South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 3 (2020), pp. 83-96

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

HYPERGEOMETRIC FORMS OF SOME COMPOSITE FUNCTIONS CONTAINING ARCCOSINE(x) USING MACLAURIN'S EXPANSION

M. I. Qureshi, Javid Majid and Aarif Hussain Bhat

Department of Applied Sciences and Humanities,
Faculty of Engineering and Technology,
Jamia Millia Islamia (A Central University), New Delhi-110025, INDIA

E-mail : miqureshi_delhi@yahoo.co.in, javidmajid375@gmail.com,
aarifsaleem19@gmail.com

(**Received:** *Feb. 25, 2020* **Accepted:** *Sep. 09, 2020* **Published:** *Dec. 30, 2020)*

Abstract: In this article, we have derived the hypergeometric forms of some composite functions containing, arccosine(x) and arccosh(x) like: $\exp(b\cos^{-1}x)$, $\frac{\exp(b\cos^{-1}x)}{\sqrt{(1-x^2)}}$, $\frac{\cos^{-1}x}{\sqrt{(1-x^2)}}$, $\frac{\sin\ (b\cos^{-1}x)}{\sqrt{(1-x^2)}}$, $\exp(a\cosh^{-1}x)$, $\frac{\exp(a\cosh^{-1}x)}{\sqrt{(x^2-1)}}$, $\frac{\cosh^{-1}x}{\sqrt{(x^2-1)}}$ and $\frac{\sin\ (a\cosh^{-1}x)}{\sqrt{(x^2-1)}}$ by using the Leibniz theorem for successive differentiation, the Maclaurin's series expansion, the Taylor's series expansion and the Euler's linear transformation, as the proof of the hypergeometric forms of the above functions is not available in the literature. Some applications of the functions are also obtained in the form of the Chebyshev polynomials and the Chebyshev functions.

Keywords and Phrases: The Gauss' Hypergeometric function, The Maclaurin's series expansion, The Taylor's series expansion, The Leibniz theorem, The Chebyshev polynomials, The Euler's linear transformation.

2010 Mathematics Subject Classification: 33C05, 34A35, 41A58, 33B10.

1. Introduction and Preliminaries

In this paper, we shall use the following standard notations:
$\mathbb{N} := \{1,2,3,\cdots\}$; $\mathbb{N}_0 := \mathbb{N}\bigcup\{0\}$; $\mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0,-1,-2,-3,\cdots\}$.
The symbols $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real

numbers, natural numbers, integers, positive and negative real numbers respectively.
The classical Pochhammer symbol $(\alpha)_p$ $(\alpha, p \in \mathbb{C})$ is defined by([11, p.22, Eq.(1), p.32, Q.N.(8) and Q.N.(9), see also [14] p.23, Eq.(22) and Eq.(23)]).
A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$ is accomplished by introducing any arbitrary number of numerator and denominator parameters [14, p.42, Eq.(1)].
Relation between inverse hyperbolic and inverse trigonometric functions:

$$\sinh^{-1}(x) = -i \sin^{-1}(ix), \quad \cosh^{-1}(x) = \pm i \cos^{-1}(x). \tag{1.1}$$

The Euler's linear transformation [11, p.60, Eq.(5), [14], p.33, Eq.(21)]:

$${}_2F_1\left[\begin{array}{ccc} \alpha, & \beta; & \\ & & z \\ & \gamma; & \end{array}\right] = (1-z)^{\gamma-\alpha-\beta} \, {}_2F_1\left[\begin{array}{cc} \gamma-\alpha, \ \gamma-\beta; & \\ & z \\ \gamma; & \end{array}\right]; \quad |z| < 1, \tag{1.2}$$

$|\arg\ (1-z)| < \pi$ and $\gamma \neq 0, -1, -2, -3, ...$
The Taylor's series of a real or complex-valued function $y(x)$ which is infinitely differentiable at a real or complex number a, is the power series:

$$\begin{aligned} y(x) = (y)_{x=a} + (x-a)(y_1)_{x=a} + \frac{(x-a)^2}{2!}(y_2)_{x=a} + \frac{(x-a)^3}{3!}(y_3)_{x=a} + \\ + \frac{(x-a)^4}{4!}(y_4)_{x=a} + \cdots = \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}\,(y_n)_{x=a} \end{aligned} \tag{1.3}$$

$$= \sum_{n=0}^{\infty} \frac{(x-a)^{2n}}{(2n)!}(y_{2n})_{x=a} + \sum_{n=0}^{\infty} \frac{(x-a)^{2n+1}}{(2n+1)!}(y_{2n+1})_{x=a}. \tag{1.4}$$

