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A NOTE ON MATHEMATICAL ANALYSIS OF ROTATING STRATIFIED BOUSSINESQ EQUATIONS

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Abstract: The mathematical analysis of the system of six coupled non-linear Ordinary Differential Equations (ODEs), which arose in the reduction of uniformly stratified fluid contained in a rotating rectangular box of dimension $L \times L \times H$ which is completely integrable if the Rayleigh number $Ra = 0$, is dealt with this paper.

Keywords and Phrases: Painlevé test, Mirror transformations, Boussinesq equations, Eigen values and Eigen vectors.

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1. Introduction

Since long back as a century, Painlevé test has been popular as the most successful technique for detecting the integrability of differential equations. This was mentioned in the Kowalevskian work. An integrability of differential equation is analogous to the characteristics of its solutions near movable singularity. The formal algebraic consequence of such a relation is exploited in the Painlevé test. It appears that the general property for passing the Painlevé test has been more or less same for the last hundred years or more.

In connection with the basin scale dynamics Maas [8] has considered the flow of fluid contained in large rectangular box of dimension $L \times L \times H$, which is temperature stratified with fixed zeroth order moment of mass and heat. The container is assumed to be steady, uniform rotation of an f-plane. With this assumptions Maas [8] has reduced the rotating stratified Boussinesq equation to a beautiful system of six coupled ODEs. Desale and Srinivasan [2] extended this work and they gave the detail mathematical analysis of reduced system of six coupled ODEs. Furthermore, Desale and Patil [3] tested the system of six coupled ODEs (1) for complete integrability using the Painlevé test. Also, in their paper [4] they investigate the case of non-integrability for $Ra \neq 0$ and thereby they have obtained weak solutions (in the form of logarithmic psi-series) for the two different branches of leading order coefficients. In continuation of this work in [5] authors have successfully implemented the mirror transformations and constructed the mirror system for the following differential equations (1) which is regular near movable singularity. Further, with the help of mirror transformation, authors has been proved that the Laurent series obtained by using the Painlevé test are convergent. In this paper we have analyse the certain aspects of the Painlevé test which was applied to investigate the singular structure of the system, that are actually missing while in applying the test but useful while constructing the mirror system. In this consequence we have investigate the relation between eigenvalues and eigenvectors of the Kowalevskian matrix K.

The structure of paper is as: Section 1 is of introduction. In section 2, eigenvalues and eigenvectors of the Kowalevskian matrix K are determined and further we investigate relations between them. The variation of balance in the direction of free parameters involved in the Painlevé test is also discussed in this section. Whereas in section 3, we have shown that the system (1) is divergence free and trace of K is equals to sum of leading exponents. Besides these, the matrix K is diagonalized by using the resonance vectors in the same section. In section 4, we conclude the work under consideration.

ODE reduction of stratified Boussinesq equations is well elaborated by the authors in their paper [4, 5]. We now begin with the same reduced system of coupled ODEs:

$$\begin{aligned}&\dot{w_1} = f'w_2 - b_2, \quad \dot{w_2} = -f'w_1 + b_1,\\ &\dot{b_1} = w_2b_3 - k_1b_2, \quad \dot{b_2} = k_1b_1 - w_1b_3 + Ra, \quad \dot{b_3} = w_1b_2 - w_2b_1.\end{aligned} \tag{1}$$

In the following section first we determined the eigenvalues and eigenvectors (that we call resonance vectors) of Kowalevskian matrix K.

