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NEW RESULTS OF bs-γ-OPEN MAPPINGS AND sb-γ-OPEN MAPPINGS IN TOPOLOGICAL SPACES

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Abstract: In this paper, we introduce the notions of b-γ-continuous, b-γ-irresolute, b-γ-open, bs-γ-open and sb-γ-open mappings in topological spaces. With this notions, we also introduce b-γ-compact, b-γ-connected and b-γ-Lindelöff spaces Also we investigate some fundamental properties. Finally, we discuss the relationship among these mappings.

Keywords and Phrases: bs-γ-open mappings, sb-γ-open mappings, b - γ - continuous, b-γ-irresolute, b-γ-open mappings.

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1. Introduction

In 1979, Kasahara [2] introduced the notion of an operation γ on topological spaces. After that the notion of γ-open sets was introduced by Ogata [3] in 1991. As a generalization of γ-open sets, Hariwan Z. Ibrahim [1] defined and investigated the notion of b-γ-open sets in general topological spaces. Recently, Sivashanmugaraja and Vadivel [5] introduced the notion of b-γ-open fuzzy sets in fuzzy topological spaces. The purpose of this paper is to introduce and investigate a new type of mappings called b-γ-continuous mappings, b-γ-irresolute, b-γ-open mappings, bs-γ-open mappings and sb-γ-open mappings. Connected and compactness are powerful tools in topology but they have many dissimilar properties. The notions of b-γ-compact, b-γ-connected and b-γ-Lindelöff spaces are also introduced. Further, we discussed some basic properties of these mappings.

2. Preliminaries

Throughout this paper, the space (X, τ), and (Y, σ) or (simply X and Y) represent a topological space.

Definition 2.1. [3] *Let X be a space and γ be an operation on τ. A subset A of X is called γ-open, if for every $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. Then, τ_γ denotes the set of all γ-open sets in X. Clearly $\tau_\gamma \subseteq \tau$. Complements of γ-open sets are called γ-closed.*

Definition 2.2. [1] *A subset A of a space X is said to be b-γ-open if $A \subseteq \tau_\gamma$-int$(cl(A)) \cup cl(\tau_\gamma$-int$(A))$.*

Remark 2.1. [4] *A subset A of X is called b-γ-closed if and only if its complement is b-γ-open. The collection of all b-γ-open and b-γ-closed sets of (X, τ) are denoted by b-$\gamma O(X)$ and b-$\gamma C(X)$ respectively.*

Definition 2.3. [4] *Let (X, τ) be a space and A be a subset of X. Then the b-γ-closure and b-γ-interior of A are defined as follows:*

(i) $bcl_\gamma(A)) = \bigcap\{B : A \subseteq B$ *and* $B \in$ b-$\gamma C(X)\}$;

(ii) $bint_\gamma(A) = \bigcup\{B : A \supseteq B$ *and* $B \in$ b-$\gamma O(X)\}$.

Definition 2.4. [4] *Let (X, τ) be a space and A be a subset of X. Then A is said to be b-γ-neighborhood of a point $x \in X$, if there exists a b-γ-open set B such that $x \in B \subseteq A$.*

The class of all *b*-γ-nbds of $x \in X$ is said to be *b*-γ-neighborhood system of x and represented by *b*-γ-N_x.

Definition 2.5. [1] *A space (X, τ) is said to be:*

(i) b-γ-T_1, *if for every $x, y \in X$ and $x \neq y$, there exists b-γ-open sets U and V such that $x \in U$, $y \notin U$ and $x \notin V$, $y \in V$;*

(ii) b-γ-T_2, *if for every $x, y \in X$ and $x \neq y$, there exists b-γ-open sets U, V and $U \cap V = \phi$ such that $x \in U$, $y \in V$.*

Proposition 2.1. [4] *Let A be a subset of a space X. Then, the following statements are hold:*

(i) *A is b-γ-closed $\Leftrightarrow$ b-$\gamma Ds(A) \subset A$;*

(ii) *A is b-γ-open $\Leftrightarrow$ A is b-γ-neighborhood for every point $x \in A$;*

(iii) $bcl_\gamma(A) = A \cup b\text{-}\gamma Ds(A)$.

3. b-γ-continuous and b-γ-irresolute mappings

Definition 3.1. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is said to be b-γ-continuous, if* $f^{-1}(A)$ *is b-γ-open in* X, *for every open set* A *of* Y.

Theorem 3.1. *For a mapping* $f : (X, \tau) \to (Y, \sigma)$, *the following statements are equivalent:*

(i) f *is b-γ-continuous;*

(ii) $f^{-1}(B)$ *is b-γ-closed in* X, *for every closed set* B *of* Y,

(iii) For every subset A *of* X, $f(bcl_\gamma(A)) \subseteq cl(f(A))$;

(iv) For every subset B *of* Y, $bcl_\gamma(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Proof. (i) ⇔ (ii) Evident.
(iii) ⇔ (iv) Let $B \subseteq Y$ and $A = f^{-1}(B)$. Then by hypothesis, we have $f(bcl_\gamma(f^{-1}(B))) \subseteq cl(f(f^{-1}(B))) = cl(B)$. Thus, $bcl_\gamma(f^{-1}(B)) \subseteq f^{-1}(cl(B))$.