The Maclaurin's series is a particular case of the Taylor's series expansion of a function, about the origin i.e, when $a = 0$ in equation (1.3), the Maclaurin's series is given as:

$$\begin{aligned} y(x) &= (y)_0 + x(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \frac{x^4}{4!}(y_4)_0 + \frac{x^5}{5!}(y_5)_0 + \cdots \\ &= \sum_{n=0}^{\infty} \frac{x^n}{n!}(y_n)_0 \end{aligned} \tag{1.5}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}(y_{2n})_0 + \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}(y_{2n+1})_0, \tag{1.6}$$

where $(y_m)_0 = \left(\frac{d^m y}{dx^m}\right)_{x=0}$.
The general Leibniz rule, which states that if $U(x)$ and $T(x)$ are n-times differentiable functions, then the product $U(x).T(x)$ is also n-times differentiable and its nth derivative is given by:

$$D^n[U(x)\ T(x)] = ({}^nC_0)(D^n\ U)(D^0\ T) + ({}^nC_1)(D^{n-1}\ U)(D^1\ T) + \\ + ({}^nC_2)(D^{n-2}\ U)(D^2\ T) + \cdots + ({}^nC_{n-1})(D^1\ U)(D^{n-1}\ T) + ({}^nC_n)(D^0\ U)(D^n\ T), \tag{1.7}$$

where $D = \frac{d}{dx}$.
The present article is organized as follows. In Section 3, we have given the proof of the hypergeometric forms of the following functions as their proofs are not available in the literature [1] - [17] So we are interested to give the proof of the hypergeometric forms of some composite functions containing arccosine (x), using the Maclaurin's expansion. In section 4, we have obtained hypergeometric forms of some more functions by using the relations between inverse trigonometric and inverse hyperbolic functions and section 5 is related to the applications involving the Chebyshev polynomials and the Chebyshev functions.

2. Main Hypergeometric Forms of Some Composite Functions

When the values of numerator, denominator parameters and arguments leading to the results which do not make sense are tacitly excluded, then each of the following hypergeometric form holds true:

$$\exp(b\cos^{-1}x) = {}_2F_1\left[\begin{matrix} -ib,\ ib; \\ \frac{1}{2}; \end{matrix}\ \frac{1-x}{2}\right];\quad \left|\frac{1-x}{2}\right| < 1. \tag{2.1}$$

$$\exp(b\cos^{-1}x) = \exp\left(b\frac{\pi}{2}\right){}_2F_1\left[\begin{matrix} \frac{-ib}{2},\ \frac{ib}{2}; \\ \frac{1}{2}; \end{matrix}\ x^2\right] - bx\exp\left(b\frac{\pi}{2}\right){}_2F_1\left[\begin{matrix} \frac{1-ib}{2},\ \frac{1+ib}{2}; \\ \frac{3}{2}; \end{matrix}\ x^2\right], \tag{2.2}$$

where $|x| < 1$.

$$\frac{\exp(b\cos^{-1}x)}{\sqrt{(1-x^2)}} = \exp\left(b\frac{\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{1-ib}{2},\ \frac{1+ib}{2}; \\ \frac{1}{2}; \end{matrix}\ x^2\right] - \\ - b\ x\ \exp\left(b\frac{\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{2-ib}{2},\ \frac{2+ib}{2}; \\ \frac{3}{2}; \end{matrix}\ x^2\right], \tag{2.3}$$

where $|x| < 1$.

$$\frac{\cos^{-1} x}{\sqrt{(1-x^2)}} = \frac{\pi}{2}\ {}_1F_0\left[\begin{matrix} \frac{1}{2}\ ; & \\ & x^2 \\ -; & \end{matrix}\right] - x\ {}_2F_1\left[\begin{matrix} 1\ ,\ 1\ ; & \\ & x^2 \\ \frac{3}{2}; & \end{matrix}\right];\quad |x| < 1. \tag{2.4}$$

$$\frac{\sin\ (b\cos^{-1} x)}{\sqrt{(1-x^2)}} = b\ {}_2F_1\left[\begin{matrix} 1-b,\ 1+b; & \\ & \frac{1-x}{2} \\ \frac{3}{2}; & \end{matrix}\right];\quad \left|\frac{1-x}{2}\right| < 1,\ |x| < 1. \tag{2.5}$$

Note: In the above hypergeometric functions, x and b can be purely real or purely imaginary or complex numbers.