2. Eigenvalues and Eigenvectors of matrix K

As according to the ad hoc nature of the Painlevé test applied to the system of ODEs (1), a rigorous foundation of the Painlevé test is provided in this section. The variation of solutions due to free parameters involved in the Painlevé test is also discussed in this section. The solution of a system (1) is obtained in the form of following power series.

$$\begin{aligned} &w_1(t) = w_{10}\tau^{m_1} + \sum_{j=1}^{\infty} w_{1j}\tau^{j+m_1}, \quad w_2(t) = w_{20}\tau^{m_2} + \sum_{j=1}^{\infty} w_{2j}\tau^{j+m_2}, \\ &b_1(t) = b_{10}\tau^{n_1} + \sum_{j=1}^{\infty} b_{1j}\tau^{j+n_1}, \quad b_2(t) = b_{20}\tau^{n_2} + \sum_{j=1}^{\infty} b_{2j}\tau^{j+n_2}, \\ &b_3(t) = b_{30}\tau^{n_3} + \sum_{j=1}^{\infty} b_{3j}\tau^{j+n_3}. \end{aligned} \tag{2}$$

A parameter, t_0, the arbitrary position of singularity which can be restored by substituting $(t - t_0)$ for τ in equation (2) has been suppressed. We can notice that the leading order coefficients w_{10}, w_{20}, b_{10}, b_{20}, b_{30} in equation (2) are important for their unusual significance from the succeeding coefficients. As according to the Painlevé algorithm, the dominant balance is the first step of the Painlevé test which determines the leading exponents (m_1, m_2, n_1, n_2, n_3) and the leading order coefficients $w_{i,0}$, $i = 1, 2$ and $b_{i,0}$, $i = 1, 2, 3$ of potential balances. As discussed in [3], the system of ODEs (1) admits the singular solution in the following case of principal dominant balance.

$$\dot{w_1} = -b_2, \quad \dot{w_2} = b_1, \quad \dot{b_1} = w_2 b_3, \quad \dot{b_2} = -w_1 b_3, \quad \dot{b_3} = w_1 b_2 - w_2 b_1. \tag{3}$$

From this dominant balance equations exponents are to be determined and these are listed as follows:

$$m_1 = m_2 = -1, \quad n_1 = n_2 = n_3 = -2. \tag{4}$$

The possible branches of leading order coefficient which involve one resonance parameter r_1 are as given below;

$$w_{10} = \pm\sqrt{-4 - r_1^2},\, w_{20} = r_1,\, b_{10} = -r_1, \quad b_{20} = \pm\sqrt{-4 - r_1^2},\, b_{30} = 2. \tag{5}$$

In [5], it has been shown that the leading exponents m_1, m_2, n_1, n_2 and n_3 of the system of ODEs (1) are Fuchsian and also we notice that a column vector $(w_{10}, w_{20}, b_{10}, b_{20}, b_{30})^T$ corresponding to the leading order coefficient is non zero.

That is why the choice of leading exponents is natural. Some unnatural exponents can be selected if they fulfil the Fuchsian condition and it is essential to define the Kowalevskian matrix. The Henon-Heiles system and the Gelfand-Dikii hierachy are examples of such unnatural leading exponents. For details one may refer [6, 7].

In the following subsection first we define the Kowalevskian matrix and then we find eigenvalues and eigenvectors of it. Consequently we find the variation of solution in the direction of these eigenvectors.

2.1. Kowalevskian Matrix

The next step of the Painlevé algorithm is to determine the subsequent coefficients in the balance (2). The computational method that need to find these coefficients is provided in the paper [3]. In equation (2), vectors $(w_{1j}, w_{2j}, b_{1j}, b_{2j}, b_{3j})^T$ for $j = 1, 2, \ldots$ are the j-th coefficient vectors. In the Painlevé test, j-th coefficient vectors are supposed to be related by specific recursive relations involving a certain Kowalevskian matrix. To make this into a rigorous case, the definition is introduced and in concern with our system in the Kowalevskian matrix K is defined as follows:

$$K = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & b_{30} & 2 & 0 & w_{20} \\ -b_{30} & 0 & 0 & 2 & -w_{10} \\ b_{20} & -b_{10} & -w_{20} & w_{10} & 2 \end{pmatrix}. \tag{6}$$

To find the eigenvalues of K, the characteristic equation $det(K - JI) = 0$ is solved and we have the eigenvalues $J = 0, -1, 2, 3, 4$. From the computations given in the paper [3], it was observed that the system (1) has infinitely many solutions. Because of that the solution contains some free parameters including the free choice of t_0. Further, the variation of balance (2) in the direction of free parameters namely t_0, r_1, r_2, r_3 and r_4 is described.