Conversely, let $A \subseteq X$ and $B = f(A)$. By hypothesis, we have, $bcl_\gamma(f^{-1}(f(A))) \subseteq f^{-1}(cl(f(A)))$. Thus, $f(bcl_\gamma(A)) \subseteq cl(f(A))$.
(ii) ⇔ (iv) Let B be any subset of Y. Since, $f^{-1}(cl(B))$ is b-γ-closed and $f^{-1}(B) \subseteq f^{-1}(cl(B))$, $bcl_\gamma(f^{-1}(B) \subseteq f^{-1}(cl(B))$.
(iv) ⇔ (ii) Let B be any closed subset Y. By hypothesis, $bcl_\gamma(f^{-1}(B) \subseteq f^{-1}(cl(B)) = f^{-1}(B)$. Thus, $f^{-1}(B)$ is b-γ-closed.

Definition 3.2. *A mapping* $f : (X, \tau) \to (Y, \sigma)$ *is called b-γ-irresolute if* $f^{-1}(V)$ *is b-γ-open in* X, *for every b-γ-open set* V *of* Y.

Theorem 3.2. *If* $f : (X, \tau) \to (Y, \sigma)$ *be a mapping, then the following are equivalent:*

(i) f *is b-γ-irresolute;*

(ii) For each subset A *of* X, $f(bcl_\gamma(A)) \subseteq bcl_\gamma(f(A))$;

(iii) $f^{-1}(K)$ *is b-γ-closed in* (X, τ), *for every b-γ-closed set* K *of* (Y, σ).

Proof. (i) ⇔ (ii) Suppose that $x_1 \in f(bcl_\gamma(A))$ and V be any b-γ-open set containing x_1. Then there exists a point $x_2 \in X$ and a b-γ-open set U such that $f(x_2) = x_1$ and $x_2 \in U$ and $f(U) \subseteq V$. Since $x_2 \in bcl_\gamma(A)$, $U \cap A \neq \phi$ and hence $\phi \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A)$. This implies $x_1 \in bcl_\gamma(f(A))$. Thus,

$f(bcl_\gamma(A)) \subseteq bcl_\gamma(f(A))$.
(ii) $\Leftrightarrow$ (iii) Let K be a b-γ-closed set in Y. Therefore, $bcl_\gamma(K) = K$. By hypothesis, we have $f(bcl_\gamma(f^{-1}(K)) \subseteq bcl_\gamma(f(f^{-1}(K)) = bcl_\gamma(K) = K$. Thus, $bcl_\gamma(f^{-1}(K) \subseteq f^{-1}(K)$. Therefore $f^{-1}(K)$ is b-γ-closed.
(iii) $\Leftrightarrow$ (i) Evident.

Theorem 3.3. *Let $f : X \to Y$ be a b-γ-continuous one-one map and Y is T_2,-space then X is b-γ-T_2 space.*
Proof. Let x and y be two distinct points in X then there exist open sets U and V in Y and $U \cap V \neq \phi$ such that $f(x) \in U$ and $f(y) \in V$. Since f is b-γ-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are b-γ-open in X containing x and y respectively. Therefore $f^{-1}(U) \cap f^{-1}(V) = \phi$. Hence, X is b-γ-T_2.

Definition 3.3. *A space $(X,\ \tau)$ is called:*

(i) *b-γ-compact if for every b-γ-open cover of X has a finite subcover;*

(ii) *b-γ-connected if it cannot be expressed as the union of two disjoint non-empty b-γ-open sets of X;*

(iii) *b-γ-Lindelöff if every b-γ-open cover of X has a countable subcover.*

Definition 3.4. *A subset A of a space X is said to be b-γ-compact relative to X if every cover of A by b-γ-open sets of X has a finite subcover.*

Example 3.1. Let $X = \{a, b, c, d\}$ with topology $\tau = \{X,\ \phi,\ \{a, b\},\ \{c, d\}\}$. Define an operation γ on τ by $\gamma(A) = A$. Then clearly the space X is b-γ-compact. Since, for every b-γ-open cover of X has a finite subcover.

Example 3.2. Let $X = \{a, b, c, \}$ with topology $\tau = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define an operation γ on τ by

$$\gamma(A) = \begin{cases} A, & \text{if } A = \{b\} \\ X, & \text{if } A \neq \{b\}. \end{cases}$$

Then the space X is b-γ-connected.

