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EXISTENCE THEOREMS FOR GENERALIZED NONLINEAR PERTURBED ABSTRACT MEASURE INTEGRODIFFERENTIAL EQUATIONS

Bapurao C. Dhage

"Kasubai", Gurukul Colony, Thodga Road,
Ahmedpur-413 515, Dist. Latur, Maharashtra, INDIA

E-mail : bcdhage@gmail.com

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Abstract: In this paper, we prove the relevance and local existence theorems for a class of generalized nonlinear perturbed abstract measure integrodifferential equations via classical hybrid fixed point theorems of Dhage (1992,2003) under mixed weaker Lipschitz and Carathéodory conditions. The existence of extremal solutions is obtained between the given lower and upper solutions under certain monotonicity conditions. Our natural hypotheses and claims have also been illustrated with a couple of numerical examples.

Keywords and Phrases: Abstract measure integrodifferential equation, Relevance theorem, Dhage fixed point principle, Existence theorem.

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1. Statement of the Problem

Let X be a real Banach space with a convenient norm $\|\cdot\|_X$ and let $x, y \in X$ be any two elements. Then the line segment $\overline{xy}$ in X is defined by

$$\overline{xy} = \{z \in X \mid z = x + r(y - x), 0 \leq r \leq 1\}. \tag{1.1}$$

Let $x_0 \in X$ be a fixed point and $z \in X$. Then for any $x \in \overline{x_0 z}$, we define the sets S_x and $\overline{S}_x$ in X by

$$S_x = \{rx \mid -\infty < r < 1\}, \tag{1.2}$$

and

$$\overline{S}_x = \{rx \mid -\infty < r \leq 1\}. \tag{1.3}$$

Let $x_1, x_2 \in \overline{xy}$ be arbitrary. We say $x_1 < x_2$ if $S_{x_1} \subset S_{x_2}$, or equivalently, $\overline{x_0 x_1} \subset \overline{x_0 x_2}$. In this case we also write $x_2 > x_1$.

Let M denote the σ-algebra of all subsets of X such that (X, M) is a measurable space. Let $ca(X, M)$ be the space of all vector measures (real signed measures) and define a norm $||\cdot||$ on $ca(X, M)$ by

$$\|p\| = |p|(X), \tag{1.4}$$

where $|p|$ is a total variation measure of p and is given by

$$|p|(X) = \sup_{\sigma} \sum_{i=1}^{\infty} \mid p(E_i)|, \; E_i \subset X, \tag{1.5}$$

where the supremum is taken over all possible partitions $\sigma = \{E_i : i \in \mathbb{N}\}$ of measurable subsets of X. It is known that $ca(X, M)$ is a Banach space with respect to the norm $\|\cdot\|$ given by (1.4). Let μ be a σ-finite positive measure on X, and let $p \in ca(X, M)$. We say p is absolutely continuous with respect to the measure μ if $\mu(E) = 0$ implies $p(E) = 0$ for some $E \in M$. In this case we also write $p << \mu$. Let $x_0 \in X$ be fixed and let M_0 denote the σ- algebra on S_{x_0}. Let $z \in X$ be such that $z > x_0$ and let M_z denote the σ-algebra of all sets containing M_0 and the sets of the form S_x, $x \in \overline{x_0 z}$. Obviously, $M_0 \subset M_z$.

The abstract measure differential and abstract measure integrodifferential equations are the generalizations of the ordinary differential and ordinary integrodifferential equations. It is a very common fact that the generalization of any idea always leads to better results with a wide range of applications. Similarly, the study of abstract measure differential equations is initiated with the prediction that they may have some nice applications to the area of control theory and optimization. Motivated by the generalizations and applications, in this paper we discuss a nonlinear abstract measure integrodifferential equation for local existence and extremality of solutions.

Given a vector measure $p \in ca(X, M)$ with $p << \mu$, consider the nonlinear abstract measure integrodifferential equation (in short AMIGDE) of the form

$$\begin{aligned}\frac{dp}{d\mu} &= f\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, p(\overline{S}_\tau)\big)\,d\mu\right) \\ &\quad + g\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\,d\mu\right) \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z},\end{aligned} \tag{1.6}$$

and

$$p(E) = q(E), \quad E \in M_0, \tag{1.7}$$

where q is a given known vector measure, $\frac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ, the maps $x \mapsto h\big(x, p(\overline{S}_x)\big)$, $x \mapsto k\big(x, p(\overline{S}_x)\big)$, $x \mapsto f\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, p(\overline{S}_\tau)\big)\,d\mu\right)$ and $x \mapsto g\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\,d\mu\right)$ are μ-integrable for each $p \in ca(S_z, M_z)$.