There are three types of free parameters. First is an arbitrary location t_0 of the singularity, second is the free parameters that appear in the leading ordered coefficients and third is the free parameters obtained in the succeeding coefficients so that the compatibility conditions get satisfied. In the present system these are involved into solutions at the level of resonances $J = 2$, 3 and 4. The cases that depict the variation of balance (2) in the direction of these free parameters for the system of ODEs (1) are described below.

Case I; Free choice of t_0 in the balance: Taking the derivative of equations (2) with respect to t_0, we obtain the variation of solution in the direction of t_0 which is given below.

$$\begin{aligned}\frac{\partial w_1}{\partial t_0} &= -w_{10}\tau^{-2} + \sum_{j=1}^{\infty}(j-1)w_{1j}\tau^{j-2},\\ \frac{\partial w_2}{\partial t_0} &= -w_{20}\tau^{-2} + \sum_{j=1}^{\infty}(j-1)w_{2j}\tau^{j-2},\\ \frac{\partial b_1}{\partial t_0} &= -2b_{10}\tau^{-3} + \sum_{j=1}^{\infty}(j-2)b_{1j}\tau^{j-3},\\ \frac{\partial b_2}{\partial t_0} &= -2b_{20}\tau^{-3} + \sum_{j=1}^{\infty}(j-2)b_{2j}\tau^{j-3},\\ \frac{\partial b_3}{\partial t_0} &= -2b_{30}\tau^{-3} + \sum_{j=1}^{\infty}(j-2)b_{3j}\tau^{j-3}.\end{aligned} \tag{7}$$

Therefore, the variation of solution (2) due to parameter t_0 is characterised by the basic resonance vector

$$(-w_{10},\ -w_{20},\ -2b_{10},\ -2b_{20},\ -2b_{30})^T.$$

Definition 2.1 (Balance). *A formal Laurent series solution (2) of (1) is called a balance.*

A balance is said to be principal if total number of free parameter including t_0 is to be n (order of the equation). The following theorems show that there is a relationship between resonance parameters and the eigenvectors of Kowalevskian matrix K for the given system of ODEs (1).

Theorem 2.2. *If the leading exponents of a balance (2) are Fuchsian and*

$$(-m_1w_{10},\ -m_2w_{20},\ -n_1b_{10},\ -n_2b_{20},\ -n_3b_{30})^T \neq 0. \tag{8}$$

Then $(-m_1w_{10},\ -m_2w_{20},\ -n_1b_{10},\ -n_2b_{20},\ -n_3b_{30})^T$ *is an eigenvector of* K *with eigenvalue -1.*

Proof. Consider the system of ODEs (1). We take the derivative of a system (1) with respect to t, we get

$$\begin{aligned}&\ddot{w}_1 = f'\dot{w}_2 - \dot{b}_2, \quad \ddot{w}_2 = -f'\dot{w}_1 + \dot{b}_1, \quad \ddot{b}_1 = \dot{w}_2b_3 + w_2\dot{b}_3 - k_1\dot{b}_2,\\ &\ddot{b}_2 = k_1\dot{b}_1 - \dot{w}_1b_3 - w_1\dot{b}_3, \quad \ddot{b}_3 = \dot{w}_1b_2 + w_1\dot{b}_2 - \dot{w}_2b_1 - w_2\dot{b}_1.\end{aligned} \tag{9}$$

Substituting the values of $(\dot{w}_1,\ \dot{w}_2,\ \dot{b}_1,\ \dot{b}_2,\ \dot{b}_3)$ and $(\ddot{w}_1,\ \ddot{w}_2,\ \ddot{b}_1,\ \ddot{b}_2,\ \ddot{b}_3)$ from the equation (2) into (9). After that equate the coefficients of $\tau^{-m_i-2}, i = 1, 2$ and $\tau^{-n_j-2}, j =$