3. Proof of Hypergeometric Forms

Proof of hypergeometric forms (2.1) and (2.2)

Let

$$y = e^{b\cos^{-1} x}. \tag{3.1}$$

Differentiating equation (3.1) with respect to x, we get

$$y_1 = \frac{-b\ e^{b\cos^{-1} x}}{\sqrt{(1-x^2)}},$$

$$(1-x^2)\ y_1^2 - b^2y^2 = 0. \tag{3.2}$$

Differentiating equation (3.2) with respect to x, we get

$$(1-x^2)\ y_2 - xy_1 - b^2y = 0. \tag{3.3}$$

Differentiating equation (3.3) with respect to x, we get

$$(1-x^2)\ y_3 - 3x\ y_2 - (b^2+1)\ y_1 = 0. \tag{3.4}$$

Differentiating equation (3.3) n-times with respect to x, we get

$$(1-x^2)\ y_{n+2} - x(2n+1)\ y_{n+1} - (n^2+b^2)\ y_n = 0;\ n \geq 2. \tag{3.5}$$

On substituting $n = 2, 3, 4, ...$ in equation (3.5), we get

$$(1-x^2)\ y_4 - 5x\ y_3 - (2^2+b^2)\ y_2 = 0, \tag{3.6}$$

$$(1-x^2)\ y_5 - 7x\ y_4 - (3^2+b^2)\ y_3 = 0, \tag{3.7}$$

$$(1-x^2)\ y_6 - 9x\ y_5 - (4^2+b^2)\ y_4 = 0, \tag{3.8}$$

$$(1-x^2)\ y_7 - 11x\ y_6 - (5^2+b^2)\ y_5 = 0, \tag{3.9}$$

$$\vdots$$

Case-I. On substituting $x = 1$ in equations (3.1), (3.3), (3.4) and equations (3.6) to (3.9), we get

$$(y)_1 = 1, \tag{3.10}$$
$$(y_1)_1 = -b^2, \tag{3.11}$$
$$(y_2)_1 = \frac{b^2\ (b^2+1)}{3.1}, \tag{3.12}$$
$$(y_3)_1 = \frac{-b^2\ (b^2+1)\ (b^2+2^2)}{5.3.1}, \tag{3.13}$$
$$(y_4)_1 = \frac{b^2\ (b^2+1)\ (b^2+2^2)\ (b^2+3^2)}{7.5.3.1}, \tag{3.14}$$
$$(y_5)_1 = \frac{-b^2\ (b^2+1)\ (b^2+2^2)\ (b^2+3^2)\ (b^2+4^2)}{9.7.5.3.1}, \tag{3.15}$$
$$\vdots$$

Using the Taylor's series expansion, we get

$$y = 1 - \frac{(x-1)}{1!}\frac{b^2}{1} + \frac{(x-1)^2}{2!}\frac{b^2\ (b^2+1)}{3.1} - \frac{(x-1)^3}{3!}\frac{b^2\ (b^2+1)\ (b^2+2^2)}{5.3.1} +$$
$$+ \frac{(x-1)^4}{4!}\frac{b^2\ (b^2+1)\ (b^2+2^2)\ (b^2+3^2)}{7.5.3.1} + \frac{(x-1)^5\ b^2\ (b^2+1)\ (b^2+2^2)}{5!} \times$$
$$\times \frac{(b^2+3^2)\ (b^2+4^2)}{9.7.5.3.1} + ...$$
$$= \sum_{n=0}^{\infty} \left[\frac{\prod_{j=1}^{n} \{b^2 + (j-1)^2\}}{\prod_{j=1}^{n} (2j-1)} \right] \frac{(1-x)^n}{n!}. \tag{3.16}$$

On further simplifying, we arrive at the result (2.1).
Case-II. On substituting $x = 0$ in equations (3.1) to (3.4) and equations (3.6) to (3.9), we get

$$(y)_0 = e^{\frac{b\pi}{2}}, \tag{3.17}$$
$$(y_1)_0 = -b\ e^{\frac{b\pi}{2}}, \tag{3.18}$$
$$(y_2)_0 = b^2\ e^{\frac{b\pi}{2}}, \tag{3.19}$$
$$(y_3)_0 = -b\ (b^2+1)\ e^{\frac{b\pi}{2}}, \tag{3.20}$$
$$(y_4)_0 = b^2\ (b^2+2^2)\ e^{\frac{b\pi}{2}}, \tag{3.21}$$

