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On some identities of Rogers-Ramanujan type and continued fractions

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Abstract: In this paper we have established certain results involving q-series identities and continued fractions.
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1. Introduction, Notations and Definitions

The q-rising factorial $(a;q)_k$ is defined as,

$$(a,q)_k = \begin{cases} 1 & \text{if } k=0; \\ (1-a)(1-aq)(1-aq^2)\ldots(1-aq^{k-1}) & \text{if } k \geq 1. \end{cases}$$

Similarly, the infinite q-rising factorial is defined by

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r), \quad \text{for} \quad |q|<1.$$

The q-generalization of 1+1+1+1+...+1=n is

$$1+q+q^2+...+q^{n-1} = \frac{1-q^n}{1-q}.$$

Similarly, Ramanujan generalized the continued fraction

$$1+\frac{1}{1+}\frac{1}{1+}\frac{1}{1+...}$$

to

$$1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+...}$$

and showed that for $|q| < 1$, this continued fraction is a ratio of very similar looking sums,

$$1+\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+\ldots}=\frac{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\ldots(1-q^k)}}{\displaystyle\sum_{k=0}^{\infty}\frac{q^{k(k+1)}}{(1-q)(1-q^2)\ldots(1-q^k)}}, \tag{1.1}$$

where

$$\sum_{k=0}^{\infty}\frac{q^{k^2}}{(1-q)(1-q^2)\ldots(1-q^k)}=\frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}, \tag{1.2}$$

and

$$\sum_{k=0}^{\infty}\frac{q^{k(k+1)}}{(1-q)(1-q^2)\ldots(1-q^k)}=\frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \tag{1.3}$$

are most famous "Series=Product" identities known as Rogers-Ramanujan identities.
Two identities on Slater's list [4], number 34 and 36, can be stated as,

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}=\frac{1}{(q^3;q^8)_\infty(q^4;q^8)_\infty(q^5;q^8)_\infty}, \tag{1.4}$$

and

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}=\frac{1}{(q;q^8)_\infty(q^4;q^8)_\infty(q^7;q^8)_\infty}. \tag{1.5}$$

These identities become better known when they were given partition interpretations by Göllnitz [2] and independently by Gordon [3].

Two more identities which are special cases of corollary 2.7 on page 21 of [1] can be stated as

$$\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}=\frac{1}{(q^2;q^8)_\infty(q^3;q^8)_\infty(q^7;q^8)_\infty}, \tag{1.6}$$

and

$$\sum_{n=0}^{\infty}\frac{(-1/q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}=\frac{1}{(q;q^8)_\infty(q^5;q^8)_\infty(q^6;q^8)_\infty}. \tag{1.7}$$

In next section we shall establish continued fractions for the ratios of (1.4), (1.5) and (1.6), (1.7).

2. Main Results

In this section we shall establish continued fraction for the ratio of (1.4) and (1.5). Our attempt will also be to find a continued fraction for the ratio of (1.6) and (1.7).

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}} = \cfrac{1}{\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}$$

$$= \cfrac{1}{1+\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}(1-q^{2n})}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}$$

$$= \cfrac{1}{1+\cfrac{\displaystyle\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_{n-1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}$$

$$= \cfrac{1}{1+\cfrac{q(1+q)\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}$$

$$= \cfrac{1}{1+\cfrac{q(1+q)}{\cfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}}} \tag{2.1}$$

Again,

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}} = 1+\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_n}\left\{(-q;q^2)_n-(-q^3;q^2)_n\right\}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}$$

$$=1+\frac{\displaystyle q\sum_{n=1}^{\infty}\frac{(-q^3;q^2)_{n-1} q^{n^2+2n}}{(q^2;q^2)_{n-1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}$$

$$=1+\frac{\displaystyle q^4\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+4n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}$$

$$=1+\frac{q^4}{\dfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+4n}}{(q^2;q^2)_n}}} \tag{2.2}$$

Combining (2.1) and (2.2) we have,

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}} = \frac{1}{1+}\,\frac{q(1+q)}{1+}\,\frac{q^4}{\dfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+4n}}{(q^2;q^2)_n}}} \tag{2.3}$$

Iterating this process finally we have

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2}}{(q^2;q^2)_n}} = \frac{1}{1+}\,\frac{q(1+q)}{1+}\,\frac{q^4}{1+}\,\frac{q^3(1+q^3)}{1+}\,\frac{q^8}{1+\ldots} \tag{2.4}$$

(b) Let us now consider the ratio

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1/q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}} = \frac{1}{\dfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1/q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}}$$

$$= \frac{1}{1+\dfrac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q^2;q^2)_n}\left\{(-1/q;q^2)_n-(-q;q^2)_n\right\}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}}$$

$$= \frac{1}{1+\dfrac{\dfrac{1}{q}\displaystyle\sum_{n=1}^{\infty}\frac{(-q;q^2)_{n-1} q^{n^2+n}}{(q^2;q^2)_{n-1}}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}}$$

$$= \frac{1}{1+\dfrac{q\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}}$$

$$\frac{1}{1+\cfrac{q}{\cfrac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}}}} \tag{2.5}$$

Since,

$$\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}} = 1+\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}(1-q^{2n})}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}},$$

$$= 1+\frac{\sum_{n=1}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_{n-1}}}{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}},$$

$$= 1+\frac{q^2(1+q)}{\cfrac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}}}. \tag{2.6}$$

Combining (2.5) and (2.6) we have,

$$\frac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-1/q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2(1+q)}{\cfrac{\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}}{\sum_{n=0}^{\infty}\frac{(-q^3;q^2)_n q^{n^2+3n}}{(q^2;q^2)_n}}}. \tag{2.7}$$

Iterating this process, we finally get,

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1/q;q^2)_n q^{n^2+n}}{(q^2;q^2)_n}} = \frac{1}{1+}\frac{q}{1+}\frac{q^2(1+q)}{1+}\frac{q^5}{1+}\frac{q^4(1+q^3)}{1+...}. \tag{2.8}$$

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