+1}) + \beta\eta(u_{n-1}, Ru_{n-1}) + \gamma\eta(u_{n+1}, Ru_{n+1}) \\ &= \alpha\eta(u_{n-1}, u_{n+1}) + \beta\eta(u_{n-1}, u_n) + \gamma\eta(u_{n+1}, u_{n+2}) \\ &\preceq \alpha[\eta(u_{n-1}, u_n) + \eta(u_n, u_{n+2}) + \eta(u_{n+2}, u_{n+3}) + \eta(u_{n+3}, u_{n+1})] \\ &+ \beta\eta(u_{n-1}, u_n) + \gamma\eta(u_{n+1}, u_{n+2}) \\ &= \alpha[\eta_{n-1} + \eta(u_n, u_{n+2}) + \eta_{n+2} + \eta_{n+1}] + \beta\eta_{n-1} + \gamma\eta_{n+1} \\ &= (\alpha+\beta)\eta_{n-1} + \alpha\eta(u_n, u_{n+2}) + (\alpha+\gamma)\eta_{n+1} + \alpha\eta_{n+2} \\ &\preceq (\alpha+\beta)\eta_{n-1} + (\alpha+\gamma)\lambda^{n+1}\eta_0 + \alpha\lambda^{n+2}\eta_0 + \alpha\eta(u_n, u_{n+2}) \\ (1-\alpha)\eta(u_n, u_{n+2}) &\preceq (\alpha+\beta)\lambda^{n-1}\eta_0 + (\alpha+\gamma)\lambda^{n+1}\eta_0 + \alpha\lambda^{n+2}\eta_0 \\ &\preceq ((\alpha+\beta) + (\alpha+\gamma)\lambda^2 + \alpha\lambda^3)\lambda^{n-1}\eta_0 \\ \eta(u_n, u_{n+2}) &\preceq \frac{(\alpha+\beta) + (\alpha+\gamma)\lambda^2 + \alpha\lambda^3}{1-\alpha}\lambda^{n-1}\eta_0, \end{aligned}$$

so,

$$\eta(u_n, u_{n+2}) \preceq \mu\lambda^{n-1}\eta_0 \forall n \geq 1,$$

where $\mu = \dfrac{(\alpha+\beta) + (\alpha+\gamma)\lambda^2 + \alpha\lambda^3}{1-\alpha} \geq 0$.
Now consider two cases for the sequence $\{u_n\}$.
Case 1. Let if $p = 2m+1$ (an odd number), then by pentagonal property and (4),

$$\begin{aligned} \eta(u_n, u_{n+2m+1}) &\preceq \eta(u_n, u_{n+1}) + \eta(u_{n+1}, u_{n+2}) + \eta(u_{n+2}, u_{n+3}) + \eta(u_{n+3}, u_{n+2m+1}) \\ &\preceq \eta(u_n, u_{n+1}) + \eta(u_{n+1}, u_{n+2}) + \eta(u_{n+2}, u_{n+3}) + \cdots \\ &+ \eta(u_{n+2m-1}, u_{n+2m}) + \eta(u_{n+2m}, u_{n+2m+1}) \\ &\preceq \lambda^n\eta_0 + \lambda^{n+1}\eta_0 + \lambda^{n+2}\eta_0 + \cdots + \lambda^{n+2m-1}\eta_0 + \lambda^{n+2m}\eta_0 \\ &\preceq (1 + \lambda + \lambda^2 + \cdots + \lambda^{2m-1} + \lambda^{2m})\lambda^n\eta_0, \end{aligned}$$

which implies that,

$$\eta(u_n, u_{n+2m+1}) \preceq \frac{\lambda^n}{1-\lambda}\eta_0. \tag{5}$$

Case 2. Let if $p = 2m$ (an even number) here $m \geq 2$, then by pentagonal property,

$$\begin{aligned}
\eta(u_n, u_{n+2m}) &\preceq \eta(u_n, u_{n+2}) + \eta(u_{n+2}, u_{n+3}) + \eta(u_{n+3}, u_{n+4}) + \eta(u_{n+4}, u_{n+2m}) \\
&\preceq \eta(u_n, u_{n+2}) + \eta(u_{n+2}, u_{n+3}) + \eta(u_{n+3}, u_{n+4}) + \cdots \\
&+ \eta(u_{n+2m-2}, u_{n+2m-1}) + \eta(u_{n+2m-1}, u_{n+2m}) \\
&= \eta(u_n, u_{n+2}) + \eta_{n+2} + \eta_{n+3} + \cdots + \eta_{n+2m-2} + \eta_{n+2m-1} \\
&\preceq \mu\lambda^{n-1}\eta_0 + \lambda^{n+2}\eta_0 + \cdots + \lambda^{n+2m-2}\eta_0 + \lambda^{n+2m-1}\eta_0 \\
&= \mu\lambda^{n-1}\eta_0 + [\lambda^2 + \cdots + \lambda^{2m-2} + \lambda^{2m-1}]\lambda^n\eta_0,
\end{aligned}$$

so,

$$\eta(u_n, u_{n+2m}) \preceq \frac{\lambda^n}{1-\lambda}\eta_0 + \mu\lambda^{n-1}\eta_0. \tag{6}$$

