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THE ADMISSIBLE MONOMIAL BASIS FOR THE POLYNOMIAL ALGEBRA OF FIVE VARIABLES IN DEGREE FOURTEEN

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Abstract: Let P_k be the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$ with the degree of each generator x_i being 1, where $\mathbb{F}_2$ denote the prime field of two elements. We study the *hit problem*, set up by Frank Peterson, of finding a minimal set of generators for the polynomial algebra P_k as a module over the mod-2 Steenrod algebra, $\mathcal{A}$. In this paper, we explicitly determine all admissible monomials for the case $k = 5$ in degree fourteen.

Keywords and Phrases: Steenrod squares, hit problem, algebraic transfer.

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1. Introduction and Statement of Results

Denote by $P_k = H^*((\mathbb{R}P^\infty)^k)$ the modulo-2 cohomology algebra of the direct product of k copies of infinite dimensional real projective spaces $\mathbb{R}P^\infty$. Then, P_k is isomorphic to the graded polynomial algebra $\mathbb{F}_2[x_1, x_2, \ldots, x_k]$ of k variables, in which each x_j is of degree 1. Here the cohomology is taken with coefficients in the prime field $\mathbb{F}_2$ of two elements.

The $\mathcal{A}$-module structure of P_k is explicitly determined by the formula

$$Sq^i(x_j) = \begin{cases} x_j, & i = 0, \\ x_j^2, & i = 1, \\ 0, & i > 1, \end{cases}$$

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and the Cartan formula $Sq^n(xy) = \sum_{i=0}^{n} Sq^i(x)Sq^{n-i}(y)$, where $x, y \in P_k$ (see Steenrod and Epstein [7]).

A polynomial f in P_k is called *hit* if it can be written as a finite sum $f = \sum_{u\geqslant 0} Sq^{2^u}(h_u)$ for suitable polynomials h_u. That means f belongs to $\mathcal{A}^+P_k$, where $\mathcal{A}^+$ denotes the augmentation ideal in $\mathcal{A}$.

The *Peterson hit problem* is to find a minimal generating set for P_k regarded as a module over the mod-2 Steenrod algebra. Equivalently, this problem is to find a basis for the vector space

$$QP_k := \mathbb{F}_2 \otimes_{\mathcal{A}} P_k \cong P_k/\mathcal{A}^+P_k$$

in each degree n, where $\mathcal{A}^+$ is an ideal of $\mathcal{A}$ generated by all Steenrod squares of positive degrees. Such a basis may be represented by a list of monomials of degree n.

This problem has first been studied by Peterson [3], Wood [15], Singer [6], Priddy [4], who pointed out its relationship with some classical problems in homotopy theory such as the cobordism theory of manifolds, the modular representation theory of linear groups, Adams spectral sequences of stable homotopy of spheres, and stable homotopy type of the classifying space of finite groups. Then, this problem was investigated by Nam [2], Silverman [5], Wood [15], Sum [8, 9, 10], Tin-Sum [12], Tin [11, 13] and others.

For a positive integer n, by $\mu(n)$ one means the smallest number r for which it is possible to write $n = \sum_{1\leqslant i\leqslant r}(2^{u_i} - 1)$, where $u_i > 0$. Wood proved the following result.

Theorem 1.1 (Wood [15]). *If $\mu(n) > k$, then $(QP_k)_n = 0$.*

From the above result of Wood, the hit problem is reduced to the case of degree n with $\mu(n) \leqslant k$.

One of our main tools is Kameko's homomorphism $\widetilde{Sq}^0_* : QP_k \to QP_k$, which is induced by an $\mathbb{F}_2$-linear map $\phi_k : P_k \to P_k$, given by

$$\phi_k(x) = \begin{cases} y, & \text{if } x = x_1x_2\ldots x_k y^2, \\ 0, & \text{otherwise,} \end{cases}$$

for any monomial $x \in P_k$. The map ϕ_k is not an $\mathcal{A}$-homomorphism. However, $\phi_k Sq^{2i} = Sq^i\phi_k$ and $\phi_k Sq^{2i+1} = 0$ for any non-negative integer i.

Theorem 1.2 (Kameko [1]). *Let d be a non-negative integer. If $\mu(2d + k) = k$, then*

$$\widetilde{Sq}^0_* : (QP_k)_{2d+k} \longrightarrow (QP_k)_d$$

is an isomorphism of $\mathbb{F}_2$-vector spaces.

Thus, the hit problem is reduced to the case of degree n of the form

$$n = r(2^s - 1) + 2^s m,$$

where r, s, m are non-negative intergers such that $0 \leqslant \mu(m) < r < k$.