1, 2, 3 to get,

$$\begin{aligned} &w_{10} = b_{20}, \quad w_{20} = -b_{10}, \quad 2b_{10} = -w_{20}b_{30}, \\ &2b_{20} = w_{10}b_{30}, \quad 2b_{30} = -w_{10}b_{20} + w_{20}b_{10}. \end{aligned} \tag{10}$$

The above equalities can be rewritten as,

$$\left\{\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & b_{30} & 2 & 0 & w_{20} \\ -b_{30} & 0 & 0 & 2 & -w_{10} \\ b_{20} & -b_{10} & -w_{20} & w_{10} & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\right\} \begin{pmatrix} -m_1w_{10} \\ -m_2w_{20} \\ -n_1b_{10} \\ -n_2b_{20} \\ -n_3b_{30} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{11}$$

that is $(K+I)(-m_1w_{10},\ -m_2w_{20},\ -n_1b_{10},\ -n_2b_{20},\ -n_3b_{30})^T = 0$.
Therefore, $(-m_1w_{10},\ -m_2w_{20},\ -n_1b_{10},\ -n_2b_{20},\ -n_3b_{30})^T$ is an eigenvector of K with eigenvalue -1.

Remark 2.3. *Due to the above theorem, we remark that t_0 is a resonance parameter corresponding to eigenvalue $J = -1$ and $E_{-1} = (-m_1w_{10},\ -m_2w_{20},\ -n_1b_{10},\ -n_2b_{20},\ -n_3b_{30})^T$ is the required resonance vector.*

Case II: Free parameter appear in the leading order coefficients for the resonance value $J = 0$.

Theorem 2.4. *If the exponents of a balance (2) are Fuchsian. Then the tangent vectors to the subvariety of required leading order coefficient vector $(w_{10},\ w_{20},\ b_{10},\ b_{20},\ b_{30})^T$ is an eigenvector of Kowalevskian matrix K with eigenvalue $J = 0$ of the system of ODEs (1).*

Proof. For the given system of ODEs (1), we have obtained leading order coefficients given by (5). We take the derivative of equations (5) with respect to free parameter r_1 and get,

$$w'_{10} = \mp\frac{r_1}{\sqrt{-4 - r_1^2}},\ w'_{20} = 1,\ b'_{10} = -1,\ b'_{20} = \mp\frac{r_1}{\sqrt{-4 - r_1^2}},\ b'_{30} = 0. \tag{12}$$

where $' = \frac{d}{dr_1}$. With further calculations using (5), we have the following equation.

$$\begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & b_{30} & 2 & 0 & w_{20} \\ -b_{30} & 0 & 0 & 2 & -w_{10} \\ b_{20} & -b_{10} & -w_{20} & w_{10} & 2 \end{pmatrix} \begin{pmatrix} w'_{10} \\ w'_{20} \\ b'_{10} \\ b'_{20} \\ b'_{30} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{13}$$

that is, $K\frac{d}{dr_1}(w_{10},\, w_{20},\, b_{10},\, b_{20},\, b_{30})^T = 0$. This shows that the vector $\frac{d}{dr_1}(w_{10},\, w_{20},\, b_{10},\, b_{20},\, b_{30})^T$ is an eigenvector of K corresponding to the eigenvalue $J = 0$. Hence, complete the proof.

Remark 2.5. *One free parameter r_1 enters in the leading order coefficients as a resonance parameter corresponding to the resonance $J = 0$ and the tangent vector to the solution curve (2) in the direction of resonance parameter r_1 is $E_0 = \frac{d}{dr_1}(w_{10},\, w_{20},\, b_{10},\, b_{20},\, b_{30})^T$ and it is the required resonance vector.*

Case III: Free parameters in the succeeding coefficients. The compatibility conditions for the remaining resonances that is $J = 2, 3, 4$ are checked, some free parameters appear in the subsequent coefficients. For our system of ODEs (1), $j = 1$ is not an eigenvalue therefore there is a unique solution for $j = 1$.