As $\mu \geq 0$, $0 \leq \lambda < 1$, it follows that $\frac{\lambda^n}{1-\lambda}\eta_0 \to \theta, \mu\lambda^{n-1}\eta_0 \to \theta$, therefore from (i) and (iv) of remark 1.1, it follows that for every $c \in E$ such that $\theta \ll c$, $\exists\ n_0 \in \mathbb{N}$ with $\eta(u_n, u_{n+2m}) \ll c, \eta(u_n, u_{n+2m+1}) \ll c\ \forall\ n > n_0$. Hence $\{u_n\}$ is a Cauchy sequence in X. Since completeness of $X \implies \exists\ k \in X$ such that

$$lim_{n\to\infty} u_n = lim_{n\to\infty} Ru_{n-1} = k.$$

Now to prove k is a fixed point of R. By (iv), we obtain $u_n \sqsubseteq k$, thus, by (i)

$$\begin{aligned}
\eta(Ru_{n-1}, Rk) &\preceq \alpha\eta(u_{n-1}, k) + \beta\eta(u_{n-1}, Ru_{n-1}) + \gamma\eta(k, Rk) \\
\eta(u_n, Rk) &\preceq \alpha\eta(u_{n-1}, k) + \beta\eta(u_{n-1}, u_n) + \gamma[\eta(k, u_{n+1}) + \eta(u_{n+1}, u_{n+2}) \\
&+ \eta(u_{n+2}, u_n) + \eta(u_n, Rk)] \\
(1-\gamma)\eta(u_n, Rk) &\preceq \alpha\eta(u_{n-1}, k) + \beta\eta_{n-1} + \gamma\eta(k, u_{n+1}) + \gamma\eta_n + \gamma\eta_{n+1}
\end{aligned}$$

By (4) and by $(1-\gamma) \geq 0$, and again by (iv) of remark 1.1, $\exists\ n_1 \in \mathbb{N}$ with for every $c \in E$ such that $\theta \ll c$, $\eta_{n-1} \ll \dfrac{(1-\gamma)}{5\beta}c$ and $\eta_n \ll \dfrac{(1-\gamma)}{5\gamma}c\ \forall\ n > n_1$. And for $u_n \to k$, $\exists\ n_2 \in \mathbb{N}$ with for every $c \in E$ such that $\theta \ll c$, $\eta(u_{n-1}, k) \ll \dfrac{(1-\gamma)}{5\alpha}c$ and $\eta(u_{n+1}, k) \ll \dfrac{(1-\gamma)}{5\gamma}c\ \forall\ n > n_2$ and also $\exists\ n_3 \in \mathbb{N}$ with for every $c \in E$ such

that $\theta \ll c$, $\eta_{n+1} \ll \dfrac{(1-\gamma)}{5\gamma} c \ \forall\ n > n_3$.
we therefore can select $n_4 \in \mathbb{N}$ with

$$\eta(u_n, Rk) \ll c \quad \forall n > n_4. \tag{7}$$

Again by pentagonal property,

$$\begin{aligned}\eta(Rk,k) &\preceq \eta(Rk,u_n) + \eta(u_n,u_{n+1}) + \eta(u_{n+1},u_{n+2}) + \eta(u_{n+2},k)\\ &= \eta(Rk,u_n) + \eta_n + \eta_{n+1} + \eta(u_{n+2},k)\end{aligned}$$

By (7), (4) and by $u_n \to k$ and also by (iii) of remark 1.1 implies that $\eta(Rk,k) = \theta$, i.e., $Rk = k$. Hence, k is a fixed point of R.
Now suppose that the set of fixed points B (say) of R is well ordered. Now to prove uniqueness of fixed point, assume $l \in B$ be another fixed point of R, i.e., $Rl = l$. Since B well ordered, for example assume that $k \sqsubseteq l$. From (i) we have,

$$\begin{aligned}\eta(k,l) &= \eta(Rk,Rl)\\ \eta(k,l) &\preceq \alpha\eta(k,l) + \beta\eta(k,Rk) + \gamma\eta(l,Rl)\\ &= \alpha\eta(k,l) + \beta\eta(k,k) + \gamma\eta(l,l)\\ &= \alpha\eta(k,l)\end{aligned}$$

Since $0 \leq \alpha < 1$, So from above inequality and (iv) of remark of 1.1 we get $k = l$, this proves the uniqueness of R. In the opposite way, if R has a unique fixed point, then B is a singleton set and hence B is well-ordered.