So far, the $\mathbb{F}_2$-vector space QP_k was explicitly calculated by Peterson [3] for $k = 1, 2$, by Kameko [1] for $k = 3$ and by Sum [9] for $k = 4$. However, for $k > 4$, it is still unsolved, even in the case of $k = 5$ with the help of computers.

In this paper, we study the hit problem for $k = 5$ and the degree fourteen. The main result of the paper is the following.

Theorem 1.3. *There exist exactly* 320 *admissible monomials of degree fourteen in* P_5. *Consequently,* $\dim(QP_5)_{14} = 320$.

We prove the above theorem by explicitly determining all admissible monomials of degree fourteen in P_5.

This paper is organized as follows. In Section 2, we recall some needed information on the weight vectors of monomials, the admissible monomials in P_k and Singer's criterion on the hit monomials. The proof of main theorem is presented in Section 3. Finally, in the appendix we list all the admissible monomials of degree fourteen in P_5^+.

2. Preliminaries

In this section, we recall some needed information from Kameko [1], Singer [6] and Sum [8], which will be used in the next section.

Notation 2.1. We denote $\mathbb{N}_k = \{1, 2, \ldots, k\}$ and

$$X_{\mathbb{J}} = X_{\{j_1, j_2, \ldots, j_s\}} = \prod_{j \in \mathbb{N}_k \setminus \mathbb{J}} x_j, \quad \mathbb{J} = \{j_1, j_2, \ldots, j_s\} \subset \mathbb{N}_k,$$

Let $\alpha_i(a)$ denote the i-th coefficient in dyadic expansion of a non-negative integer a. That means $a = \alpha_0(a)2^0 + \alpha_1(a)2^1 + \alpha_2(a)2^2 + \ldots$, for $\alpha_i(a) = 0$ or 1 with $i \geqslant 0$.

Let $x = x_1^{a_1} x_2^{a_2} \ldots x_k^{a_k} \in P_k$. Denote $\nu_j(x) = a_j, 1 \leqslant j \leqslant k$. Set

$$\mathbb{J}_t(x) = \{j \in \mathbb{N}_k : \alpha_t(\nu_j(x)) = 0\},$$

for $t \geqslant 0$. Then, we have $x = \prod_{t \geqslant 0} X_{\mathbb{J}_t(x)}^{2^t}$.

Definition 2.2. *For a monomial x in P_k, define two sequences associated with x by*

$$\omega(x) = (\omega_1(x), \omega_2(x), \ldots, \omega_i(x), \ldots), \quad \sigma(x) = (\nu_1(x), \nu_2(x), \ldots, \nu_k(x)),$$

where $\omega_i(x) = \sum_{1\leqslant j\leqslant k} \alpha_{i-1}(\nu_j(x)) = \deg X_{\mathbb{J}_{i-1}(x)}$, $i \geqslant 1$. *The sequence* $\omega(x)$ *is called the weight vector of* x.

Let $\omega = (\omega_1, \omega_2, \ldots, \omega_i, \ldots)$ *be a sequence of non-negative integers. The sequence* ω *is called the weight vector if* $\omega_i = 0$ *for* $i \gg 0$.

The sets of all the weight vectors and the exponent vectors are given the left lexicographical order.

For a weight vector ω, we define $\deg\omega = \sum_{i>0} 2^{i-1}\omega_i$. Denote by $P_k(\omega)$ the subspace of P_k spanned by all monomials y such that $\deg y = \deg\omega$, $\omega(y) \leqslant \omega$, and by $P_k^-(\omega)$ the subspace of P_k spanned by all monomials $y \in P_k(\omega)$ such that $\omega(y) < \omega$.

Definition 2.3. *Let* ω *be a weight vector and* f, g *two polynomials of the same degree in* P_k.

i) $f \equiv g$ *if and only if* $f - g \in \mathcal{A}^+P_k$. *If* $f \equiv 0$ *then* f *is called hit.*

ii) $f \equiv_\omega g$ *if and only if* $f - g \in \mathcal{A}^+P_k + P_k^-(\omega)$.

Obviously, the relations $\equiv$ and $\equiv_\omega$ are equivalence ones. Denote by $QP_k(\omega)$ the quotient of $P_k(\omega)$ by the equivalence relation $\equiv_\omega$ and $QP_k := \mathbb{F}_2 \otimes_{\mathcal{A}} P_k$. Then, we have

$$QP_k(\omega) = P_k(\omega)/((\mathcal{A}^+P_k \cap P_k(\omega)) + P_k^-(\omega)).$$

Definition 2.4. *Let* x, y *be monomials of the same degree in* P_k. *We say that* $x < y$ *if and only if one of the following holds:*

i) $\omega(x) < \omega(y)$;

ii) $\omega(x) = \omega(y)$ *and* $\sigma(x) < \sigma(y)$.