4. b-γ-open and b-γ-closed mappings

Definition 4.1. *A mapping $f : (X,\ \tau) \to (Y,\ \sigma)$ is said to be*

(i) *b-γ-open, if $f(U)$ is b-γ-open in Y, for every open set U of X;*

(ii) *b-γ-closed, if $f(U)$ is b-γ-closed in Y, for every closed set U of X.*

Theorem 4.1. *For an one-one and onto mapping $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) f^{-1} *is b-γ-continuous;*

(ii) f *is b-γ-open;*

(iii) f *is b-γ-closed.*

Proof. Evident.

Definition 4.2. *Let (X, τ) be a topological space and A be a subset of X. Then the b-γ-border$(A) = A \setminus bint_\gamma(A)$. It is denoted by b-$\gamma Br(A)$.*

Theorem 4.2. *For a mapping $f : (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:*

(i) f *is b-γ-open;*

(ii) For every $x \in X$ and every neighborhood U of x, there exists b-γ-open set V in Y containing $f(x)$ such that $V \subseteq f(U)$;

(iii) For every subset A of X, $f(int(A)) \subseteq bint_\gamma(f(A))$;

(iv) For every subset B of Y, $int(f^{-1}(B)) \subseteq f^{-1}(bint_\gamma(B))$;

(v) For every subset B of Y, $f^{-1}(b\text{-}\gamma Br(B)) \subseteq Br(f^{-1}(B))$;

(vi) For every subset B of Y, $f^{-1}(bcl_\gamma(B)) \subseteq cl(f^{-1}(B))$.

Proof. (i) $\Rightarrow$ (ii) Let $x \in X$ and U be neighborhood of x. Then there exists an open set K such that $x \in K \subseteq U$ and hence $f(x) \in f(K) \subseteq f(U)$. Since f is b-γ-open, then $f(K)$ is b-γ-open in Y. Take $f(K) = V$, we have $f(x) \in V \subseteq f(U)$.
(ii) $\Rightarrow$ (i) Let $x \in X$ and U be an open set containing x. Then U is neighborhood of every $x \in U$. By hypothesis, there exists a b-γ-open set V in Y such that $f(x) \in V \subseteq f(U)$. Hence, $f(U)$ is b-γ-neighborhood of each $f(x) \in f(U)$. By Proposition 2.1, $f(U)$ is b-γ-open in Y. Thus, f is b-γ-open mapping.
(i) $\Rightarrow$ (iii) Let $A \subseteq X$. Since $int(A) \subseteq A \subseteq X$, which is open.By hypothesis, $f(int(A))$ is b-γ-open in Y. Thus, $f(int(A)) \subseteq bint_\gamma(f(A))$, Hence $f(int(A)) \subseteq bint_\gamma(f(A)) \subseteq f(A)$.
(iii) $\Rightarrow$ (iv) Let $A = f^{-1}(B)$. Then by hypothesis, $f(int(f^{-1}(B)) \subseteq bint_\gamma(f(f^{-1}(B))$. Therefore $int(f^{-1}(B)) \subseteq f^{-1}(bint_\gamma(f(f^{-1}(B)))) \subseteq f^{-1}(bint_\gamma(B))$.
(iv) $\Rightarrow$ (i) Let A be an open set in X. Then $f(A) \subseteq Y$ and by hypothesis,

$int(f^{-1}(f(A))) \subseteq f^{-1}(bint_\gamma(f(A)))$. This implies that, $int(A) \subseteq f^{-1}(bint_\gamma(f(A)))$. Thus $f(int(A)) \subseteq bint_\gamma(f(A))$. Therefore, f is b-γ-open.
(iv) $\Rightarrow$ (v) Let $B \subseteq Y$. Then by hypothesis, $f^{-1}(B) \setminus f^{-1}(bint_\gamma(B)) \subseteq f^{-1}(B) \setminus int(f^{-1}(B))$. Therefore, $f^{-1}(b\text{-}\gamma Br(B)) \subseteq Br(f^{-1}(B))$.
(v) $\Rightarrow$ (iv) Let $B \subseteq Y$. Then $f^{-1}(B \setminus bint_\gamma(B)) \subseteq f^{-1}(B) \setminus int(f^{-1}(B))$ and hence $f^{-1}(B) \setminus f^{-1}(bint_\gamma(B)) \subseteq f^{-1}(B) \setminus int(f^{-1}(B))$. Therefore, $int(f^{-1}(B)) \subseteq f^{-1}(bint_\gamma(B))$.
(i) $\Rightarrow$ (vi) Let B be any subset of Y and $x \in f^{-1}(bcl_\gamma(B))$. Then $f(x) \in bcl_\gamma(B)$. Suppose that U is an open set containing x. By hypothesis, $f(U)$ is b-γ-open in Y. Hence, $B \cap f(U) \neq \phi$. Thus $U \cap f^{-1}(B) \neq \phi$. Thus, $x \in cl(f^{-1}(B))$. So, $f^{-1}(bcl_\gamma(B)) \subseteq cl(f^{-1}(B))$.
(vi) $\Rightarrow$ (i) Let B be any subset of Y. Then $(Y \setminus B) \subseteq Y$. By hypothesis, $f^{-1}(bcl_\gamma(Y \setminus B)) \subseteq cl(f^{-1}(Y \setminus B))$ and hence $X \setminus f^{-1}(bint_\gamma(B)) \subseteq X \setminus int(f^{-1}(B))$ that implies $int(f^{-1}(B)) \subseteq f^{-1}(bint_\gamma(B))$. Then by (iv), f is b-γ-open.

