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STUDY ON k-GAUSS SECOND SUMMATION THEOREMS AND k-KUMMER'S TRANSFORMATION

Ekta Mittal and Sunil Joshi*

Department of Mathematics,
IIS Deemed to be University, Jaipur - 302020, INDIA

E-mail : ekta.mittal@iisuniv.ac.in

*Department of Mathematics and Statistics,
Manipal University Jaipur, Jaipur - 303007, INDIA

E-mail : sunil.joshi@jaipur.manipal.edu

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Abstract: The aim of the present investigation is to create some summation theorems like Gauss, Bailey, and Kummer in the form of k- hypergeometric function. Further, we develop a new class of Kummer's differential equation of k-parameter and Kummer's transformations formulae in terms k- confluent hypergeometric function.

Keywords and Phrases: k-Gamma function, k-Beta function, k-hypergeometric functions, k-pochhammer symbols.

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1. Introduction and Preliminaries

Recently, the extension of the special functions has been painstaking by numerous authors. The generalization of the gamma and beta functions presented by number of researchers (See [2, 3, 5, 8]) in the form of a new parameter k, where $k > 0$, called k-gamma and k-beta functions respectively.
The k -Pochhammer symbol and k -Gamma function demarcated as

$$\Gamma_k(x) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{x}{k}-1}}{x_{n,k}}, \quad k > 0, x \in C\backslash kZ^-, \tag{1}$$

where $(x)_{n,k}$ is the k-pochhammer symbol and given by

$$(x)_{n,k} = x(x+k)...(x+(n-1)k = \frac{\Gamma_k(x+nk)}{\Gamma_k(x)}, \quad x \in C, k \in R, n \in N^+.$$

we use following relation of k-Gamma Function
$\Gamma_k(x+k) = x\Gamma_k(x), \Gamma_k(k) = 1$ and $\Gamma_k(x)\Gamma_k(k-x) = \frac{\pi}{Sin(\pi x/k)}$
The connection between k-gamma and k- beta functions is assumed by

$$B_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad Re(x), Re(y) > 0. \tag{2}$$

Mubeen et al. [7] demarcated the k-hypergeometric function and k-confluent hypergeometric function which are as follows:

$${}_2F_{1,k}\,(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}(b)_{n,k}}{(c)_{n,k}} \frac{z^n}{n!}, \quad k > 0, |z| > 0, c \neq 0, -1, .. \tag{3}$$

and

$${}_1F_{1,k}\,(a;b;z) = \sum_{n=0}^{\infty} \frac{(a)_{n,k}}{(b)_{n,k}} \frac{z^n}{n!}, \quad k > 0, |z| > 0, b \neq 0, -1, .. \tag{4}$$

Mubeen and Habibullah [6] also presented integral representation of k - hypergeometric function and k-Gauss theorem such as

$${}_2F_{1,k}\,(a,b;c;z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(1-kzt)^{-\frac{a}{k}}dt \tag{5}$$

and

$${}_1F_{1,k}\left(a,b;c;\frac{1}{k}\right) = \frac{\Gamma_k(c)\,\Gamma_k(c-b-a)}{\Gamma_k(c-a)\Gamma_k(c-b)} \tag{6}$$

Numerous authors (see [1, 2, 4, 9]) offered the well-known summation theorems for the series ${}_2F_1(-)$ such as of Gauss, Bailey and Kummer. This paper is divide into two sections as follows.

2. Some summation theorem in terms of ${}_1F_1(-)$

In this section we proving some known familiar summation theorem for k- Gauss hypergeometric function and these results convert the original summation Theorems, when $k \to 1$.