Definition 2.5. *A monomial* x *is said to be inadmissible if there exist monomials* $y_1, y_2, \ldots, y_m$ *such that* $y_t < x$ *for* $t = 1, 2, \ldots, m$ *and* $x - \sum_{t=1}^m y_t \in \mathcal{A}^+P_k$.

A monomial x *is said to be admissible if it is not inadmissible.*

Obviously, the set of all the admissible monomials of degree n in P_k is a minimal set of $\mathcal{A}$-generators for P_k in degree n.

Theorem 2.6 (See Kameko [1], Sum [8]). *Let* x, y, w *be monomials in* P_k *such that* $\omega_i(x) = 0$ *for* $i > r > 0$, $\omega_s(w) \neq 0$ *and* $\omega_i(w) = 0$ *for* $i > s > 0$.

i) *If* w *is inadmissible, then* xw^{2^r} *is also inadmissible.*

ii) *If* w *is strictly inadmissible, then* wy^{2^s} *is also strictly inadmissible.*

Now, we recall a result of Singer [6] on the hit monomials in P_k.

Definition 2.7. *A monomial* z *in* P_k *is called a spike if* $\nu_j(z) = 2^{t_j} - 1$ *for* t_j *a non-negative integer and* $j = 1, 2, \ldots, k$. *If* z *is a spike with* $t_1 > t_2 > \ldots > t_{r-1} \geqslant t_r > 0$ *and* $t_j = 0$ *for* $j > r$, *then it is called the minimal spike.*

In [6], Singer showed that if $\mu(n) \leqslant k$, then there exists uniquely a minimal spike of degree n in P_k.

The following is a criterion for the hit monomials in P_k.

Theorem 2.8 (See Singer [6]). *Suppose $x \in P_k$ is a monomial of degree n, where $\mu(n) \leqslant k$. Let z be the minimal spike of degree n. If $\omega(x) < \omega(z)$, then x is hit.*

Now, we recall some notations and definitions in [9], which will be used in the next sections. We set

$$P_k^0 = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \ : \ a_1a_2\ldots a_k = 0\}\rangle,$$
$$P_k^+ = \langle\{x = x_1^{a_1}x_2^{a_2}\ldots x_k^{a_k} \ : \ a_1a_2\ldots a_k > 0\}\rangle.$$

It is easy to see that P_k^0 and P_k^+ are the $\mathcal{A}$-submodules of P_k. Furthermore, we have the following.

Proposition 2.9. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $QP_k = QP_k^0 \oplus QP_k^+$. Here $QP_k^0 = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^0$ and $QP_k^+ = \mathbb{F}_2 \otimes_{\mathcal{A}} P_k^+$.*

Definition 2.10. *For any $1 \leqslant i \leqslant k$, define the homomorphism $f_i : P_{k-1} \to P_k$ of algebras by substituting*

$$f_i(x_j) = \begin{cases} x_j, & \text{if } 1 \leqslant j < i, \\ x_{j+1}, & \text{if } i \leqslant j < k. \end{cases}$$

Then, f_i is a homomorphism of $\mathcal{A}$-modules.

For a subset $B \subset P_k$, we denote $[B] = \{[f] : f \in B\}$. If $B \subset P_k(\omega)$, then we set $[B]_\omega = \{[f]_\omega : f \in B\}$. From Theorem 2.8, we see that if ω is the weight vector of a minimal spike in P_k, then $[B]_\omega = [B]$. Obviously, we have

Proposition 2.11. *If B is a minimal set of generators for $\mathcal{A}$-module P_{k-1} in degree n, then $f(B) = \bigcup_{i=1}^{k} f_i(B)$ is a minimal set of generators for $\mathcal{A}$-module P_k^0 in degree n.*

From now on, we denote by $B_k(n)$ the set of all admissible monomials of degree n in P_k, $B_k^0(n) = B_k(n) \cap P_k^0$, $B_k^+(n) = B_k(n) \cap P_k^+$. For a weight vector ω of degree n, we set $B_k(\omega) = B_k(n) \cap P_k(\omega)$, $B_k^+(\omega) = B_k^+(n) \cap P_k(\omega)$.

Then, $[B_k(\omega)]_\omega$ and $[B_k^+(\omega)]_\omega$, are respectively the basses of the $\mathbb{F}_2$-vector spaces $QP_k(\omega)$ and $QP_k^+(\omega) := QP_k(\omega) \cap QP_k^+$.