Theorem 4.3. *Let $f : (X, \tau) \to (Y, \sigma)$ be a b-γ-closed mapping. Then the following are hold:*

(i) If f is an onto and $f^{-1}(B)$, $f^{-1}(C)$ have disjoint neighborhoods of X, then B and C are disjoint of Y;

(ii) For every subset A of X, $bint_\gamma(bcl_\gamma(f(A))) \subseteq f(cl(A))$.

Proof. (i) Let P and Q be two disjoint neighborhoods of $f^{-1}(B)$ and $f^{-1}(C)$. Then there exists two b-γ-open sets U and V such that $f^{-1}(B) \subseteq U \subseteq P$, $f^{-1}(C) \subseteq V \subseteq Q$. But, f is an onto map, then $f(f^{-1}(B)) = B \subseteq f(U) \subseteq f(P)$, $f(f^{-1}(C)) = C \subseteq f(V) \subseteq f(Q)$. Since P and Q are disjoint, $f(P \cap Q) = \phi$ and hence $B \cap C \subseteq f(U \cap V) \subseteq f(P \cap Q) = \phi$. Therefore, B and C are disjoint of Y.
(ii) Since $A \subseteq cl(A) \subseteq X$ and f is a b-γ-closed mapping, $f(cl(A))$ is b-γ-closed in Y. Thus, $f(A) \subseteq bcl_\gamma(f(A)) \subseteq f(cl(A))$. Hence, $bint_\gamma(bcl_\gamma(f(A))) \subseteq f(cl(A))$.

Theorem 4.4. *For a mapping $f : (X, \tau) \to (Y, \sigma)$, then the following are equivalent:*

(i) f is b-γ-closed;

(ii) For each subset A of X, $bcl_\gamma(f(A)) \subseteq f(cl(A))$;

(iii) If f is an onto, then for every subset B of Y and each open set U in X containing $f^{-1}(B)$, there exists a b-γ-open set V of Y containing B such that $f^{-1}(V) \subseteq U$.

Proof. (i) $\Rightarrow$ (ii) Let $cl(A)$ be closed subset of X. Since f is b-γ-closed, $f(cl(A)) \in b\text{-}\gamma C(Y)$. Thus, $bcl_\gamma(f(A)) \subseteq f(cl(A))$.
(ii) $\Rightarrow$ (i) Let A be a closed subset of X. By hypothesis, $bcl_\gamma(f(A)) \subseteq f(cl(A)) = f(A)$. Thus, $f(A) \in b\text{-}\gamma C(Y)$. Hence, f is b-γ-closed.
(i) $\Rightarrow$ (iii) Let $V = Y \setminus (f(X \setminus U))$ and U is an open set of X containing $f^{-1}(B)$. Since f is b-γ-closed, V is b-γ-open in Y. But, $f^{-1}(B) \subseteq U$, then B is a subset of $f(U)$ and $f(X \setminus U) \subseteq Y \setminus B$, that is, B is a subset of V. and $f^{-1}(V) \subseteq U$.
(iii) $\Rightarrow$ (i) Let F be a closed subset of X and $y \in Y \setminus f(F)$. Then $f^{-1}(y) \in X \setminus F$, which is open in X. Hence by hypothesis, there exists a b-γ-open set V containing y such that $f^{-1}(V) \subseteq X \setminus F$. But f is an onto, then $y \in V \subseteq Y \setminus f(F)$ and $Y \setminus f(F)$ is the union of b-γ-open sets and hence, $f(F)$ is b-γ-closed. Thus, f is b-γ-closed.

Remark 4.1. *The restriction of b-γ-open mapping is may not be b-γ-open as shown in the following example.*

Example 4.1. Let $X = Y = \{a, b, c, d\}$ with topologies $\tau = \{X, \phi, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}\}$ and $\sigma = \{Y, \phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$. Define an operation γ on σ by

$$\gamma(A) = \begin{cases} int(cl(A)), & \text{if } A \neq \{a\} \\ cl(A), & \text{if } A = \{a\}. \end{cases}$$

Also a mapping $f : (X,\ \tau) \to (Y,\ \sigma)$ is defined as $f(a) = b, f(b) = d, f(c) = c$ and $f(d) = a$. Then clearly f is b-γ-open. Take $A = \{b, d\} \subseteq X$. Then $f_A : (A, \tau_A) \to (Y, \sigma)$ is not b-γ-open. Since $\{b, d\} \in \tau_A$ but $f(\{b, d\}) = \{a, d\} \notin b\text{-}\gamma O(Y)$.

