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RATIONAL TYPE FIXED POINT THEOREM IN 2-METRIC SPACE

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Abstract: In this paper we have established a fixed point theorem in 2-metric space, using a more general rational type contraction. Here we have extended the result of Olatinwo et. al [14] in 2-metric space.

Keywords and Phrases: Generalized contraction principle, 2-metric, fixed point, Cauchy sequence, convergent sequence.

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1. Introduction

One of the generalization of a metric space is 2-metric space. Gahler [7], [8] introduced the concept of 2-metric space. Geometrically in plane,2-metric function abstracts the properties of the area function for Euclidean triangle just as a metric function abstracts the length function for Euclidean segment.
It is precisely defined as follows

Definition 1.1. [19] *Let X be a non-empty set and $d : X \times X \times X \to \mathbb{R}^+$. If for all $x, y, z,$ and u in X we have*
$(d_1) d(x, y, z) = 0$ *if at least two of x, y, z are equal.*
(d_2) *for all $x \neq y$, there exists a point z in x such that $d(x, y, z) \neq 0$.*
$(d_3) d(x, y, z) = d(x, z, y) = d(y, z, x) = ...$ *and so on*
$(d_4) d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$.
then d is called a metric on X and the pair (X, d) is said to be a 2-metric space.

In 1968 [12] Kannan established a fixed point theorem for $T : X \to X$ in the context of a metric space (X, d) where T satisfies a generalized Banach's contraction condition:

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)] \forall x, y \in X, where a \in [0, 1) \tag{1}$$

Later Chatterjee [4] also used another generalized contractive condition on $T : X \to X$ in the context of 2-metric space namely:

$$d(Tx, Ty) \leq b[d(x, Ty) + d(y, Tx)] \forall x, y \in X, where b \in [0, 1) \tag{2}$$

In 1975, Das and Gupta [5] introduced the concepts of a new contractive condition named rational type of contractive condition' and studied its relationship with continuity of the self map in reference. Later, a notable work of Jaggi and Das [11] appeared which requires mention for a better understanding of the nature of works in reference. Any continuous map $T : X \to X$ on a complete metric space (X, d) satisfying the rational type of contractive condition:

$$d(Tx, Ty) \leq \alpha \frac{d(x, Tx).d(y, Ty)}{d(x, y) + d(x, Ty) + d(y, Tx)} + \beta d(x, y) \tag{3}$$

for all $x, y \in X$ has a unique fixed point in X where $\alpha, \beta \in [0, 1)$ such that $\alpha + \beta < 1$.

Recently in 2018, Olatinwo and Ishola [14] introduced a more general form of rational type of contractive condition and found.

Theorem 1.2. *Let (X, d) be a complete metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty) \leq \alpha \frac{[p + d(x, Tx)][d(y, Ty)]^r [d(y, Tx)]^q}{1 + \mu d(y, Tx) + \eta d(x, Ty) + d(x, y)} + \beta d(x, y).$$

for all $x, y \in X$ where $\alpha, p, q, r, \mu, \eta \in \mathbb{R}^+$ and $\beta \in [0, 1)$. Then T has a unique fixed point in X.

This result involves six parameters α, p, q, r, μ and η. We have extended the above result for 2-metric space with more parameters.

It is helpful to recall some definitions in the context of a 2- metric space.

Definition 1.3. [19] *A sequence $\{x_n\}_{n \in \mathbb{N}}$in a 2-metric space (X, d) is said to be a Cauchy sequence if $\lim_{m,n \to \infty} d(x_m, x_n, a) = 0$ for all $a \in X$.*

Definition 1.4. [5] *A sequence $\{x_n\}_{n \in \mathbb{N}}$ in a 2-metric space (X, d) is said to be a convergent at $x \in X$ if $\lim_{n \to \infty} d(x_n, x, a) = 0$ for all $a \in X$. The point x is called the limit of the sequence.*

Definition 1.5. [17] *A 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.*

2. Main Result

Theorem 2.1. *Let (X, d) be a complete 2-metric space and $T : X \to X$ a mapping satisfying*

$$d(Tx, Ty, a) \leq \alpha \frac{[p + d(x, Tx, a)][d(y, Ty, a)]^r [d(y, Tx, a)]^q}{1 + \mu d(y, Tx, a) + \eta d(x, Ty, a) + d(x, y, a)} + \beta d(x, y, a) + $$
$$+\gamma[d(x, Tx, a) + d(y, Ty, a)] + \delta[d(y, Tx, a) + d(x, Ty, a)]$$

for all $x, y, a \in X$, where $\alpha, \beta, \gamma, p, q, r, \mu, \eta \in \mathbb{R}^+$ and $\beta + 2\gamma + 2\delta \leq 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be any arbitrary point in X and we define a sequence $\{x_n\}$ as $x_1 = Tx_0$, $x_2 = Tx_1$, ..., $x_{2n+1} = Tx_{2n}, x_{2n+2} = Tx_{2n+1}$, and so on. We may assume that $x_{2n+1} \neq x_{2n}$ for some n, otherwise x_{2n} will be a fixed point. Here first we show that $d(x_{2n}, x_{2n+1}, x_{2n+2}) = 0$. So,