3. Proof of Theorem 1.3

In this section, we will prove Theorem 1.3 by explicitly determining all admissible monomials of degree fourteen in P_5. First, we give a direct summand decomposition of the $\mathbb{F}_2$-vector spaces $(QP_5)_{14}$ as follows.

Proposition 3.1. *We have a direct summand decomposition of the $\mathbb{F}_2$-vector spaces*

$$(QP_5)_{14} = (QP_5^0)_{14} \oplus QP_5^+(4,3,1) \oplus QP_5^+(4,5) \oplus QP_5^+(2,4,1) \oplus QP_5^+(2,2,2).$$

Proof. Since $P_k = \oplus_{d\geqslant 0}(P_k)_d$ is the graded polynomial algebra and using Proposition 2.9, we obtain $(QP_5)_{14} = (QP_5^0)_{14} \oplus (QP_5^+)_{14}$.

Suppose x is an admissible monomial of degree 14 in P_5^+. Observe that $z = x_1^7x_2^7$ is the minimal spike of degree 14 in P_5 and $\omega(z) = (2,2,2)$. Since the degree of (x) is even, using Theorem 2.8, we obtain either $\omega_1(x) = 2$ or $\omega_1(x) = 4$.

If $\omega_1(x) = 2$ then $x = x_ix_jv^2$ with v an admissible monomial of degree six in P_5. It is easy to check that $\omega(v) = (4,1)$ or $\omega(v) = (2,2)$. Therefore, $\omega(x) = (2,4,1)$ or $\omega(x) = (2,2,2)$.

If $\omega_1(x) = 4$ then we have $x = x_ix_jx_\ell x_tu^2$ with u an admissible monomial of degree five in P_5. It is easy to see that $\omega(u) = (5,0)$ or $\omega(u) = (3,1)$. Thus, $\omega(x) = (4,5)$ or $\omega(x) = (4,3,1)$.

So, $\omega(x)$ is one of the following sequences:

$$\omega_{(1)} = (4,5),\ \omega_{(2)} = (4,3,1),\ \omega_{(3)} = (2,2,2),\ \omega_{(4)} = (2,4,1).$$

On the other hand, we have

$$QP_k(\omega) \cong QP_k^\omega := \langle\{[x] \in QP_k : x \text{ is admissible and } \omega(x) = \omega\}\rangle.$$

Hence, one get

$$(QP_k)_n = \bigoplus_{\deg\omega=n} QP_k^\omega \cong \bigoplus_{\deg\omega=n} QP_k(\omega), \quad \text{(see Walker-Wood [14]).}$$

And therefore, $(QP_5^+)_{14} = QP_5^+(\omega_{(1)}) \oplus QP_5^+(\omega_{(2)}) \oplus QP_5^+(\omega_{(3)}) \oplus QP_5^+(\omega_{(4)})$. The proposition is proved.

Recall that $(QP_4)_{14}$ is an $\mathbb{F}_2$-vector space of dimension 35 with a basis consisting of all the classes represented by the monomials $w_j,\ 1 \leqslant j \leqslant 35$. Consequently, $|B_4(14)| = 35$, (see Sum [9]).

Using Proposition 2.11, we obtain

$$\dim(QP_5^0)_{14} = |f(B_4(14))| = |\bigcup_{i=1}^{5} f_i(B_4(14))| = 190.$$

So, $[B_5^0(14)] = \{[a_t] :\ a_t \in \bigcup_{i=1}^5 f_i(B_4(14)), 1 \leqslant t \leqslant 190\}$ is a basis of the $\mathbb{F}_2$-vector space $(QP_5^0)_{14}$.

Next, we explicitly determine $(QP_5^+)_{14}$ by proving some propositions as follows.

Proposition 3.2. $B_5^+(\omega_{(1)}) = \emptyset$. *That means* $QP_5^+(\omega_{(1)}) = 0$.

Proof. Let x be an monomial in P_5^+ such that $\omega(x) = \omega_{(1)}$. Then $x = x_ix_jx_\ell x_t.v^2$

with $v \in B_5(5)$ and $1 \leqslant i < j < \ell < t \leqslant 5$. By direct computation, using Theorem 2.6 we see that x is a permutation of one of the monomials: $x_i^2x_j^3x_\ell^3x_t^3x_s^3$. Here (i, j, ℓ, t, s) is a permutation of $(1, 2, 3, 4, 5)$.