Remark 4.2. *The composition of two b-γ-open mappings may not be b-γ-open as shown in the following example.*

Example 4.2. Let $X = Y = Z = \{a, b, c\}$ with topologies $\tau_X = \{X, \phi, \{a, c\}\},\ \tau_Y$ is an indiscrete topology and $\tau_Z = \{Z, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$. Define an operation γ on τ_Y and τ_Z by $\gamma(A) = A$ and

$$\gamma(A) = \begin{cases} A, & \text{if } A = \{b\} \\ X, & \text{if } A \neq \{b\}. \end{cases}$$

respectively. Also, $f : (X,\ \tau_X) \to (Y,\ \tau_Y)$ and $g : (Y, \tau_Y) \to (Z, \tau_Z)$ are identity mappings. Clearly, f and g are b-γ open but $(g \circ f)$ is not b-γ-open. Since $\{a, c\} \subseteq X$ is an open set of X, but $(g \circ f)(\{a, c\}) = \{a, c\} \notin b\text{-}\gamma O(Z)$.

Theorem 4.5. *Let $f : X \to Y$ and $g : Y \to Z$ be two mappings. Then the following are hold:*

(i) If f is an open and g is a b-γ-open mappings, then the composite map $g \circ f$ is b-γ-open;

(ii) if f is an onto continuous map and the composite map $g \circ f$ is a b-γ-open mapping, then the map g is b-γ-open, ;

(iii) If the composite map $g \circ f$ is an open and g is an one-one b-γ-continuous map, then the map f is b-γ-open, .

Proof. (i) Let U be an open set in X. Since f is an open, $f(U)$ is an open in Y. But g is a b-γ-open map, then $g(f(U))$ is b-γ-open set on Z. Hence, $g \circ f$ is b-γ-open.
(ii) Let U be an open set in Y and f be a continuous map. Then $f^{-1}(U)$ is open in X. But $g \circ f$ is a b-γ-open map, then $(g \circ f)(f^{-1}(U))$ is b-γ-open in Z. Since f is onto, $g(U)$ is b-γ-open in Z. Thus, g is b-γ-open.
(iii) Let U be an open set in X. and $g \circ f$ be an open map. Then $(g \circ f)(U) = g(f(U))$ is open in Z. Since g is an onto b-γ-continuous map, $f(U)$ is b-γ-open in Y. Thus, f is b-γ-open.

Theorem 4.6. *Let $f : X \to Y$ be a bijective b-γ-open mapping. Then the following statements are hold:*

(i) If X is a T_i-space, then Y is b-γ-T_i where $i = 1, 2$;

(ii) If Y is a b-γ-compact (b-γ-Lindelöff) space, then X is compact (Lindelöff).

Proof. (i) We prove that for the case of T_1-space. Let $y_1,\ y_2 \in Y$ and $y_1 \neq y_2$. Then there exists $x_1,\ x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a T_1-space, then there exists two open sets $U,\ V$ of X such that $x_1 \in U,\ x_2 \notin U$ and $x_2 \in V,\ x_1 \notin V$. But, f is a b-γ-open map, then $f(U)$ and $f(V)$ are b-γ-open sets of Y with $y_1 \in f(U)$, $y_2 \notin f(U)$ and $y_2 \in f(V)$, $y_1 \notin f(V)$. Thus, Y is b-γ-T_1.
(ii) We prove that the theorem for b-γ-compact. Let $\{U_i : i \in I\}$ be a family of open cover of X and f be a onto b-γ-open mapping. Then $\{f(U_i) : i \in I\}$ is a b-γ-open cover of Y. But, Y is b-γ-compact space, hence there exists a finite subset I_o of I such that $Y = \cup\{f(U_i) : i \in I_0\}$ Then by one-one of f, $\{U_i : i \in I_0\}$ is a finite subfamily of X. Hence, X is compact.

Theorem 4.7. *If $f : X \to Y$ is a onto b-γ-open mapping and Y is b-γ-connected space, then X is connected.*
Proof. Assume that X is a disconnected space. Then there exists two non-empty sets $U,\ V$ of X and $U \cap V = \phi$ such that $X = U \cup V$. But f is a onto b-γ-open map, then $f(U)$ and $f(V)$ are non-empty b-γ-open sets of Y and $f(U) \cap f(V) = \phi$

with $Y = f(U) \cup f(V)$, which is a contradiction to our assumption that Y is b-γ-connected.

5. sb-γ-open and sb-γ-closed mappings

Definition 5.1. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is called:*

(i) super b-γ-open (shortly, sb-γ-open), if the image of b-γ-open set of (X, τ) is open in (Y, σ);

(ii) super b-γ-closed (shortly, sb-γ-closed), if the image of b-γ-closed set of (X, τ) is closed in (Y, σ).