Theorem 2.1. *If* $R(a) > 0, R(b) > 0, k > 0,$ *then*

$${}_2F_{1,k}\left(a,b;\frac{(a+b+k)}{2};\frac{1}{2k}\right) = \sqrt{\frac{\pi}{k}} \frac{\Gamma_k\frac{(a+b+k)}{2}}{\Gamma_k\frac{(a+k)}{2}\,\Gamma_k\frac{(b+k)}{2}} \tag{7}$$

Proof. Using the equation (5) in left side of equation (7), we have

$$ {}_2F_{1,k}\left[a, b; c; \frac{1}{2k}\right] = \frac{2^{\frac{a}{k}}\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}(2-t)^{-\frac{a}{k}} dt \tag{8} $$

After substituting $u = 1 - t$ in equation(8), we get

$$ {}_2F_{1,k}\left[a, b; c; \frac{1}{2k}\right] = \frac{2^{\frac{a}{k}}\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)} \int_0^1 (1-u)^{\frac{b}{k}-1}u^{\frac{c-b}{k}-1}(1+u)^{-\frac{a}{k}} du \tag{9} $$

Let

$$ \int_0^1 (1-u)^{\frac{b}{k}-1}u^{\frac{c-b}{k}-1}(1+u)^{-\frac{a}{k}} du = H, $$

in above equation (9) replacing $u = Tan^2(\frac{\theta}{2})$, therefore $du = Tan(\frac{\theta}{2})Sec^2(\frac{\theta}{2})d\theta$ and $Tan(\frac{\theta}{2}) = \frac{Sin(\theta)}{1+Cos(\theta)}$, So H becomes

$$ \begin{aligned} H &= \int_0^{\frac{\pi}{2}} \left(\frac{2Cos(\theta)}{1+Cos(\theta)}\right)^{\frac{b}{k}-1} \left(\frac{Sin(\theta)}{1+Cos(\theta)}\right)^{\frac{2c-2b}{k}-1} \left(\frac{2}{1+Cos(\theta)}\right)^{\frac{-a}{k}+1} d\theta \\ &= \int_0^{\frac{\pi}{2}} \left(\frac{(Cos(\theta))^{\frac{b}{k}-1}(Sin(\frac{\theta}{2})Cos(\frac{\theta}{2}))^{\frac{(2c-2b)}{k}-1}}{(Cos^2(\frac{\theta}{2}))^{\frac{(2c-b-a)}{k}-1}}\right) d\theta \\ &= \int_0^{\frac{\pi}{2}} (Cos(\theta))^{\frac{b}{k}-1}(Sin(\frac{\theta}{2}))^{\frac{(2c-2b)}{k}-1}(Cos(\frac{\theta}{2}))^{\frac{(-2c+2a)}{k}+1} d\theta \end{aligned} \tag{10} $$

Again put $c = \frac{(a+b+k)}{2}$ in equation (10), we get

$$ H = \int_0^{\frac{\pi}{2}} (Cos(\theta))^{\frac{b}{k}-1}(Sin(\theta))^{\frac{(a-b)}{k}} d\theta $$

Using k-Beta function property and k-Gamma function property in above equation, we get

$$ H = 2^{\frac{(b-a)}{k}} \frac{\Gamma\left(\frac{a-b+k}{2k}\right)\Gamma\left(\frac{b}{2k}\right)}{2\Gamma\left(\frac{a+k}{2k}\right)} = 2^{\frac{(b-a)}{k}-1} k \frac{\Gamma_k\left(\frac{a-b+k}{2}\right)\Gamma_k\left(\frac{b}{2}\right)}{\Gamma_k\left(\frac{a+k}{2}\right)} \tag{11} $$

Combining equations (11), (9) after putting the value c, we obtain

$$ {}_2F_{1,k}\left[a, b; \frac{(a+b+k)}{2}; \frac{1}{2k}\right] = \frac{2^{\frac{b}{k}-1}\Gamma_k\left(\frac{(a+b+k)}{2}\right)\Gamma_k\left(\frac{b}{2}\right)}{\Gamma_k(b)\Gamma_k\left(\frac{a+k}{2}\right)} \tag{12} $$

Using duplication formula $\Gamma_k(2x) = \sqrt{\frac{k}{\pi}} 2^{\frac{2x}{k}-1}\Gamma_k(x)\Gamma_k(x+\frac{k}{2})$ by substituting in above equation (12) and simplify we obtain the desired result.