A simple computation shows that

$$x = x_i^2x_j^3x_\ell^3x_t^3x_s^3 = Sq^1(x_ix_j^3x_\ell^3x_t^3x_s^3) + x_ix_j^4x_\ell^3x_t^3x_s^3 + \\ + x_ix_j^3x_\ell^4x_t^3x_s^3 + x_ix_j^3x_\ell^3x_t^4x_s^3 + x_ix_j^3x_\ell^3x_t^3x_s^4,$$

and $\omega(x_ix_j^4x_\ell^3x_t^3x_s^3) = \omega(x_ix_j^3x_\ell^4x_t^3x_s^3) = \omega(x_ix_j^3x_\ell^3x_t^4x_s^3) = \omega(x_ix_j^3x_\ell^3x_t^3x_s^4) < \omega(x)$ These relations imply that x is inadmissible. The proposition is proved.

Proposition 3.3. *The set* $\{[a_t] : 191 \leqslant t \leqslant 290\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $QP_5^+(\omega_{(2)})$. *Here, the monomials* $a_t, 191 \leqslant t \leqslant 290$, *are determined as in Appendix.*

We prove the proposition by showing that $B_5^+(\omega_{(2)}) = \{a_t : \ 191 \leqslant t \leqslant 290\}$, where the monomials $a_t, 191 \leqslant t \leqslant 290$, are listed in Appendix. We need some lemmas for the proof of this proposition.

Lemma 3.4. *The monomials* $x_i^3x_j^3x_\ell^2x_tx_s^5, x_i^3x_j^3x_\ell^6x_tx_s, x_i^3x_j^7x_\ell^2x_tx_s$ *are inadmissible. Here* (i, j, ℓ, t, s) *is a permutation of* $(1, 2, 3, 4, 5)$, *with* $\ell < t, s$.

Proof. We prove this lemma for the monomial $x = x_1^3x_2^7x_3^2x_4x_5$. The others can be proved by similar computations. It is easy to see that

$$x = Sq^1(x_1^3x_2^7x_3x_4x_5) + x_1^4x_2^7x_3x_4x_5 + x_1^3x_2^8x_3x_4x_5 + x_1^3x_2^7x_3x_4^2x_5 + x_1^3x_2^7x_3x_4x_5^2$$

and

$$\omega(x) \geqslant \max\{\omega(x_1^4x_2^7x_3x_4x_5),\ \omega(x_1^3x_2^8x_3x_4x_5),\ \omega(x_1^3x_2^7x_3x_4^2x_5),\ \omega(x_1^3x_2^7x_3x_4x_5^2)\}$$

Hence, x is an inadmissible monomial. The lemma is proved.

By direct calculation, one gets the following lemma.

Lemma 3.5. *The monomials* $x_1^2x_ix_j^3x_\ell^3x_t^5,\ x_1^2x_ix_jx_\ell^3x_t^7,\ x_1^4x_ix_j^3x_\ell^3x_t^3,\ x_1^6x_ix_jx_\ell^3x_t^3$ *are inadmissible. Here* (i, j, ℓ, t) *is a permutation of* $(2, 3, 4, 5)$.

Proof of Proposition 3.3. Let x be an admissible monomial in P_5^+ such that $\omega(x) = \omega_{(2)}$. Then $x = x_ix_jx_\ell x_t.w^2$ with $w \in B_5(3, 1)$ and $1 \leqslant i < j < \ell < t \leqslant 5$.

By direct computation, using Theorem 2.6 we see that if $z \in P_5^+(\omega_{(2)})$ and $z \notin B_5^+(\omega_{(2)})$ then z is one of the monomials which is given in one of Lemmas 3.4 and 3.5. And therefore, the $\mathbb{F}_2$-vector space $QP_5^+(\omega_{(4)})$ is generated by the set $\{[a_t] :\ 191 \leqslant t \leqslant 290\}$.

We now prove the set $\{[a_t] :\ 191 \leqslant t \leqslant 290\}$ is linearly independent in QP_5. For any $1 \leqslant i < j \leqslant 5$, we define the homomorphism $p_{(i;j)} : P_5 \to P_4$ of algebras by substituting

$$p_{(i;j)}(x_u) = \begin{cases} x_u, & \text{if } 1 \leqslant u < i, \\ x_{j-1}, & \text{if } u = i, \\ x_{u-1}, & \text{if } i < u \leqslant 5. \end{cases}$$

Remarkably, these homomorphisms are also $\mathcal{A}$-modules homomorphisms. We use them to prove that a certain set of monomials is actually the set of admissible monomials in P_5 by showing these monomials are linearly independent in QP_5.