Example 5.1. Let $X = Y = \{a, b, c\}$ with topologies $\tau_X = \{X, \phi, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}$ and τ_Y be the discrete topology. Define an operation γ on τ_X by

$$\gamma(A) = \begin{cases} int(cl(A)), & \text{if } a \in A \\ cl(A), & \text{if } a \notin A. \end{cases}$$

Also the map $f : (X, \tau_X) \to (Y, \tau_Y)$ is defined as $f(a) = b,\ f(b) = c$ and $f(c) = a$ is sb-γ-open.

Theorem 5.1. *If $f : (X,\ \tau) \to (Y,\ \sigma)$ is a mapping, then the following are equivalent:*

(i) f is sb-γ-open;

(ii) for every $x \in X$ and each b-γ-neighborhood U of x, there exists a neighborhood V of $f(x)$ such that $V \subseteq f(U)$;

(iii) For every subset A of X, $f(bint_\gamma(A)) \subseteq int(f(A))$;

(iv) For every subset B of Y, $bint_\gamma(f^{-1}(B)) \subseteq f^{-1}(int(B))$;

(v) For every subset B of Y, $f^{-1}(Br(B)) \subseteq b\text{-}\gamma Br(f^{-1}(B))$;

(vi) For every subset B of Y, $f^{-1}(cl(B)) \subseteq bcl_\gamma(f^{-1}(B))$;

(vii) If f is onto, then for every subset B of Y and for any set $F \in b\text{-}\gamma C(X)$ containing $f^{-1}(B)$, there exists a closed subset H of Y containing B such that $f^{-1}(H) \subseteq F$.

Proof. (i)⇒ (ii) Let $x \in X$ and U be a b-γ-neighborhood of x. Then there exists $K \in b\text{-}\gamma O(X)$ such that $x \in K \subseteq U$ and hence $f(x) \in f(K) \subseteq f(U)$. Hence by hypothesis, $f(K) \in \sigma$ and containing $f(x)$. Take $f(K) = V$, then $f(x) \in V \subseteq f(U)$.
(ii)⇒ (i) Let $x \in X$ and U be a b-γ-open set of X containing x. Then $f(x) \in f(U)$. Hence by hypothesis, there exists $V \in \sigma$ containing $f(x)$ such that $f(x) \in V \subseteq f(U)$. Therefore, $f(U)$ is neighborhood for $f(x) \in f(U)$. Hence $f(U)$ is open in Y and therefore f is sb-γ-open.
(i) ⇒ (iii) Let $A \subseteq X$. Since $bint_\gamma(A) \subseteq A \subseteq X$ is b-γ-open set and by hypothesis, $f(bint_\gamma(A)) \subseteq f(A)$ is open in Y. Thus, $f(bint_\gamma(A)) \subseteq int(f(A))$.
(iii) ⇒ (iv) Let $A = f^{-1}(B)$. Then by hypothesis, $f(bint_\gamma(f^{-1}(B))) \subseteq int\,(f(f^{-1}(B))) \subseteq int(B)$. Thus, $bint_\gamma(f^{-1}(B)) \subseteq f^{-1}(int(B))$.
(iv) ⇒ (v) Let B be a subset of Y. Then by hypothesis and Definition 2.4, we have $f^{-1}(B) \setminus f^{-1}(int(B)) \subseteq f^{-1}(B) \setminus bint_\gamma(f^{-1}(B))$ and therefore, $f^{-1}(Br(B)) \subseteq b\text{-}\gamma Br(f^{-1}(B))$.
(v) ⇒ (iv) Let B be a subset of Y. Then by hypothesis and Definition 2.4, we have $f^{-1}(B \setminus int(B)) \subseteq f^{-1}(B) \setminus bint_\gamma(f^{-1}(B))$ and hence $f^{-1}(B) \setminus f^{-1}(int(B)) \subseteq f^{-1}(B) \setminus bint_\gamma(f^{-1}(B))$. Thus, $bint_\gamma(f^{-1}(B)) \subseteq f^{-1}(int(B))$.
(iv) ⇒ (vi) Let B be a subset of Y. Then $Y \setminus B \subseteq Y$, hence by hypothesis, we have $bint_\gamma(f^{-1}(Y \setminus B)) \subseteq f^{-1}(int(Y \setminus B))$ and hence $X \setminus bcl_\gamma(f^{-1}(B)) \subseteq X \setminus f^{-1}(cl(B))$. Thus, $f^{-1}(cl(B)) \subseteq bcl_\gamma(f^{-1}(B))$.
(vi) ⇒ (iv) Let B be a subset of Y. Then $Y \setminus B \subseteq Y$. So by hypothesis, we have $f^{-1}(cl(Y \setminus B)) \subseteq bcl_\gamma(f^{-1}(Y \setminus B))$ and hence $X \setminus f^{-1}(int(B)) \subseteq X \setminus bint_\gamma(f^{-1}(B))$. Thus, $bint_\gamma(f^{-1}(B) \subseteq f^{-1}(int(B))$.
(iv) ⇒ (i) Let A be a b-γ-open set in X. Then $f(A) \subseteq Y$ and by hypothesis, $bint_\gamma(f^{-1}(f(A))) \subseteq f^{-1}(int(f(A)))$. This gives that, $bint_\gamma(A) \subseteq f^{-1}(int(f(A)))$. Thus $f(bint_\gamma(A)) \subseteq int(f(A))$. Hence by (iii), f is sb-γ-open.
(i) ⇒ (vii) Let $H = Y \setminus f(X \setminus F)$ and F be a b-γ-closed set of X containing $f^{-1}(B)$. Then $X \setminus F$ is a b-γ-open set. But f is a sb-γ-open mapping, then $f(X \setminus F)$ is open in Y. Therefore, H is a closed set of Y and $f^{-1}(H) = X \setminus f^{-1}f(X \setminus F) \subseteq X \setminus (X \setminus F) = F$.
(vii) ⇒ (i) Let U be a b-γ-open set in X and put $B = Y \setminus f(U)$. Then $X \setminus U$ is b-γ-closed with $f^{-1}(B) \subseteq X \setminus U$. By hypothesis, there exists a closed set M of Y such that $B \subseteq M$ and $f^{-1}(M) \subseteq X \setminus U$. Hence, $f(U) \subseteq Y \setminus M$ and since $B \subseteq M$, then $Y \setminus M \subseteq Y \setminus B = f(U)$. Therefore $f(U) = Y \setminus M$ which is open. Thus, f is sb-γ-open.