Theorem 2.2. *If* $R(\frac{b}{2}) > R(a) > 0, k > 0$ *then*

$$ {}_2F_{1,k}\left[a, b; k-a+b; \frac{-1}{k}\right] = \frac{\Gamma_k(k-a+b)\Gamma_k\left(\frac{b}{2}+k\right)}{\Gamma_k(b+k)\,\Gamma_k\left(\frac{k-a+b}{2}\right)} \tag{13} $$

Proof. Using the integral representation of k-Gauss Hypergeometric function by putting $c = k-a+b$ and $z = \frac{-1}{k}$ in equation (5), we have

$$ {}_2F_{1,k}\left[a, b; k-a+b; \frac{-1}{k}\right] = \frac{\Gamma_k(k-a-b)}{k\Gamma_k(b)\Gamma_k(k-a)} \int_0^1 t^{\frac{b}{k}-1}(1-t^2)^{-\frac{a}{k}}dt, \tag{14} $$

Put $t^2 = u$, in the right hand side of equation (14), we get

$$ \begin{aligned} {}_2F_{1,k}\left[a, b; k-a+b; \frac{-1}{k}\right] &= \frac{\Gamma_k(k-a-b)}{k\Gamma_k(b)\Gamma_k(k-a)} \int_0^1 u^{\frac{b}{2k}-1}(1-u)^{-\frac{a}{k}}du \\ &= \frac{\Gamma_k(k-a-b)}{k\Gamma_k(b)\Gamma_k(k-a)} kB_k\left(\frac{b}{2}, k-a\right), \end{aligned} $$

Using k-Beta function property, we get desired result.

Theorem 2.3. *If* $R(\frac{c}{2}) > R(\frac{a}{2}) > 0, k > 0$ *then*

$$ {}_2F_{1,k}\left[a, k-a; c; \frac{1}{2k}\right] = \frac{\Gamma_k\left(\frac{c+k}{2}\right)\Gamma_k\left(\frac{c}{2}+k\right)}{\Gamma_k\left(\frac{c+a}{2}\right)\,\Gamma_k\left(\frac{c-a+k}{2}\right)} \tag{15} $$

Proof. Using equation (5) and put $z = \frac{1}{2k}$ and $b = k-a$,then the resulting integral can be evaluated (by putting in $(1-t) = u$, after using k -beta function, we have

$$ {}_2F_{1,k}\left[a, k-a; c; \frac{1}{2k}\right] = 2^{\frac{a}{k}}\ {}_2F_{1,k}\left[a, c+a-k; c; \frac{-1}{k}\right], $$

Applying equation (13) in right hand side of above given equation, we have

$$ {}_2F_{1,k}\left[a, k-a; c; \frac{1}{2k}\right] = 2^{\frac{a}{k}} \frac{\Gamma_k(c)\,\Gamma_k\left(\frac{c+a+k}{2}\right)}{\Gamma_k(c+a)\,\Gamma_k\left(\frac{c-a+k}{2}\right)}, $$

Finally, using the duplication formula$\Gamma_k(2x) = \sqrt{\frac{k}{\pi}} 2^{\frac{2x}{k}-1}\Gamma_k(x)\Gamma_k(x+\frac{k}{2})$, we get desired result.

3. Kummer's Differential Equation in k–parameter and Transformations

In this section we prove k- Kummers differential equation and k-first transformation and k-second transformation. These results also convert original results as $k \to 1$ The differential equation of k-Gauss Hypergeometric function defined by S. Mubeen [7] as

$$kz(1-kz)\frac{d^2u}{dz^2} + (c-(a+b+k)kz)\frac{du}{dz} - abu = 0 \tag{16}$$

by replacing $z \to \frac{z}{b}$ in above equation and taking $b \to \infty$, where $k > 0$.