Remark 3.1. *In an ordered cone pentagonal metric space above result is a generalization and extension of Abba Auwalu [2] and Garg et. al. [7] (on putting $\alpha = 0$, $\beta = \gamma$ and $\beta = \gamma = 0$ respectively) in aspect of contractive conditions which we have used here.*

For validity of our main result, we construct an example below here.
Example 3.1. Let $X = \{1,2,3,4,5\}$, $E = C^1_{\mathbb{R}}[0,1]$, $\| u(t) \| = \| u(t) \|_\infty + \| u'(t) \|_\infty$ and the cone $P = \{u(t) : u(t) \geq 0 \forall t \in [0,1]\}$. Consider a mapping $\eta : X \times X \to E$ by

$$\begin{gathered}\eta(1,3) = \eta(3,1) = \eta(2,3) = \eta(3,2) = \eta(1,4) = \eta(4,1) = \eta(3,4) = \eta(4,3) =\\ \eta(2,4) = \eta(4,2) = e^t\\ \eta(1,2) = \eta(2,1) = 4e^t\\ \eta(1,5) = \eta(5,1) = \eta(2,5) = \eta(5,2) = \eta(3,5) = \eta(5,3) = \eta(4,5) = \eta(5,4) = 3e^t\\ \eta(u,v) = \theta \text{ if } u = v.\end{gathered}$$

It is clear that (X, η) is a complete non-normal cone pentagonal metric space. Consider self mapping $R : X \to X$ with " $\sqsubseteq$ " (partial order) on X as given below:

$$R1 = 2, R2 = 1, R3 = 3, R4 = 5, R5 = 4$$

and $\sqsubseteq = \{(1,1),(2,2),(3,3),(4,4),(5,5),(1,3),(2,5),(1,5)\}$.
By definition of $\sqsubseteq$, it is clear that R is ordered Reich type contraction in $(X, \sqsubseteq, \eta)$ with $\alpha = \beta = \frac{2}{7}, \gamma = \frac{1}{4}$.
Now we have to check this only for $(u, v) = (1,3), (2,5), (1,5)$
Let if $(u, v) = (1,5)$, then

$$\eta(R1, R5) = \eta(2,4) = e^t$$

and

$$\alpha\eta(2,5) + \beta\eta(2, R2) + \gamma\eta(5, R5) = \alpha 3e^t + \beta 4e^t + \gamma 3e^t$$

which holds for $\alpha = \beta = \frac{2}{7}, \gamma = \frac{1}{4}$. In similar manner, (3) holds for $(u, v) = \{(1,3),(2,5)\}$ with the same value of α, β, γ. Rest of the settings of theorem 3.1 are satisfied and 3 is the unique fixed point of R.
Here it is notice that, R is not a Reich type contraction in cone pentagonal metric space which is non-ordered, Let us assume that if $u = 2$,$v = 1$ $\eta(R2, R1) = \eta(1,2) = 4e^t$ and $\alpha\eta(2,1)+\beta\eta(2, R2)+\gamma\eta(1, R1) = \alpha 4e^t+\beta 4e^t+\gamma 4e^t = 4e^t(\alpha+\beta+\gamma) = 3.28$ for $\alpha = \beta = \frac{2}{7}, \gamma = \frac{1}{4}$,this implies that (3) is not satisfies here.

Below is an example which proves that fixed point may not be unique in above obtained results if the given set of fixed point is not in B.
Example 3.2. Let $X = \{1,2,3,4,5\}$, $E = \mathbb{R}^2$, $P = \{(u, v) : u, v \geq 0\}$ is a normal cone in E, and $\eta : X \times X \to E$ be a mapping such that

$$\begin{gathered}\eta(1,3) = \eta(3,1) = \eta(2,3) = \eta(3,2) = \eta(1,4) = \eta(4,1) = \eta(3,4) = \eta(4,3) = \\ \eta(2,4) = \eta(4,2) = (1,2) \\ \eta(1,2) = \eta(2,1) = (4,8) \\ \eta(1,5) = \eta(5,1) = \eta(2,5) = \eta(5,2) = \eta(3,5) = \eta(5,3) = \eta(4,5) = \eta(5,4) = (3,6) \\ .\ \eta(u, v) = \theta \text{ if } u = v.\end{gathered}$$

Then (X, d) is a complete cone pentagonal metric space.
Consider self mapping $R : X \to X$ with $\sqsubseteq X$ such that

$$R(u) = \begin{cases} u, & \text{if } u \in \{1,4\}; \\ 1, & \text{if } u = 2; \\ 5, & \text{if } u = 3; \\ 3, & \text{if } u = 5; \end{cases}$$

and $\sqsubseteq = \{(1,1),(2,2),(3,3),(4,4),(5,5),(2,5),(1,5)\}$.
Now it is clear that $\eta(Ru, Rv) \preceq \alpha\eta(u,v)$ $\forall$ $u, v \in X$ with $u \sqsubseteq v$ for $\alpha \in [\frac{3}{4}, 1)$. Hence R is an ordered Banach contraction on X. Remaining settings of theorem 3.1 are satisfied if we replace the first condition by ordered Banach contraction and with the exception of the set of fixed points of self-map is well ordered. Thus 1 and $4 \in X$ are the two fixed points. Notice $(1,4),(4,1) \notin \sqsubseteq$. Rather if $u = 1, v = 4$, we can find no α with $0 \leq \alpha < 1$ and $\eta(Ru, Rv) \preceq \alpha\eta(u,v)$, Hence, R is not a Banach contraction on X.

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