Suppose there is a linear relation

$$\mathcal{S} = \sum_{t=191}^{290} \gamma_t a_t \equiv 0,$$

where $\gamma_t \in \mathbb{F}_2$. Using the results in [9], we explicitly compute $p_{(i;j)}(\mathcal{S})$ in terms of the admissible monomials in $P_4(\text{mod}(\mathcal{A}^+P_4))$. By direct computation from the relations $p_{(i;j)}(\mathcal{S}) \equiv 0$, one gets $\gamma_t = 0$ for all $191 \leqslant t \leqslant 290$. The proposition follows.

Proposition 3.6. *The set* $\{[a_t] : 291 \leqslant t \leqslant 305\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $QP_5^+(\omega_{(3)})$. *Here, the monomials* $a_t, 291 \leqslant t \leqslant 305$, *are determined as in Appendix.*

We prove the proposition by showing that $B_5^+(\omega_{(3)}) = \{a_t : \ 291 \leqslant t \leqslant 305\}$, where the monomials $a_t, 291 \leqslant t \leqslant 305$, are listed in Appendix. We need some lemmas for the proof of this proposition.

Lemma 3.7. *The following monomials are inadmissible:* $x_1^3x_j^2x_\ell^4x_tx_s^4$ *with* $j < t, s$; $x_1x_j^2x_\ell^2x_t^4x_s^5, x_1x_j^2x_\ell^6x_t^4x_s$ *with* $j, \ell < s$. *Here* (j, ℓ, t, s) *is a permutation of* $(2, 3, 4, 5)$.

Proof. It is easy to check that

$$x = x_1x_2^2x_3^2x_4^4x_5^5 = Sq^2(x_1x_2x_3x_4^4x_5^5) + Sq^1(x_1^2x_2x_3x_4^4x_5^5) \ (\text{mod } P_5^-(\omega_{(3)})).$$

This equality shows that $[x_1x_2^2x_3^2x_4^4x_5^5]_{\omega_{(3)}} = 0$. Hence, x is inadmissible monomial. The others can be proved by the similar computations. And therefore, the lemma is proved.

By a simple computation, one gets the following lemma.

Lemma 3.8. *The monomials* $x_1^2x_ix_jx_\ell^4x_t^6$, $x_1^2x_ix_j^2x_\ell^4x_t^5$, $x_1^2x_ix_j^3x_\ell^4x_t^4$, $x_1^6x_ix_jx_\ell^2x_t^4$ *are inadmissible. Here* (i, j, ℓ, t) *is a permutation of* $(2, 3, 4, 5)$.

Proof of Proposition 3.6. Let x be an admissible monomial in P_5^+ such that $\omega(x) = \omega_{(3)}$. Then $x = x_ix_j.v^2$ with $v \in B_5(2, 2)$ and $1 \leqslant i < j \leqslant 5$.

By direct calculation, using Theorem 2.6 we see that if $z \in P_5^+(2, 2, 2)$ and $z \notin B_5^+(\omega_{(3)})$ then z is a one of the monomials which is given in one of Lemmas 3.7 and 3.8. This implies $B_5^+(\omega_{(3)}) \subset \{a_t : \ 291 \leqslant t \leqslant 305\}$.

We now prove the set $\{[a_t] :\ 291 \leqslant t \leqslant 305\}$ is linearly independent in QP_5. Suppose there is a linear relation

$$\mathcal{U} = \sum_{t=291}^{305} \gamma_t a_t \equiv 0,$$

where $\gamma_t \in \mathbb{F}_2$. Using the results in [9], we explicitly compute $p_{(i;j)}(\mathcal{U})$ in terms of the admissible monomials in $P_4(\mathrm{mod}(\mathcal{A}^+ P_4))$. By the direct computation from the relations $p_{(i;j)}(\mathcal{U}) \equiv 0$, one gets $\gamma_t = 0$ for all $291 \leqslant t \leqslant 305$. The proposition follows.

By a similar computation as in Proposition 3.6, we get the following.

Proposition 3.9. *The set* $\{[a_t] : 306 \leqslant t \leqslant 320\}$ *is a basis of the* $\mathbb{F}_2$*-vector space* $QP_5^+(\omega_{(4)})$. *Here, the monomials* $a_t, 306 \leqslant t \leqslant 320$, *are determined as in Appendix.*

In summary, $\dim(QP_5^+)_{14} = 130$. And therefore, Theorem 1.3 is completely proved.

4. Appendix

In this section, we list all admissible monomials, $a_t, 191 \leqslant t \leqslant 320$, in $(P_5^+)_{14}$.