Definition 1.1. *Given an initial real measure q on M_0, a vector $p \in ca(S_z, M_z)$ $(z > x_0)$ is said to be a solution of the perturbed AMIGDE (1.6)-(1.7) if*

(i) $p(E) = q(E)$, $E \in M_0$,

(ii) $p << \mu$ *on* $\overline{x_0 z}$, *and*

(iii) p *satisfies (1.6)-(1.7) a.e.* $[\mu]$ *on* $\overline{x_0 z}$.

The following result from measure theory is often times used for transforming the abstract measure differential equation into an equivalent abstract measure integral equation.

Theorem 1.1 (Radon-Nikkodym theorem). *Let λ and μ be two σ-finite measures defined on a measurable space (X, M) such that $\lambda << \mu$. Then there exists a M-measurable function $f : X \to [0, \infty)$ such that*

$$\lambda(E) = \int_E f\,d\mu \tag{1.8}$$

for any $E \in M$. The function f is unique upto the set of measure zero.

Note that the function f in the expression (1.8) is called the Radon-Nikodym derivative of the measure λ with respect to the measure μ and in this case we write

$$\frac{d\lambda}{d\mu} = f \quad \text{a.e. } [\mu] \quad \text{on} \quad X. \tag{1.9}$$

A few details of Radon-Nikodym derivative and its integral representation also appear in Ruddin [18], Sharma [19, 20], Dhage [1] and the references therein.

Remark 1.1. *By an application of Radon-Nikodym theorem given in Theorem 1.1, the AMIGDE (1.6)-(1.7) is equivalent to the generalized abstract measure integral equation (in short AMIGDE)*

$$\begin{aligned} p(E) = & \int_E f\left(x, p(\overline{S}_x), \int_{\overline{S}_x - S_{x_0}} h\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\right) d\mu \\ & + \int_E g\left(x, p(\overline{S}_x), \int_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\right) d\mu, \end{aligned} \tag{1.10}$$

if $E \in M_z$, $E \subset \overline{x_0 z}$. and

$$p(E) = q(E) \quad \textit{if } E \in M_0. \tag{1.11}$$

A solution p of the AMIGDE (1.6)-(1.7) on $\overline{x_0 z}$ will be denoted by $p(\overline{S}_{x_0}, q)$.

The existence theorem for the AMIGDE (1.6)-(1.7) is an open problem raised in Dhage [8] and in this paper we prove a local existence result under some generalized natural Lipschitz and compactness type conditions. The study of abstract measure differential equations (in short AMDEs) is initiated by Sharma [19, 20] as the generalizations of the ordinary differential equations in which ordinary derivative is replaced with the Radon-Nykodym derivative of vector measures in an abstract space, whereas the study of nonlinear AMIGDEs as the generalization of the ordinary integrodifferential equations is initiated by Dhage [1, 2, 3]. The existence results of Sharma [19, 20] and Dhage [1, 2, 3] are not of local nature whereas the results of the present paper are local and obtained in a closed ball in the Banach space $ca(X, M)$ centered at the given initial vector measure q. In the present paper we discuss the relevance and existence theorems to the AMIGDE (1.6)-(1.7) under suitable natural conditions via a Dhage's hybrid fixed point technique from nonlinear functional analysis. In the following section 2 we prove the relevance theorem for the AMIGDE (1.6)-(1.7) by relating it to an ordinary integrodifferential equations. Section 3 deals with the fixed point results needed in the subsequent sections of the paper. The existence result is proved in section 4 and the existence result for extremal solutions is proved in section 5.

2. Relevance Results

In this section we prove the relevance theorem for the AMIGDE (1.6)-(1.7) and it is shown that the AMIGDE (1.6)-(1.7) reduces to an ordinary integrodifferential equation, viz.,

$$\left.\begin{aligned} y'(x) &= f\Big(x, y(x), \int_{x_0}^{x} h(\tau, y(\tau))\, d\tau\Big) \\ &\quad + g\Big(x, y(x), \int_{x_0}^{x} k(\tau, y(\tau))\, d\tau\Big), \quad x \geq x_0, \\ y(x_0) &= y_0, \end{aligned}\right\} \tag{2.1}$$

under certain suitable natural conditions, where f and g are Carathéodory real-valued functions on $[x_0, x_0 + T] \times \mathbb{R} \times \mathbb{R}$ into $\mathbb{R}$.