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}, x_{2n}) = d(Tx_{2n}, Tx_{2n+1}, x_{2n})\\
&\leq \alpha \frac{[p + d(x_{2n}, Tx_{2n}, x_{2n})][d(x_{2n+1}, Tx_{2n+1}, x_{2n})]^r [d(x_{2n+1}, Tx_{2n}, x_{2n})]^q}{1 + \mu d(x_{2n+1}, Tx_{2n}, x_{2n}) + \eta d(x_{2n}, Tx_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n+1}, x_{2n})}\\
&+\beta d(x_{2n}, x_{2n+1}, x_{2n}) + \gamma[d(x_{2n}, Tx_{2n}, x_{2n}) + d(x_{2n+1}, Tx_{2n+1}, x_{2n})] +\\
&\delta[d(x_{2n+1}, Tx_{2n}, x_{2n}) + d(x_{2n}, Tx_{2n+1}, x_{2n})]\\
&= \alpha \frac{[p + d(x_{2n}, Tx_{2n}, x_{2n})][d(x_{2n+1}, Tx_{2n+1}, x_{2n})]^r [d(x_{2n+1}, Tx_{2n}, x_{2n})]^q}{1 + \mu d(x_{2n+1}, Tx_{2n}, x_{2n}) + \eta d(x_{2n}, Tx_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n+1}, x_{2n})}\\
&+\beta d(x_{2n}, x_{2n+1}, x_{2n}) + \gamma[d(x_{2n}, x_{2n+1}, x_{2n}) + d(x_{2n+1}, x_{2n+2}, x_{2n})]\\
&+\delta[d(x_{2n+1}, x_{2n+1}, x_{2n}) + d(x_{2n}, x_{2n+2}, x_{2n})]
\end{aligned}$$

or,

$$d(x_{2n+1}, x_{2n+2}, x_{2n}) \leq \gamma d(x_{2n+1}, x_{2n+2}, x_{2n}),$$

which is a contradiction and hence $d(x_{2n+1}, x_{2n+2}, x_{2n}) = 0$.

Now,

$$\begin{aligned}
&d(x_{2n+1}, x_{2n+2}, a) = d(Tx_{2n}, Tx_{2n+1}, a)\\
&\leq \alpha \frac{[p + d(x_{2n}, Tx_{2n}, a)][d(x_{2n+1}, Tx_{2n+1}, a)]^r [d(x_{2n+1}, Tx_{2n}, a)]^q}{1 + \mu d(x_{2n+1}, Tx_{2n}, a) + \eta d(x_{2n}, Tx_{2n+1}, a) + d(x_{2n}, x_{2n+1}, a)} + \beta d(x_{2n}, x_{2n+1}, a)\\
&+\gamma[d(x_{2n}, Tx_{2n}, a) + d(x_{2n+1}, Tx_{2n+1}, a)] + \delta[d(x_{2n+1}, Tx_{2n}, a) + d(x_{2n}, Tx_{2n+1}, a)]\\
&= \alpha \frac{[p + d(x_{2n}, x_{2n+1}, a)][d(x_{2n+1}, x_{2n+2}, a)]^r [d(x_{2n+1}, x_{2n+1}, a)]^q}{1 + \mu d(x_{2n+1}, x_{2n+1}, a) + \eta d(x_{2n}, x_{2n+2}, a) + d(x_{2n}, x_{2n+1}, a)} + \beta d(x_{2n}, x_{2n+1}, a)\\
&+\gamma[d(x_{2n}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n+2}, a)] + \delta[d(x_{2n+1}, x_{2n+1}, a) + d(x_{2n}, x_{2n+2}, a)]
\end{aligned}$$

$$\leq \beta d(x_{2n}, x_{2n+1}, a) + \gamma[d(x_{2n}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n+2}, a)]$$
$$+\delta[d(x_{2n}, x_{2n+2}, x_{2n+1}) + d(x_{2n}, x_{2n+1}, a) + d(x_{2n+1}, x_{2n+2}, a)]$$

or,

$$d(x_{2n+1}, x_{2n+2}, a) \leq \frac{\beta + \gamma + \delta}{1 - (\gamma + \delta)} d(x_{2n}, x_{2n+1}, a)$$