$$\frac{du}{dz} = b\frac{dw}{dz} \implies \frac{d^2u}{dz^2} = b^2\frac{d^2w}{dz^2},$$

then we get

$$kz\frac{d^2w}{dz^2} + (c-kz)\frac{dw}{dz} - aw = 0 \tag{17}$$

This is the required differential equation for k -parameter. Mubeen et al. [6] defined integral representation in k–parameter where $R(c) > R(b) > 0$ then for all finite z

$${}_1F_{1,k}(b;c;z) = \frac{\Gamma_(c)}{k\Gamma_(b)\Gamma_(c-b)}\int_0^1 t^{\frac{b}{k}-1}\,(1-t)^{\frac{c-b}{k}-1}\,e^{zt}dt \tag{18}$$

Theorem 3.1. *If c is the neither zero nor a negative integer, then*

$${}_1F_{1,k}(b;c;z) = e^z\,{}_1F_{1,k}(c-b;c;-z) \tag{19}$$

Proof. With the help of integral representation for$k$-Parameter, it follows that

$${}_1F_{1,k}(b;c;z) = \frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}e^{zt}dt \tag{20}$$

using property of definite integral, we get

$${}_1F_{1,k}(b;c;z) = e^z\,\frac{\Gamma_k(c)}{k\Gamma_k(b)\Gamma_k(c-b)}\int_0^1 t^{\frac{b}{k}-1}(1-t)^{\frac{c-b}{k}-1}e^{-zt}dt$$

$${}_1F_{1,k}(b;c;z) = e^z\,{}_1F_{1,k}(c-b;c;-z)$$

Theorem 3.2. *If $2a$ is not an odd integer less than zero then*

$$e^{-z}\,{}_1F_{1,k}(a;2a;z) = {}_0F_{1,k}\left(-;a+\frac{1}{2};\frac{z^2}{4}\right) \tag{21}$$

Proof. Using Kummer's first transformation

$$ {}_1F_{1,k}(a;2a;z) = e^{z}\ {}_1F_{1,k}(a;2a;-z) $$

If we multiply both side by $e^{\frac{-z}{2}}$, we obtain

$$ e^{\frac{-z}{2}}\ {}_1F_{1,k}(a;2a;z) = e^{\frac{z}{2}}\ {}_1F_{1,k}(a;2a;-z) $$

i.e. $e^{\frac{-z}{2}}\ {}_1F_{1,k}(a;2a;z)$ is even function of z and therefore $e^{-z}\ {}_1F_{1,k}(a;2a;2z)$ is an even function of z (Replacing z by 2z)
Again we consider

$$ e^{-z}\ {}_1F_{1,k}(a;2a;2z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!} \sum_{m=0}^{\infty} \frac{(a)_{n,k}(2z)^m}{(2a)_{n,k}\ k^m\ m!} \tag{22} $$

$$ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(a)_{n,k}(-1)^{n-m}\ z^n\ 2^m}{(2a)_{n,k}\ k^m\ (n-m)!\, m!} $$

$$ = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{(a)_{n,k}(-n)_m\ (-z)^n\ 2^m}{(2a)_{n,k}\ k^m\ n!\ m!} $$

$$ e^{-z}\ {}_1F_{1,k}(a;2a;2z) = \sum_{n=0}^{\infty}\ {}_2F_{1,k}\left(-n;a;2a;\frac{2}{k}\right) \frac{(-z)^n}{n!} $$

Since left hand side be an even function of z so right-hand side should be even function of z so odd terms will be vanishes.

$$ {}_2F_{1,k}(-2n-1;a;2a;\frac{2}{k}) = 0 \tag{23} $$

then

$$ e^{-z}\ {}_1F_{1,k}(a;2a;2z) = \sum_{s=0}^{\infty}\ {}_2F_{1,k}(-2s;a;2a;\frac{2}{k})\frac{z^{2s}}{2s!} \tag{24} $$