$$(y_5)_0 = -b\ (b^2+1)\ (b^2+3^2)\ e^{\frac{b\pi}{2}}, \tag{3.22}$$

$$(y_6)_0 = b^2\ (b^2+2^2)\ (b^2+4^2)\ e^{\frac{b\pi}{2}}, \tag{3.23}$$

$$(y_7)_0 = -b\ (b^2+1)\ (b^2+3^2)\ (b^2+5^2)\ e^{\frac{b\pi}{2}}, \tag{3.24}$$

$$\vdots$$

Using the Maclaurin's series expansion, we get

$$y = e^{\frac{b\pi}{2}} + \frac{x^2}{2!}\ b^2\ e^{\frac{b\pi}{2}} + \frac{x^4}{4!}\ b^2\ (b^2+2^2)\ e^{\frac{b\pi}{2}} + \frac{x^6}{6!}\ b^2\ (b^2+2^2)\ (b^2+4^2)\ e^{\frac{b\pi}{2}} + ...$$

$$-bxe^{\frac{b\pi}{2}} - \frac{x^3}{3!}b(b^2+1)e^{\frac{b\pi}{2}} - \frac{x^5}{5!}b(b^2+1)(b^2+3^2)e^{\frac{b\pi}{2}} - \frac{x^7}{7!}b(b^2+1)(b^2+3^2)(b^2+5^2)e^{\frac{b\pi}{2}} - ...$$

$$= e^{\frac{b\pi}{2}}\left[1 + \frac{x^2}{2!}\ b^2 + \frac{x^4}{4!}\ b^2\ (b^2+2^2) + \frac{x^6}{6!}\ b^2\ (b^2+2^2)\ (b^2+4^2) + ...\right] - b\ e^{\frac{b\pi}{2}} \times$$

$$\times\left[x + \frac{x^3}{3!}(b^2+1) + \frac{x^5}{5!}(b^2+1)(b^2+3^2) + \frac{x^7}{7!}(b^2+1)(b^2+3^2)\ (b^2+5^2) + ...\right]$$

$$= e^{\frac{b\pi}{2}}\left[\sum_{n=0}^{\infty}\frac{x^{2n}}{(2n)!}\ \prod_{j=1}^{n}\left\{(2j-2)^2+b^2\right\}\right] - b\ e^{\frac{b\pi}{2}}\left[\sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}\ \prod_{j=1}^{n}\left\{(2j-1)^2+b^2\right\}\right]$$

$$= e^{\frac{b\pi}{2}}\ \sum_{n=0}^{\infty}2^{2n}\left\{\prod_{j=1}^{n}\left(\frac{ib}{2}+j-1\right)\prod_{j=1}^{n}\left(\frac{-ib}{2}+j-1\right)\right\}\frac{x^{2n}}{(2n)!} -$$

$$-b\ e^{\frac{b\pi}{2}}\ \sum_{n=0}^{\infty}2^{2n}\left\{\ \prod_{j=1}^{n}\left(\frac{ib}{2}+j-\frac{1}{2}\right)\ \prod_{j=1}^{n}\left(\frac{-ib}{2}+j-\frac{1}{2}\right)\right\}\frac{x^{2n+1}}{(2n+1)!}.$$

On further simplifying, we arrive at the result (2.2).

Proof of hypergeometric forms (2.3), (2.4) and (2.5)

The proof of hypergeometric forms (2.3) and (2.4) can be given by following the same procedure as that of the above and making use of the Maclaurin's series expansion. Similarly the proof of hypergeometric form (2.5) can be given by proceeding as above and making use of the Taylor's series expansion. So we omit the details here.

4. Some other Trigonometric and Hyperbolic Functions as Special Cases

Put $b = ia$ in equation (2.1), we get

$$\exp\left(a\cosh^{-1}x\right) = {}_2F_1\left[\begin{matrix} -a,\ a; & \\ & \frac{1-x}{2} \\ \frac{1}{2}; & \end{matrix}\right]. \tag{4.1}$$

Put $b = ia$ in equation (2.2), we get

$$\exp\left(a\cosh^{-1}x\right) = \exp\left(\frac{ia\pi}{2}\right)\left\{ {}_2F_1\left[\begin{array}{c} \frac{-a}{2}, \frac{a}{2}; \\ \\ \frac{1}{2}; \end{array} x^2\right] - iax\ {}_2F_1\left[\begin{array}{c} \frac{1-a}{2}, \frac{1+a}{2}; \\ \\ \frac{3}{2}; \end{array} x^2\right]\right\}. \tag{4.2}$$

Put $b = ia$ in equation (2.3), we get

$$\frac{\exp\left(a\cosh^{-1}x\right)}{\sqrt{(x^2-1)}} = \exp\left(\frac{ia\pi}{2}\right)\left\{ ax{}_2F_1\left[\begin{array}{c} \frac{2-a}{2}, \frac{2+a}{2}; \\ \\ \frac{3}{2}; \end{array} x^2\right] + i{}_2F_1\left[\begin{array}{c} \frac{1-a}{2}, \frac{1+a}{2}; \\ \\ \frac{1}{2}; \end{array} x^2\right]\right\}. \tag{4.3}$$