Theorem 2.6. *If the leading exponents of a balance (2) are Fuchsian. Then for $j = 1$, the affine space is trivial and for the eigenvalues $J = 2, 3, 4$, the j-th coefficient vectors form an affine space parallel to the eigen space of K of the ODEs (1).*
Proof. To prove this theorem we first find the eigenvectors of Kowalevskian matrix K for $J > 0$.
For $j = 1$: Since $j = 1$ is not an eigenvalue of K, the j-th coefficient vector is trivial to determine. Therefore, the affine space for $j = 1$ is trivial.
For $j = 2$: Since $J = 2$ is an eigenvalue for the matrix K and corresponding eigenvector is

$$\begin{aligned} E_2 &= (e_{20},\, e_{21},\, e_{22},\, e_{23},\, e_{24})^T \\ &= \left(\mp\frac{\sqrt{-4-r_1^2}}{2},\, -\frac{r_1}{2},\, -\frac{r_1}{2},\, \pm\frac{\sqrt{-4-r_1^2}}{2},\, 1\right)^T. \end{aligned} \tag{14}$$

For $j = 3$: Since $J = 3$ is an eigenvalue of matrix K, the corresponding eigenvector is

$$\begin{aligned} E_3 &= (e_{30},\, e_{31},\, e_{32},\, e_{33},\, e_{34})^T \\ &= \left(-\frac{1}{2},\, \pm\frac{\sqrt{-4-r_1^2}}{2r_1},\, \pm\frac{\sqrt{-4-r_1^2}}{r_1},\, 1,\, 0\right)^T. \end{aligned} \tag{15}$$

For $j = 4$: Also we have $J = 4$ is an eigenvalue of the matrix K and corresponding eigenvector is given below.

$$\begin{aligned} E_4 &= (e_{40},\, e_{41},\, e_{42},\, e_{43},\, e_{44})^T \\ &= \left(-\frac{1}{3},\, \mp\frac{r_1}{3\sqrt{-4-r_1^2}},\, \mp\frac{r_1}{\sqrt{-4-r_1^2}},\, 1,\, \mp\frac{4}{3\sqrt{-4-r_1^2}}\right)^T. \end{aligned} \tag{16}$$

Now we look at the j-th coefficient vector for $j = 2, 3, 4$ from the computations that we did in deployment of Painlevé test are now import here for ready reference (c.f. [3]). Firstly for $j = 2$

$$\begin{aligned} w_{12} &= \frac{\sqrt{-4 - r_1^2}}{2}[-r_2 + f'k_1] = \mp e_{20}[-r_2 + f'k_1], \\ w_{22} &= \frac{-r_2 r_1 + f' r_1 k_1}{2} = e_{21}[r_2 - f'k_1], \\ b_{12} &= \frac{-r_2 r_1 + f'^2 r_1}{2} = \pm e_{22}[r_2 - f'^2], \\ b_{22} &= \frac{\sqrt{-4 - r_1^2}}{2}[r_2 - f'^2] = e_{23}[r_2 - f'^2], \\ b_{32} &= e_{24} r_2. \end{aligned} \tag{17}$$

For $j = 3$

$$\begin{aligned} w_{13} &= e_{30} r_3 + \frac{1}{4} f' r_1 (f'k_1 - r_2), \\ w_{23} &= \pm e_{31} r_3 + \frac{\sqrt{-4 - r_1^2}}{2}(f' r_2 - f'^2 r_1), \\ b_{13} &= \pm e_{32} r_3, \quad b_{23} = e_{33} r_3, \quad b_{33} = e_{34}. \end{aligned} \tag{18}$$