1) $B_5^+(4,3,1)$ is the set of 100 monomials as follows:

$x_1^1x_2^1x_3^2x_4^3x_5^7$	$x_1^1x_2^1x_3^2x_4^7x_5^3$	$x_1^1x_2^1x_3^3x_4^2x_5^7$	$x_1^1x_2^1x_3^3x_4^7x_5^2$	$x_1^1x_2^1x_3^7x_4^2x_5^3$
$x_1^1x_2^1x_3^7x_4^3x_5^2$	$x_1^1x_2^2x_3^1x_4^3x_5^7$	$x_1^1x_2^2x_3^1x_4^7x_5^3$	$x_1^1x_2^2x_3^3x_4^1x_5^7$	$x_1^1x_2^2x_3^3x_4^7x_5^1$
$x_1^1x_2^2x_3^7x_4^1x_5^3$	$x_1^1x_2^2x_3^7x_4^3x_5^1$	$x_1^1x_2^3x_3^1x_4^2x_5^7$	$x_1^1x_2^3x_3^1x_4^7x_5^2$	$x_1^1x_2^3x_3^2x_4^1x_5^7$
$x_1^1x_2^3x_3^2x_4^7x_5^1$	$x_1^1x_2^3x_3^7x_4^1x_5^2$	$x_1^1x_2^3x_3^7x_4^2x_5^1$	$x_1^1x_2^7x_3^1x_4^2x_5^3$	$x_1^1x_2^7x_3^1x_4^3x_5^2$
$x_1^1x_2^7x_3^2x_4^1x_5^3$	$x_1^1x_2^7x_3^2x_4^3x_5^1$	$x_1^1x_2^7x_3^3x_4^1x_5^2$	$x_1^1x_2^7x_3^3x_4^2x_5^1$	$x_1^3x_2^1x_3^1x_4^2x_5^7$
$x_1^3x_2^1x_3^1x_4^7x_5^2$	$x_1^3x_2^1x_3^2x_4^1x_5^7$	$x_1^3x_2^1x_3^2x_4^7x_5^1$	$x_1^3x_2^1x_3^7x_4^1x_5^2$	$x_1^3x_2^1x_3^7x_4^2x_5^1$
$x_1^3x_2^7x_3^1x_4^1x_5^2$	$x_1^3x_2^7x_3^1x_4^2x_5^1$	$x_1^7x_2^1x_3^1x_4^2x_5^3$	$x_1^7x_2^1x_3^1x_4^3x_5^2$	$x_1^7x_2^1x_3^2x_4^1x_5^3$
$x_1^7x_2^1x_3^2x_4^3x_5^1$	$x_1^7x_2^1x_3^3x_4^1x_5^2$	$x_1^7x_2^1x_3^3x_4^2x_5^1$	$x_1^7x_2^3x_3^1x_4^1x_5^2$	$x_1^7x_2^3x_3^1x_4^2x_5^1$
$x_1^1x_2^1x_3^3x_4^3x_5^6$	$x_1^1x_2^1x_3^3x_4^6x_5^3$	$x_1^1x_2^1x_3^6x_4^3x_5^3$	$x_1^1x_2^3x_3^1x_4^3x_5^6$	$x_1^1x_2^3x_3^1x_4^6x_5^3$
$x_1^1x_2^3x_3^3x_4^1x_5^6$	$x_1^1x_2^3x_3^3x_4^6x_5^1$	$x_1^1x_2^3x_3^6x_4^1x_5^3$	$x_1^1x_2^3x_3^6x_4^3x_5^1$	$x_1^1x_2^6x_3^1x_4^3x_5^3$
$x_1^1x_2^6x_3^3x_4^1x_5^3$	$x_1^1x_2^6x_3^3x_4^3x_5^1$	$x_1^3x_2^1x_3^1x_4^3x_5^6$	$x_1^3x_2^1x_3^1x_4^6x_5^3$	$x_1^3x_2^1x_3^3x_4^1x_5^6$
$x_1^3x_2^1x_3^3x_4^6x_5^1$	$x_1^3x_2^1x_3^6x_4^1x_5^3$	$x_1^3x_2^1x_3^6x_4^3x_5^1$	$x_1^3x_2^3x_3^1x_4^1x_5^6$	$x_1^3x_2^3x_3^1x_4^6x_5^1$
$x_1^1x_2^2x_3^3x_4^3x_5^5$	$x_1^1x_2^2x_3^3x_4^5x_5^3$	$x_1^1x_2^2x_3^5x_4^3x_5^3$	$x_1^1x_2^3x_3^2x_4^3x_5^5$	$x_1^1x_2^3x_3^2x_4^5x_5^3$
$x_1^1x_2^3x_3^3x_4^2x_5^5$	$x_1^1x_2^3x_3^3x_4^5x_5^2$	$x_1^1x_2^3x_3^5x_4^2x_5^3$	$x_1^1x_2^3x_3^5x_4^3x_5^2$	$x_1^3x_2^1x_3^2x_4^3x_5^5$
$x_1^3x_2^1x_3^2x_4^5x_5^3$	$x_1^3x_2^1x_3^3x_4^2x_5^5$	$x_1^3x_2^1x_3^3x_4^5x_5^2$	$x_1^3x_2^1x_3^5x_4^2x_5^3$	$x_1^3x_2^1x_3^5x_4^3x_5^2$
$x_1^3x_2^3x_3^1x_4^2x_5^5$	$x_1^3x_2^3x_3^1x_4^5x_5^2$	$x_1^3x_2^3x_3^5x_4^1x_5^2$	$x_1^3x_2^3x_3^5x_4^2x_5^1$	$x_1^3x_2^5x_3^1x_4^2x_5^3$
$x_1^3x_2^5x_3^1x_4^3x_5^2$	$x_1^3x_2^5x_3^2x_4^1x_5^3$	$x_1^3x_2^5x_3^2x_4^3x_5^1$	$x_1^3x_2^5x_3^3x_4^1x_5^2$	$x_1^3x_2^5x_3^3x_4^2x_5^1$
$x_1^1x_2^3x_3^3x_4^3x_5^4$	$x_1^1x_2^3x_3^3x_4^4x_5^3$	$x_1^1x_2^3x_3^4x_4^3x_5^3$	$x_1^3x_2^1x_3^3x_4^3x_5^4$	$x_1^3x_2^1x_3^3x_4^4x_5^3$
$x_1^3x_2^1x_3^4x_4^3x_5^3$	$x_1^3x_2^3x_3^1x_4^3x_5^4$	$x_1^3x_2^3x_3^1x_4^4x_5^3$	$x_1^3x_2^3x_3^3x_4^1x_5^4$	$x_1^3x_2^3x_3^3x_4^4x_5^1$
$x_1^3x_2^3x_3^4x_4^1x_5^3$	$x_1^3x_2^3x_3^4x_4^3x_5^1$	$x_1^3x_2^4x_3^1x_4^3x_5^3$	$x_1^3x_2^4x_3^3x_4^1x_5^3$	$x_1^3x_2^4x_3^3x_4^3x_5^1$