i.e. $d(x_{2n+1}, x_{2n+2}, a) \leq pd(x_{2n}, x_{2n+1}, a)$ where $p = \frac{\beta+\gamma+\delta}{1-(\gamma+\delta)} < 1$
Again,

$$d(x_{2n}, x_{2n+1}, a) \leq d(Tx_{2n-1}, x_{2n}, a)$$
$$\leq \alpha \frac{[p + d(x_{2n-1}, Tx_{2n-1}, a)][d(x_{2n}, Tx_{2n}, a)]^r [d(x_{2n}, Tx_{2n-1}, a)]^q}{1 + \mu d(x_{2n}, Tx_{2n-1}, a) + \eta d(x_{2n-1}, Tx_{2n}, a) + d(x_{2n-1}, x_{2n}, a)}$$
$$+\beta d(x_{2n-1}, x_{2n}, a) + \gamma[d(x_{2n-1}, Tx_{2n-1}, a) + d(x_{2n}, Tx_{2n}, a)]$$
$$+\delta[d(x_{2n}, Tx_{2n-1}, a) + d(x_{2n-1}, Tx_{2n}, a)]$$
$$= \alpha \frac{[p + d(x_{2n-1}, x_{2n}, a)][d(x_{2n}, x_{2n+1}, a)]^r [d(x_{2n}, x_{2n}, a)]^q}{1 + \mu d(x_{2n}, x_{2n}, a) + \eta d(x_{2n-1}, x_{2n+1}, a) + d(x_{2n-1}, x_{2n}, a)}$$
$$+\beta d(x_{2n-1}, x_{2n}, a) + \gamma[d(x_{2n-1}, x_{2n}, a) + d(x_{2n}, x_{2n+1}, a)]$$
$$+\delta[d(x_{2n}, x_{2n}, a) + d(x_{2n-1}, x_{2n+1}, a)]$$
$$\leq \beta d(x_{2n-1}, x_{2n}, a) + \gamma[d(x_{2n-1}, x_{2n}, a) + d(x_{2n}, x_{2n+1}, a)]$$
$$+\delta[d(x_{2n-1}, x_{2n+1}, x_{2n}) + d(x_{2n-1}, x_{2n}, a) + d(x_{2n+1}, x_{2n}, a)]$$

or,

$$d(x_{2n}, x_{2n+1}, a) \leq \frac{\beta + \gamma + \delta}{1 - (\gamma + \delta)} d(x_{2n-1}, x_{2n}, a)$$

i.e. $d(x_{2n}, x_{2n+1}, a) \leq pd(x_{2n-1}, x_{2n}, a)$
Thus, we get

$$d(x_{2n+1}, x_{2n+2}, a) \leq pd(x_{2n}, x_{2n+1}, a)$$
$$\leq p^2 d(x_{2n-1}, x_{2n}, a)$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$\leq p^n d(x_0, x_1, a)$$

Using triangle inequality we have,

$$d(x_n, x_{n+m}, a) \leq d(x_n, x_{n+1}, x_{n+m}) + d(x_n, x_{n+1}, a)$$
$$+d(x_{n+1}, x_{n+2}, x_{n+m}) + d(x_{n+1}, x_{n+2}, a)$$
$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$+\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$+d(x_{n+m-2},x_{n+m-1},x_{n+m})+d(x_{n+m-1},x_{n+m},a)$$
$$\leq \sum_{k=1}^{n+m-2} d(x_k,x_{k+1},x_{n+m})+\sum_{k=1}^{n+m-1} d(x_k,x_{k+1},a)$$

Now we have,

$$d(x_n,x_{n+1},x_{n+m}) \leq pd(x_n,x_{n-1},x_{n+m})$$
$$\leq p^2 d(x_{n-1},x_{n-2},x_{n+m})$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leq p^n d(x_0,x_1,x_{n+m})$$

And also,

$$d(x_n,x_{n+1},a) \leq pd(x_n,x_{n-1},a)$$
$$\leq p^2 d(x_{n-1},x_{n-2},a)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\leq p^n d(x_0,x_1,a)$$

Thus,

$$\sum_{k=1}^{n+m-2} d(x_k,x_{k+1},x_{n+m}) \leq [p^{n+m-2}+p^{n+m-1}+...+...+p^n]d(x_0,x_1,x_{n+m})$$
$$\leq [\frac{p^{n+m-2}}{1-p}]d(x_0,x_1,x_{n+m}) \to 0 \text{ as n } \to \infty.$$

and,

$$\sum_{k=1}^{n+m-1} d(x_k,x_{k+1},a) \leq [p^{n+m-1}+p^{n+m}+...+...+p^n]d(x_0,x_1,a)$$
$$\leq [\frac{p^{n+m-1}}{1-p}]d(x_0,x_1,a) \to 0 \text{ as n } \to \infty.$$