and for $j = 4$,

$$\begin{aligned} w_{14} &= e_{40} r_4 - \frac{\sqrt{-4 - r_1^2}}{12 r_1}[-f'^2 r_1 r_2 - 2f' r_3 + f'^3 r_1 k_1], \\ w_{24} &= \mp e_{41} r_4 + \frac{f'^2 r_1 r_2 + 2f' r_3 - f'^3 r_1 k_1}{12}, \\ b_{14} &= \mp e_{42} r_4, \quad b_{24} = e_{43} r_4, \\ b_{34} &= \mp e_{44} r_4 - \frac{f'^2 r_1 r_2 - 3 r_1 r_2^2 + 2f' r_3 - f'^3 r_1 k_1 + 3f' r_1 r_2 k_1 - 6 r_3 k_1}{6 r_1}. \end{aligned} \tag{19}$$

This shows that the j-th coefficient vectors for both the branches of leading order coefficients form an affine space parallel to the Eigen space of K with eigenvalues $J = 2, 3, 4$.

In the next section, it is shown that the matrix K is diagonalizable.

3. Diagonalization of Kowalevskian matrix K

We may diagonalize the Kowalevskian matrix K with the Eigenvectors(resonance vectors) which are analytically dependent on t_0 and the leading order coefficients. It is known that the matrix K is called diagonalizable if there exist an invertible matrix R such that $R^{-1}KR$ is a diagonal matrix say D. Our system of ODEs (1) has the balance (2) which consist of the resonance parameters t_0, r_1, r_2, r_3, r_4 with corresponding resonances $J = -1, 0, 2, 3, 4$ that are nothing but the distinct eigenvalues of matrix K. Further, the spectrum and an indicial locus of the Kowalevskian matrix K are defined as follows.

Definition 3.1 (Spectrum). *The set of eigenvalues of matrix K is called as the spectrum of matrix K.*

We also notice from literature that if a free parameter r enters at the r-th step of mirror transformations, then r is belongs to the spectrum of Kowalevskian matrix K.

Definition 3.2 (Indicial Locus). *The set of leading order coefficients of (2) which are determined by using the principal dominant balance is called as indicial locus.*

The Kowalevskian matrix K will only be evaluated at elements of indicial locus [1]. An important property of the Kowalevskian matrix K that we noticed is presented in the form of following proposition.

Proposition 3.3. *If the exponents of balance (2) are Fuchsian and the vector field defined by the system (1) is divergence free then the trace of matrix K is given by*

$$Trace\,(K) = -\left(\sum_{i=1}^{2} m_i + \sum_{j=1}^{3} n_j\right).$$

Proof. In their paper Desale and Patil [5] has shown that the exponents of balance (2) are Fuchsian. The derivative of vector field

$$(w_1,\, w_2,\, b_1,\, b_2,\, b_3) \longmapsto (f'w_2 - b_2, -f'w_1 + b_1, w_2b_3 - k_1b_2, k_1b_1 - w_1b_3 + Ra, w_1b_2 - w_2b_1)$$

is given by the following matrix

$$\begin{pmatrix} 0 & f' & 0 & -1 & 0 \\ -f' & 0 & 1 & 0 & 0 \\ 0 & b_3 & 0 & -k_1 & w_2 \\ -b_3 & 0 & k_1 & 0 & -w_1 \\ b_2 & -b_1 & -w_2 & w_1 & 0 \end{pmatrix}. \tag{20}$$

The trace of above matrix is zero and hence the system (1) is divergence free. By definition of the matrix K,

$$K = \begin{pmatrix} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & b_{30} & 2 & 0 & w_{20} \\ -b_{30} & 0 & 0 & 2 & -w_{10} \\ b_{20} & -b_{10} & -w_{20} & w_{10} & 2 \end{pmatrix} \tag{21}$$