2) $B_5^+(2,2,2)$ is the set of 15 monomials as follows:

$$\begin{array}{lllll} x_1^1x_2^1x_3^2x_4^4x_5^6 & x_1^1x_2^1x_3^2x_4^6x_5^4 & x_1^1x_2^1x_3^6x_4^2x_5^4 & x_1^1x_2^2x_3^1x_4^4x_5^6 & x_1^1x_2^2x_3^1x_4^6x_5^4 \\ x_1^1x_2^2x_3^4x_4^1x_5^6 & x_1^1x_2^6x_3^1x_4^2x_5^4 & x_1^1x_2^2x_3^5x_4^2x_5^4 & x_1^1x_2^2x_3^3x_4^4x_5^4 & x_1^1x_2^2x_3^4x_4^3x_5^4 \\ x_1^1x_2^3x_3^2x_4^4x_5^4 & x_1^1x_2^3x_3^4x_4^2x_5^4 & x_1^3x_2^1x_3^2x_4^4x_5^4 & x_1^3x_2^1x_3^4x_4^2x_5^4 & x_1^3x_2^4x_3^1x_4^2x_5^4 \end{array}$$

3) $B_5^+(2,4,1)$ is the set of 15 monomials as follows:

$$\begin{array}{lllll} x_1^1x_2^2x_3^2x_4^2x_5^7 & x_1^1x_2^2x_3^2x_4^7x_5^2 & x_1^1x_2^2x_3^7x_4^2x_5^2 & x_1^1x_2^7x_3^2x_4^2x_5^2 & x_1^7x_2^1x_3^2x_4^2x_5^2 \\ x_1^1x_2^2x_3^2x_4^3x_5^6 & x_1^1x_2^2x_3^3x_4^2x_5^6 & x_1^1x_2^2x_3^3x_4^6x_5^2 & x_1^1x_2^3x_3^2x_4^2x_5^6 & x_1^1x_2^3x_3^2x_4^6x_5^2 \\ x_1^1x_2^3x_3^6x_4^2x_5^2 & x_1^3x_2^1x_3^2x_4^2x_5^6 & x_1^3x_2^1x_3^2x_4^6x_5^2 & x_1^3x_2^1x_3^6x_4^2x_5^2 & x_1^3x_2^5x_3^2x_4^2x_5^2 \end{array}$$

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