Thus x_n is a Cauchy sequence. Since X is a complete 2-metric space, then there exists a point z in X such that $\lim_{n\to\infty} x_n = z$.
Now we have to show that z is a fixed point of T.

$$d(Tz,z,a) \leq d(Tz,z,x_{2n})+d(Tz,x_{2n},a)+d(x_{2n},z,a)$$
$$\leq d(Tz,z,x_{2n})+d(x_{2n},z,a)+d(Tz,Tx_{2n-1},a)$$

$$\leq d(Tz, z, x_{2n}) + d(x_{2n}, z, a)$$
$$+\alpha \frac{[p + d(z, Tz, a)][d(x_{2n-1}, Tx_{2n-1}, a)]^r [d(x_{2n-1}, Tz, a)]^q}{1 + \mu d(x_{2n-1}, Tz, a) + \eta d(z, Tx_{2n-1}, a) + d(z, x_{2n-1}, a)}$$
$$+\beta d(z, x_{2n-1}, a) + \gamma[d(z, Tz, a) + d(x_{2n-1}, Tx_{2n-1}, a)]$$
$$+\delta[d(x_{2n-1}, Tz, a) + d(z, Tx_{2n-1}, a)].$$

$$= d(Tz, z, x_{2n}) + d(x_{2n}, z, a)$$
$$+\alpha \frac{[p + d(z, Tz, a)][d(x_{2n-1}, x_{2n}, a)]^r [d(x_{2n-1}, Tz, a)]^q}{1 + \mu d(x_{2n-1}, Tz, a) + \eta d(z, x_{2n}, a) + d(z, x_{2n-1}, a)} + \beta d(z, x_{2n-1}, a)$$
$$+\gamma[d(z, Tz, a) + d(x_{2n-1}, x_{2n}, a)] + \delta[d(x_{2n-1}, Tz, a) + d(z, x_{2n}, a)].$$

when $n \to \infty, x_{2n} \to z$ and $x_{2n-1} \to z$.
So, we have

$$d(Tz, z, a) \leq (\gamma + \delta)d(z, Tz, a)$$

or, $1 - (\gamma + \delta)d(z, Tz, a) \leq 0$, which gives $d(Tz, z, a) = 0$.
Hence Tz =z. i.e. z is a fixed point of T. Now we show that z is a unique fixed point of T. If possible let w is the another fixed point of T. then

$$d(z, w, a) = d(Tz, Tw, a)$$
$$\leq \alpha \frac{[p + d(z, Tz, a)][d(w, Tw, a)]^r [d(w, Tz, a)]^q}{1 + \mu d(w, Tz, a) + \eta d(z, Tw, a) + d(z, w, a)}$$
$$+\beta d(z, w, a) + \gamma[d(z, Tz, a) + d(w, Tw, a)] + \delta[d(w, Tz, a) + d(z, Tw, a)].$$

Or,

$$d(z, w, a) \leq (\beta + 2\delta)d(z, w, a)$$

which is a contradiction.
So, d(z,w,a)=0 i.e. z = w. Thus z is a unique fixed point of T.

3. Remarks

In the above theorem that we proved, if we put

1. $\alpha = \beta = \delta = 0$, we get an analogue of Kannan [12]- type of contraction in 2-metric space.

2. $\alpha = \beta = \gamma = 0$, we get an analogue of Chatterjee [4] in 2-metric space.

3. $\alpha = \gamma = \delta = 0$, we get an analogue of Banach's fixed point theorem in 2-metric space.

4. $\alpha = 0$, we get Ciric-type of contraction mapping in 2-metric space.

5. $\gamma = \delta = 0$, we get Olatinwo and Ishola [14] type of contraction mapping in 2-metric space.

6. $p = \gamma = 1, q = \gamma = \delta = 0, \mu d(y, Tx, a) + \eta d(x, Ty, a) = 0$ such that $\alpha, \beta \in [0, 1), \alpha + \beta < 1$, we get Das and Gupta [5].

7. $p = q = \mu = \eta = \delta = 0, r = 1, \alpha, \beta \in [0, 1), \alpha + \beta < 1$ and $x \neq y$, we get Jaggi [10].

8. $p = q = \gamma = \delta = 0, \mu = \eta = \gamma = 1, \alpha, \beta \in [0, 1), \alpha + \beta < 1$, we get Jaggi and Das[10] [11].

The theorem generalizes similarly several other contractive conditions.

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