This implies

$$K = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & b_{30} & 0 & 0 & w_{20} \\ -b_{30} & 0 & 0 & 0 & -w_{10} \\ b_{20} & -b_{10} & -w_{20} & w_{10} & 0 \end{pmatrix} + \begin{pmatrix} -m_1 & 0 & 0 & 0 & 0 \\ 0 & -m_2 & 0 & 0 & 0 \\ 0 & 0 & -n_1 & 0 & 0 \\ 0 & 0 & 0 & -n_2 & 0 \\ 0 & 0 & 0 & 0 & -n_3 \end{pmatrix} \tag{22}$$

Therefore,

$$\text{Trace}\,(K) = 0 - (m_1 + m_2 + n_1 + n_2 + n_3)$$

Hence,

$$\text{Trace}\,(K) = -\left(\sum_{i=1}^{2} m_i + \sum_{j=1}^{3} n_j\right).$$

The diagonalization of matrix K is provided in the following proposition.

Proposition 3.4. *If the exponents of balance (2) are Fuchsian, then the resonance vectors E_{-1}, E_0, E_2, E_3 and E_4 form a basis of $\mathbb{C}^5$ and the Kowalevskian matrix K is diagonalizable.*

Proof. From section 2, we saw that the resonance vectors that we have determined are E_{-1}, E_0, E_2, E_3 and E_4 are corresponding the distinct eigenvalues of matrix K. Let us denote R to be a resonance matrix whose columns are the eigenvectors of K. That is,

$$
\begin{aligned}
R &= (E_{-1},\, E_0,\, E_2,\, E_3,\, E_4) \\
&= \begin{pmatrix}
-\sqrt{-4-r_1^2} & -\dfrac{r_1}{\sqrt{-4-r_1^2}} & -\dfrac{1}{2}\sqrt{-4-r_1^2} & -\dfrac{1}{2} & -\dfrac{1}{3} \\
-r_1 & 1 & -\dfrac{r_1}{2} & \dfrac{\sqrt{-4-r_1^2}}{2r_1} & -\dfrac{r_1}{3\sqrt{-4-r_1^2}} \\
2r_1 & -1 & -\dfrac{r_1}{2} & \dfrac{\sqrt{-4-r_1^2}}{r_1} & -\dfrac{r_1}{\sqrt{-4-r_1^2}} \\
-2\sqrt{-4-r_1^2} & -\dfrac{r_1}{\sqrt{-4-r_1^2}} & \dfrac{1}{2}\sqrt{-4-r_1^2} & 1 & 1 \\
-4 & 0 & 1 & 0 & -\dfrac{4}{3\sqrt{-4-r_1^2}}
\end{pmatrix}
\end{aligned}
\tag{23}
$$

Further, we see that $\det(R) = \dfrac{120}{r_1(4+r_1^2)} \neq 0$, so that the vectors E_{-1}, E_0, E_2, E_3 and E_4 are linearly independent vectors belonging to $\mathbb{C}^5$ and dim $\mathbb{C}^5$ is five. Therefore, these vectors forms the basis for $\mathbb{C}^5$. Since matrix K has five linearly independent resonance vectors (Eigenvectors) in $\mathbb{C}^5$ so it is diagonalizable and diagonal form of matrix K is given by:

$$
D = R^{-1}KR = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 3 & 0 \\
0 & 0 & 0 & 0 & 4
\end{pmatrix}
\tag{24}
$$

Thus it complete the proof.

In following section we conclude the work under consideration.

4. Conclusion

The exponents given by (4) of balance (2) are principal for both branches of leading order coefficients. Also, we conclude that the Kowalevskian matrix K fulfils the following conditions:

1. -1 is the eigenvalue of matrix K which is essential for the solution (2) have movable singularity.

2. The eigen space corresponding to the eigenvalue -1 is of dimension one.

3. Matrix K is diagonalizable.

4. The number of resonance parameters including t_0 an arbitrary position of singularity are to be equal to the order of system.

Furthermore, the resonance vectors E_{-1}, E_0, E_2, E_3 and E_4 form a basis for $\mathbb{C}^5$ which is the essential condition for constructing the mirror system for given system of differential equations.

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