On using the relation (1.1) in equation (2.4), we get

$$\frac{\cosh^{-1}x}{\sqrt{(x^2-1)}} = x\ {}_2F_1\left[\begin{array}{c} 1\ ,\ 1\ ; \\ \\ \frac{3}{2}; \end{array} x^2\right] - \frac{\pi}{2}\ {}_1F_0\left[\begin{array}{c} \frac{1}{2}\ ; \\ \\ -; \end{array} x^2\right]. \tag{4.4}$$

Put $b = ia$ in equation (2.5), we get

$$\frac{\sin\ \left(a\cosh^{-1}x\right)}{\sqrt{(x^2-1)}} = \frac{\sinh\ \left(a\cos^{-1}x\right)}{\sqrt{(1-x^2)}} = a\ {}_2F_1\left[\begin{array}{c} 1-ia,\ 1+ia; \\ \\ \frac{3}{2}; \end{array} \frac{1-x}{2}\right]. \tag{4.5}$$

Replacing a by id in equation (4.5), we get

$$\frac{\sinh\ \left(d\cosh^{-1}x\right)}{\sqrt{(x^2-1)}} = d\ {}_2F_1\left[\begin{array}{c} 1-d,\ 1+d; \\ \\ \frac{3}{2}; \end{array} \frac{1-x}{2}\right]. \tag{4.6}$$

Suppose $x \in \mathbb{R}$ and b is purely imaginary in equation (2.1), then put $b = ia$, where a is purely real, we get

$$\exp\left(ia\cos^{-1}x\right) = {}_2F_1\left[\begin{array}{c} -a,\ a; \\ \\ \frac{1}{2}; \end{array} \frac{1-x}{2}\right]. \tag{4.7}$$

Using the Euler's formula on the left hand side of equation (4.7) and further on equating real and imaginary parts, we get

$$\cos\ \left(a\cos^{-1}x\right) = {}_2F_1\left[\begin{array}{c} -a,\ a; \\ \\ \frac{1}{2}; \end{array} \frac{1-x}{2}\right], \tag{4.8}$$

$$\sin\ (a\cos^{-1}x) = 0. \tag{4.9}$$

Remark 4.1. *We can't expand* $\sin\ (a\cos^{-1}x)$ *in positive-integral powers of* $x-1$, *since the function* $\sin\ (a\cos^{-1}x)$ *and its higher order derivatives vanish at* $x=1$. *So,* $\sin\ (a\cos^{-1}x)$ *can't be expressed in hypergeometric form.*
Put $a = id$ in equation (4.8), where d is purely imaginary, we get

$$\cosh\ (d\cos^{-1}x) = \cos\ (d\cosh^{-1}x) = {}_2F_1\left[\begin{matrix} -id,\ id; & \\ & \frac{1-x}{2} \\ \frac{1}{2}; & \end{matrix}\right]. \tag{4.10}$$

Replacing d by ig in equation (4.10), where g is purely real, we get

$$\cosh\ (g\cosh^{-1}x) = {}_2F_1\left[\begin{matrix} -g,\ g; & \\ & \frac{1-x}{2} \\ \frac{1}{2}; & \end{matrix}\right]. \tag{4.11}$$

Suppose $x \in \mathbb{R}$ and b is purely imaginary in equation (2.2), then put $b = ia$, where a is purely real, we get

$$\exp\left(ia\cos^{-1}x\right) = \exp\left(\frac{ia\pi}{2}\right)\left\{ {}_2F_1\left[\begin{matrix} \frac{-a}{2}, \frac{a}{2}; & \\ & x^2 \\ \frac{1}{2}; & \end{matrix}\right] - iax\ {}_2F_1\left[\begin{matrix} \frac{1-a}{2}, \frac{1+a}{2}; & \\ & x^2 \\ \frac{3}{2}; & \end{matrix}\right]\right\}. \tag{4.12}$$

Using the Euler's formula in equation (4.12) and further on equating real and imaginary parts, we get

$$\cos(a\cos^{-1}x) = \cos\left(\frac{a\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{-a}{2}, \frac{a}{2}; & \\ & x^2 \\ \frac{1}{2}; & \end{matrix}\right] + ax\sin\left(\frac{a\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{1-a}{2}, \frac{1+a}{2}; & \\ & x^2 \\ \frac{3}{2}; & \end{matrix}\right], \tag{4.13}$$

$$\sin(a\cos^{-1}x) = \sin\left(\frac{a\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{-a}{2}, \frac{a}{2}; & \\ & x^2 \\ \frac{1}{2}; & \end{matrix}\right] - ax\cos\left(\frac{a\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{1-a}{2}, \frac{1+a}{2}; & \\ & x^2 \\ \frac{3}{2}; & \end{matrix}\right]. \tag{4.14}$$