Theorem 5.2. *Let $f : (X, \tau) \to (Y, \sigma)$ be an one-one and onto sb-γ-open mapping. Then the following are hold:*

(i) If X is a b-γ-T_i-space, then Y is T_i, where $i = 1,\ 2$;

(ii) If Y is a compact (Lindelöff) space, then X is b-γ-compact (b-γ-Lindelöff).

Proof. (i) We prove that for the case of b-γ-T_2-space. Let $y_1,\ y_2 \in Y$ and $y_1 \neq y_2$. Then there exists $x_1,\ x_2 \in X$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Since X is a b-γ-T_2-space, then there exists two b-γ-open sets $U,\ V$ of X and $U \cap V = \phi$ such that $x_1 \in U$ and $x_2 \in V$. But, f is sb-γ-open map, then $f(U),\ f(V)$ are open sets of Y with $y_1 \in f(U),\ y_2 \in f(V)$, and $f(U) \cap f(V) = \phi$. Thus, Y is T_2.
(ii) We prove that the theorem for b-γ-Lindelöff space. Let $\{U_i : i \in I\}$ be a family of b-γ-open cover of X and f be a onto sb-γ-open mapping. Then $\{f(U_i) : i \in I\}$ is an open cover of Y. But, Y is a Lindelöff space, hence there exists a countable subset I_0 of I such that $Y = \cup\{f(U_i) : i \in I_0\}$. Then by one-one of f, $\{U_i : i \in I_0\}$ is a countable subfamily of X. Therefore, X is b-γ-Lindelöff.

Theorem 5.3. *If $f : (X, \tau) \to (Y, \sigma)$ is an onto sb-γ-open mapping and Y is a connected space, then X is b-γ-connected.*
Proof. Obvious.

6. bs-γ-open and bs-γ-closed mappings

Definition 6.1. *A mapping $f : (X, \tau) \to (Y, \sigma)$ is said to be:*

(i) b-star-γ-open (shortly, bs-γ-open), if the image of b-γ-open set of (X, τ) is b-γ-open in $(Y,\ \sigma)$;

(ii) b-star-γ-closed (shortly, bs-γ-closed), if the image of b-γ-closed set of (X, τ) is b-γ-closed in $(Y,\ \sigma)$.

Theorem 6.1. *Let $f : X \to Y$ be an 1-1 and onto mapping. Then the following statements are equivalent:*

(i) f is bs-γ-closed;

(ii) f is bs-γ-open;

(iii) f^{-1} is b-γ-irresolute.

Proof. Evident.

Example 6.1. Let $X = Y = \{a, b, c\}$ with topologies $\tau = \{X, \phi,\ \{a\},\ \{b\},\ \{a, b\},\ \{b, c\}\}$ and $\sigma = \{Y, \phi, \{a, c\}\}$. Define an operation γ on τ and σ by $\gamma(A) = A$. Also a mapping $f : (X, \tau) \to (Y, \sigma)$ which defined by $f(a) = c,\ f(b) = a$ and $f(c) = b$. Then f is bs-γ-open.