Again the $u =_1 F_{1,k}(a;c;z)$ is one solution of the Kummer's differential equation

$$ kz\frac{d^2u}{dz^2} + (c-kz)\frac{du}{dz} - au = 0 \tag{25} $$

when $c = 2a$ and $z = 2z$ then

$$ kz\frac{d^2u}{dz^2} + 2(a-kz)\frac{du}{dz} - 2au = 0 \tag{26} $$

If we put $u = e^z w$ then $\frac{du}{dz} = e^z \ \frac{dw}{dz} + e^z w$

$$\frac{d^2u}{dz^2} = \frac{d}{dz}\left(\frac{du}{dz}\right) = e^z\frac{d^2w}{dz^2} + 2e^z\frac{dw}{dz} + e^z w \tag{27}$$

using the value of u and $\frac{du}{dz}$ and equation (27) in equation (26), we have

$$kz\left(e^z\frac{d^2w}{dz^2} + 2e^z\frac{dw}{dz} + e^z w\right) + 2(a-kz)\left(e^z\ \frac{dw}{dz} + e^z w\right) - 2ae^z w = 0$$

After some simplification, we obtain

$$kz\frac{d^2w}{dz^2} + 2a\frac{dw}{dz} - kzw = 0 \tag{28}$$

$u = {}_1F_{1,k}(a;c;z)$ solution of (25) then $u = e^z w$ is also solution of (28) or $w = e^{-z}u = e^{-z}\ {}_1F_{1,k}(a;2a;2z)$ is also satisfy the equation (28) , if we put $kdz = t^{-\frac{1}{2}}dt$ then $\frac{dw}{dz} = \frac{dw}{dt}\frac{dt}{dz} = t^{\frac{1}{2}}k\frac{dw}{dt}$ and $\frac{d^2w}{dz^2} = k^2\left(\frac{1}{2}\frac{dw}{dt} + t\frac{d^2w}{dt^2}\right)$ in equation (28), we get

$$2t^{\frac{1}{2}}k^2\left(\frac{1}{2}\frac{dw}{dt} + t\frac{d^2w}{dt^2}\right) + 2at^{\frac{1}{2}}k\frac{dw}{dt} - 2t^{\frac{1}{2}}w = 0$$

After simplifying it, we get

$$tk^2\left(\frac{d^2w}{dt^2}\right) + k\left(a + \frac{k}{2}\right)\frac{dw}{dt} - w = 0 \tag{29}$$

this is differential equation of ${}_0F_{1,k}(-;a;t)$,So the solution of above equation is

$$\therefore\ w = A_0F_{1,k}(-;a+\frac{k}{2};t) + Bt^{1-\left(a+\frac{k}{2}\right)}\ {}_0F_{1,k}\left(-;2-\left(a+\frac{k}{2}\right);t\right)$$

$$\therefore\ w = A_0F_{1,k}(-;a+\frac{k}{2};\frac{z^2}{4}) + B\left(\frac{z^2}{4}\right)^{1-\left(a+\frac{k}{2}\right)}\ {}_0F_{1,k}\left(-;2-\left(a+\frac{k}{2}\right);\frac{z^2}{4}\right) \tag{30}$$

Where $a + \frac{k}{2}$ is not a positive integer and $2a$ is not an odd integer, where A and B are constants. When $z = 0 \implies A = 1$ then

$$\begin{aligned} w = e^{-z}\ {}_1F_{1,k}(a;2a;2z) &= {}_0F_{1,k}\left(-;a+\tfrac{k}{2};\tfrac{z^2}{4}\right) + B\left(\tfrac{z^2}{4}\right)^{1-\left(a+\frac{k}{2}\right)} \times \\ &\times \qquad {}_0F_{1,k}\left(-;2-\left(a+\tfrac{k}{2}\right);\tfrac{z^2}{4}\right) \end{aligned} \tag{31}$$

The left hand member and first term of (31) are analytic at but second term is not analytic at $z = 0$ so $B = 0$

$$w = e^{-z}\ {}_1F_{1,k}(a; 2a; 2z) = {}_0F_{1,k}\left(-; a + \frac{k}{2}; \frac{z^2}{4}\right) \tag{32}$$

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