Replacing a by id in equations (4.13) and (4.14), where d is purely imaginary, we get

$$\cosh\ (d\cos^{-1}x) = \cos\ (d\cosh^{-1}x) = \cosh\ \left(\frac{d\pi}{2}\right)\ {}_2F_1\left[\begin{matrix} \frac{-id}{2}, \frac{id}{2}; & \\ & x^2 \\ \frac{1}{2}; & \end{matrix}\right] -$$

$$-\,d\;x\;\sinh\;\left(\frac{d\pi}{2}\right)\;{}_2F_1\left[\begin{array}{c}\frac{1-id}{2},\;\frac{1+id}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right], \tag{4.15}$$

$$\sinh(d\cos^{-1}x)=\sinh\left(\frac{d\pi}{2}\right)\;{}_2F_1\left[\begin{array}{c}\frac{-id}{2},\frac{id}{2};\\ \\ \frac{1}{2};\end{array}\;x^2\right]-dx\cosh\left(\frac{d\pi}{2}\right)\;{}_2F_1\left[\begin{array}{c}\frac{1-id}{2},\frac{1+id}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right]. \tag{4.16}$$

Replacing d by ig, in equation (4.15), where g is purely real, we get

$$\cosh(g\cosh^{-1}x)=\cos\left(\frac{g\pi}{2}\right)\;{}_2F_1\left[\begin{array}{c}\frac{-g}{2},\frac{g}{2};\\ \\ \frac{1}{2};\end{array}\;x^2\right]+gx\sin\left(\frac{g\pi}{2}\right)\;{}_2F_1\left[\begin{array}{c}\frac{1-g}{2},\frac{1+g}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right]. \tag{4.17}$$

Suppose $x\in\mathbb{R}$ and b is purely imaginary in equation (2.3), then put $b=ia$, where a is purely real, we get

$$\frac{\exp\left(ia\cos^{-1}x\right)}{\sqrt{(1-x^2)}}=\exp\left(\frac{ia\pi}{2}\right)\left\{{}_2F_1\left[\begin{array}{c}\frac{1-a}{2},\frac{1+a}{2};\\ \\ \frac{1}{2};\end{array}\;x^2\right]-iax\,{}_2F_1\left[\begin{array}{c}\frac{2-a}{2},\frac{2+a}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right]\right\}. \tag{4.18}$$

Using the Euler's formula in equation (4.18) and further on equating real and imaginary parts, we get

$$\frac{\cos(a\cos^{-1}x)}{\sqrt{(1-x^2)}}=\cos\left(\frac{a\pi}{2}\right){}_2F_1\left[\begin{array}{c}\frac{1-a}{2},\frac{1+a}{2};\\ \\ \frac{1}{2};\end{array}\;x^2\right]+ax\sin\left(\frac{a\pi}{2}\right){}_2F_1\left[\begin{array}{c}\frac{2-a}{2},\frac{2+a}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right], \tag{4.19}$$

$$\frac{\sin(a\cos^{-1}x)}{\sqrt{(1-x^2)}}=\sin\left(\frac{a\pi}{2}\right){}_2F_1\left[\begin{array}{c}\frac{1-a}{2},\frac{1+a}{2};\\ \\ \frac{1}{2};\end{array}\;x^2\right]-ax\cos\left(\frac{a\pi}{2}\right){}_2F_1\left[\begin{array}{c}\frac{2-a}{2},\frac{2+a}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right]. \tag{4.20}$$

Put $a=id$ in equations (4.19) and (4.20), where d is purely imaginary, we get

$$\frac{\cosh\;(d\cos^{-1}x)}{\sqrt{(1-x^2)}}=\frac{\cos\;(d\cosh^{-1}x)}{\sqrt{(1-x^2)}}=\cosh\left(\frac{d\pi}{2}\right)\;{}_2F_1\left[\begin{array}{c}\frac{1-id}{2},\;\frac{1+id}{2};\\ \\ \frac{1}{2};\end{array}\;x^2\right]-$$

$$-\,dx\;\sinh\left(\frac{d\pi}{2}\right){}_2F_1\left[\begin{array}{c}\frac{2-id}{2},\;\frac{2+id}{2};\\ \\ \frac{3}{2};\end{array}\;x^2\right], \tag{4.21}$$

$$\frac{\sin\ (d\cosh^{-1}x)}{\sqrt{(x^2-1)}} = \frac{\sinh\ (d\cos^{-1}x)}{\sqrt{(1-x^2)}} = \sinh\left(\frac{d\pi}{2}\right)\ {}_2F_1\left[\begin{array}{c} \frac{1-id}{2},\ \frac{1+id}{2}; \\ \frac{1}{2}; \end{array} x^2\right] -$$

$$- dx\ \cosh\left(\frac{d\pi}{2}\right) {}_2F_1\left[\begin{array}{c} \frac{2-id}{2},\ \frac{2+id}{2}; \\ \frac{3}{2}; \end{array} x^2\right]. \qquad (4.22)$$