Theorem 6.2. *For a mapping $f : X \to Y$ the following statements are equivalent:*

(i) f *is bs-γ-open;*

(ii) *For each* $x \in X$ *and each b-γ-neighborhood* U *of* x*, there exists* $V \in b\text{-}\gamma O(Y)$ *containing* $f(x)$ *such that* $V \subseteq f(U)$*;*

(iii) *For every subset* A *of* X*,* $f(bint_\gamma(A)) \subseteq bint_\gamma(f(A))$*;*

(iv) *For every subset* B *of* Y*,* $bint_\gamma(f^{-1}(B)) \subseteq f^{-1}(bint_\gamma(B))$*;*

(v) *For every subset* B *of* Y*,* $f^{-1}(b\text{-}\gamma Br(B)) \subseteq b\text{-}\gamma Br(f^{-1}(B))$*;*

(vi) *For every subset* B *of* Y*,* $f^{-1}(bcl_\gamma(B)) \subseteq bcl_\gamma(f^{-1}(B))$*.*

Proof. It is similar to that of Theorem 5.1.

Theorem 6.3. *If* $f : X \to Y$ *is an onto bs-γ-closed mapping and* $f^{-1}(M)$*,* $f^{-1}(N)$ *have disjoint b-γ-neighborhoods of* X*, then* M *and* N *are disjoint of* Y*.*
Proof. Evident.

Theorem 6.4. *For a mapping* $f : X \to Y$*, then the following statements are equivalent:*

(i) f *is bs-γ-closed;*

(ii) *For every subset* A *of* X*,* $bcl_\gamma(f(A)) \subseteq f(bcl_\gamma(A))$*;*

(iii) *If* f *is an onto, then for every subset* B *of* Y *and for each b-γ-open set* U *of* X *containing* $f^{-1}(B)$*, there exists a b-γ-open set* V *of* Y *containing* B *such that* $f^{-1}(V) \subseteq U$*.*

Proof. Evident.

Theorem 6.5. *Let* $f : (X, \tau_X) \to (Y, \tau_Y)$ *and* $g : (Y, \tau_Y) \to (Z, \tau_Z)$ *be two mappings. Then the following statements are hold:*

(i) *If* f *and* g *are bs-γ-open, then the composite map* $g \circ f$ *is a bs-γ-open mapping;*

(ii) *If* f *is a onto b-γ-continuous mapping and the composite map* $g \circ f$ *is bs-γ-open, then* g *is b-γ-open.*

Proof. (i) Let U be a b-γ-open in X and f be a bs-γ-open mapping. Then $f(U)$ is b-γ-open in Y. Since g is bs-γ-open, $g(f(U))$ is b-γ-open in Z. Thus, $g \circ f$ is bs-γ-open.
(ii) Let U be an open set in Y and f be a bs-γ-continuous mapping. Then $f^{-1}(U) \in$

b-$\gamma O(X)$. Since, $g \circ f$ is bs-γ-open, $(g \circ f)(f^{-1}(U))$ is b-γ-open in Z. Also, by onto of f, $g(U)$ is b-γ-open in Z. Thus, g is b-γ-open.

Theorem 6.6. *Let $f : X \to Y$ and $g : Y \to Z$ be two mappings.*

(i) If g is an one-one bs-γ-open mapping and $g \circ f$ is b-γ-irresolute, then f is b-γ-irresolute;

(ii) If f is an onto bs-γ-open mapping and $g \circ f$ is b-γ-irresolute, then g is b-γ-irresolute,

Proof. (i) Let U be a b-γ-open in Y. Then $g(U)$ is b-γ-open in Z. Since, $g \circ f$ is b-γ-irresolute, $(g \circ f)^{-1}(g(U))$ is b-γ-open in X. Since g is an one-one map, $f^{-1}(U)$ is b-γ-open in X. Thus, f is b-γ-irresolute.
(ii) Let V be a b-γ-open in Z. Then $(g \circ f)^{-1}(V)$ is b-γ-open in X. Since, f is a bs-γ-open mapping, $f((g \circ f)^{-1}(V))$ is b-γ-open in Y. Since f is a onto map, then $g^{-1}(V)$ is b-γ-open in Y. Thus, g is b-γ-irresolute.

Theorem 6.7. *Let $f : (X, \tau) \to (Y, \sigma)$ be an one-one and onto b-γ-open mapping. Then the following statements are hold:*

(i) If X is a b-γ-T_i-space, then Y is b-γ-T_i, where $i = 1,\ 2$;

(ii) If Y is a b-γ-compact (b-γ-Lindelöff) space, then X is b-γ-compact (b-γ-Lindelöff).

Proof. Evident.

Theorem 6.8. *If $f : (X, \tau) \to (Y, \sigma)$ is a onto bs-γ-open mapping and Y is a b-γ-connected space, then X is b-γ-connected.*
Proof. Evident.

7. Conclusion

In this paper, we introduced and investigated b-γ-continuous, b-γ-irresolute, b-γ-open, bs-γ-open and sb-γ-open mappings. These maps are stated to be independent of each other. Similarly b-γ-connected and b-γ-compact have different notions. We have also discussed the relationships between these mappings in topological spaces. Applications of b-γ-connected and b-γ-compact will be discussed in my future work. There is a scope to study and extend these newly defined mappings.

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