Replacing d by ig in equations (4.21) and (4.22), where g is purely real, we get

$$\frac{\cosh(g\cosh^{-1}x)}{\sqrt{(1-x^2)}} = \cos\left(\frac{g\pi}{2}\right) {}_2F_1\left[\begin{array}{c} \frac{1-g}{2}, \frac{1+g}{2}; \\ \frac{1}{2}; \end{array} x^2\right] + gx\sin\left(\frac{g\pi}{2}\right) {}_2F_1\left[\begin{array}{c} \frac{2-g}{2}, \frac{2+g}{2}; \\ \frac{3}{2}; \end{array} x^2\right], \qquad (4.23)$$

$$\frac{\sinh(g\cosh^{-1}x)}{\sqrt{(x^2-1)}} = \sin\left(\frac{g\pi}{2}\right) {}_2F_1\left[\begin{array}{c} \frac{1-g}{2}, \frac{1+g}{2}; \\ \frac{1}{2}; \end{array} x^2\right] - gx\cos\left(\frac{g\pi}{2}\right) {}_2F_1\left[\begin{array}{c} \frac{2-g}{2}, \frac{2+g}{2}; \\ \frac{3}{2}; \end{array} x^2\right]. \qquad (4.24)$$

5. Applications in the Chebyshev Polynomials and the Chebyshev Functions

When a is a positive integer (suppose $a = m$), then from equation (4.8), we obtain the hypergeometric form of the Chebyshev polynomials of first kind $T_m(x)$:

$$T_m(x) = \cos\ (m\cos^{-1}x) = {}_2F_1\left[\begin{array}{c} -m,\ m; \\ \frac{1}{2}; \end{array} \frac{1-x}{2}\right];\ m \in \mathbb{N}_0. \qquad (5.1)$$

If $a = 2m$, where m is positive integer, then from equations (4.13) and (4.14), we obtain the hypergeometric forms of the Chebyshev polynomials of first kind $T_{2m}(x)$ and the Chebyshev functions of second kind $U^*_m(x)$:

$$T_{2m}(x) = \cos\ (2m\cos^{-1}x) = (-1)^m\ {}_2F_1\left[\begin{array}{c} -m,\ m; \\ \frac{1}{2}; \end{array} x^2\right];\quad m \in \mathbb{N}_0, \qquad (5.2)$$

$$U^*_m(x) = \sin(2m\cos^{-1}x) = 2mx(-1)^{m+1}\ {}_2F_1\left[\begin{array}{c} \frac{1-2m}{2}, \frac{1+2m}{2}; \\ \frac{3}{2}; \end{array} x^2\right];\ m \in \mathbb{N}_0. \qquad (5.3)$$

Applying the Euler's linear transformation (1.2), in equation (5.3), we get the hypergeometric form of the Chebyshev polynomials of second kind $U_{2m}(x)$:

$$U_{2m}(x) = \frac{\sin(2m\cos^{-1}x)}{\sqrt{(1-x^2)}} = 2mx(-1)^{m+1}{}_2F_1\left[\begin{array}{c}-m+1, m+1;\\ \\ \frac{3}{2};\end{array} x^2\right];\ m \in \mathbb{N}_0. \quad (5.4)$$

If $a = 2m+1$, where m is positive integer, then from equations (4.13) and (4.14), we obtain the hypergeometric forms of the Chebyshev polynomials of first kind $T_{2m+1}(x)$ and the Chebyshev functions of second kind $U^*_{2m+1}(x)$:

$$T_{2m+1}(x) = \cos\left\{(2m+1)\cos^{-1}x\right\} = (2m+1)x(-1)^m\,{}_2F_1\left[\begin{array}{c}-m, m+1;\\ \\ \frac{3}{2};\end{array} x^2\right]; m \in \mathbb{N}_0, \quad (5.5)$$

$$U^*_{2m+1}(x) = \sin\left\{(2m+1)\cos^{-1}x\right\} = (-1)^m\,{}_2F_1\left[\begin{array}{c}\frac{-2m-1}{2}, \frac{2m+1}{2};\\ \\ \frac{1}{2};\end{array} x^2\right];\ m \in \mathbb{N}_0. \quad (5.6)$$

Put $b = m+1$ in equation (2.5), where m is positive integer, we get the hypergeometric forms of the Chebyshev polynomials of second kind $U_m(x)$:

$$U_m(x) = \frac{\sin\left((m+1)\cos^{-1}x\right)}{\sqrt{(1-x^2)}} = (m+1){}_2F_1\left[\begin{array}{c}-m, m+2;\\ \\ \frac{3}{2};\end{array} \frac{1-x}{2}\right]; m \in \mathbb{N}_0. \quad (5.7)$$

6. Conclusion

In this paper, we have derived the hypergeometric forms of some functions involving arccosine(x) and arccosh(x) by using the Maclaurin's expansion. We conclude this presentation with the remark that the hypergeometric forms of some more functions can be derived in an analogous manner. Moreover the results deduced above are quite significant and these are expected to lead some potential applications in several fields of Applied Mathematics or Mathematical Physics.

7. Acknowledgment: The authors are very thankful to the anonymous referees for their valuable suggestions to improve the paper in its present form.

References

[1] Abramowitz, M. and Stegun, I. A., Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables, Reprint of the 1972 Edition, Dover Publications, Inc., New York, 1992.

[2] Andrews, G. E., Askey, R. and Roy, R., Special Functions, Cambridge University Press, Cambridge, UK, 1999.

[3] Andrews, L. C., Special Functions for Engineers and Applied Mathematicians, Macmillan Publishing Company, New York, 1985.

[4] Andrews, L. C., Special Functions of Mathematics for Engineers, Reprint of the 1992 Second Edition, SPIE Optical Engineering Press, Bellingham, W. A., Oxford University Press, Oxford, 1998.

[5] Erdélyi, A., Magnus, W., Oberhettinger, F. and Tricomi, F. G., Higher Transcendental Functions, Vol. I, McGraw-Hill Book Company, New York, Toronto and London, 1955.

[6] Gradshteyn, I. S. and Ryzhik, I. M., Table of Integrals, Series and Products, Fifth Edition, Academic Press, New York, 1994.

[7] Magnus, W., Oberhettinger, F. and Soni, R. P., Some Formulas and Theorems for the Special Functions of Mathematical Physics, Third Enlarged Edition, Springer-Verlag, New York, 1966.

[8] Prudnikov, A. P., Brychknov, Yu. A. and Marichev, O. I., Integrals and Series, Vol. III, More special functions, Nauka Moscow, 1986 (in Russian), (Translated from the Russian by G. G. Gould), Gordon and Breach Science Publishers, New York, Philadelphia London, Paris, Montreux, Tokyo, Melbourne, 1990.

[9] Qi, F., Nisar, K. S. and Rahman, G., Convexity and inequalities related to extended beta and confluent hypergeometric functions, AIMS Mathematics, 4 (5) (2019), 1499–1508.

[10] Qureshi, M. I., Porwal, S., Ahamad, D. and Quraishi, K. A., Successive differentiations of tangent, cotangent, secant, cosecant functions and related hyperbolic functions (A hypergeometric approach), International Journal of Mathematics Trends and Technology, 65 (7(4)) (2019), 354-367.

[11] Rainville, E. D., Special Functions, The Macmillan Co. Inc., New York 1960; Reprinted by Chelsea publ. Co., Bronx, New York, 1971.

[12] Safdar, M., Rahman, G., Ullah, Z., Ghaffar, A. and Nisar, K. S., A New Extension of the Pochhammer Symbol and Its Application to Hypergeometric Functions, Int. J. Appl. Comput. Math, 5 (2019), 151.

[13] Srivastava, H. M. and Karlsson, P. W., Multiple Gaussian Hypergeometric Series, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and sons, New York, Chichester, Brisbane and Toronto, 1985.

[14] Srivastava, H. M. and Manocha, H. L., A Treatise on Generating Functions, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1984.

[15] Srivastava, H. M., Rahman, G. and Nisar, K. S., Some Extensions of the Pochhammer Symbol and the Associated Hypergeometric Functions, Iran J. Sci. Technol. Trans. Sci., https://doi.org/10.1007/s40995-019-00756-8, (2019).

[16] Srivastava, H. M., Tassaddiq, A., Rahman, G., Nisar, K. S. and Khan, I., A New Extension of the τ-Gauss Hypergeometric Function and Its Associated Properties, Mathematics, 7 (2019), 996.

[17] Szegö, G., Orthogonal Polynomials, Vol. XXIII, Amer. Math. Soc., Providence, Rhode Island, Colloquium publ., New York, 1939.