Let $X = \mathbb{R}$, $\mu = m$, the Lebesgue measure on $\mathbb{R}$, $\overline{S}_x = (-\infty, x]$, $x \in \mathbb{R}$, and q a given real Borel measure on M_0. Then equations (1.6)-(1.7) take the form

$$\left.\begin{aligned} \frac{d}{dm} p((-\infty, x]) &= f\Big(x, p(-\infty, x], \int\limits_{[x_0, x]} h\big(\tau, p(-\infty, \tau]\big)\, dm\Big) \\ &\quad + g\Big(x, p(-\infty, x], \int\limits_{[x_0, x]} k\big(\tau, p(-\infty, \tau]\big)\, dm\Big), \\ p(E) &= q(E), \quad E \in M_0. \end{aligned}\right\} \tag{2.2}$$

It will now be shown that the equations (2.1) and (2.2) are equivalent in the sense of the following theorem.

Theorem 2.1. *Let $q : M_0 \to \mathbb{R}$ be a given initial measure such that $q(E) = 0$ for all $E \in M_0$ and $q(\{x_0\}) = 0$. Then,*

(a) for every solution $p = p(\overline{S}_{x_0}, q)$ of (2.2) existing on $[x_0, x_1)$, there corresponds a solution y of (2.1) satisfying $y(x_0) = y_0$.

(b) Conversely, for every solution $y(x)$ of (2.1), there corresponds a solution $p(\overline{S}_{x_0}, q)$, of (2.2) existing on $[x_0, x_1)$ with a suitable initial measure q provided f satisfies the relation $f(x_0, 0) = 0$.

Proof. (a) Let $p = p(\overline{S}_{x_0}, q)$ be a solution of (2.2), existing on $[x_0, x_1)$. Define a real Borel measure p_1 on $\mathbb{R}$ as follows.

$$p_1((-\infty, x)) = \begin{cases} 0, & \text{if } x \le x_0, \\ p((-\infty, x]) - p((-\infty, x_0]), & \text{if } x_0 < x < x_1 \\ p((-\infty, x_1)), & \text{if } x \ge x_1, \end{cases} \tag{2.3}$$

and

$$p_1(-\infty, x_0]) = p(-\infty, x_0]).$$

Define the functions $y_1(x)$ and $y(x)$ by

$$\begin{aligned} y_1(x) &= p_1((-\infty, x)), && x \in \mathbb{R} \\ y(x) &= y_1(x) + p((-\infty, x_0]), && x \in [x_0, x_1). \end{aligned} \tag{2.4}$$

The condition $q(\{x_0\}) = 0$, the definition of the solution p, and the definition of $y(x)$ together imply that

$$p_1(\{x_0\}) = p(\{x_0\}) = 0.$$

Now for each $x \in [x_0, x_1)$ we obtain from (2.2) and the definition of $y(x)$

$$\begin{aligned} y(x) &= y_1(x) + p((-\infty, x_0]) \\ &= p_1((-\infty, x)) + p((-\infty, x_0]) \\ &= p(\overline{S}_x). \end{aligned} \tag{2.5}$$

Since p is a solution of (2.2) we have $p << m$ on $[x_0, x_1)$. Hence $y(x)$ is absolutely continuous on $[x_0, x_1)$. The details concerning these arguments appear in Rudin [18, pages 163-165]. This shows that $y'(x)$ exists a. e. on $[x_0, x_1)$. Now for each $x \in [x_0, x_1)$, we have, by virtue of (2.3) and (2.4)

$$p([x_0, x]) = \int_{[x_0,x]} \frac{d}{dm} p((-\infty, t])\, dm.$$

Therefore,

$$p((-\infty, x]) - p((-\infty, x_0]) = \int_{[x_0,x]} \frac{d}{dm} p((-\infty, t])\, dm.$$

This further implies that

$$
\begin{aligned}
p(\overline{S}_x) = p(\overline{S}_{x_0}) &+ \int_{x_0}^{x} f\Big(t, p(\overline{S}_t), \int_{x_0}^{t} h\big(\tau, p(\overline{S}_\tau)\big)\, dm\Big)\, dm \\
&+ \int_{x_0}^{x} g\Big(t, p(\overline{S}_t \int_{x_0}^{t} g\big(\tau, p(\overline{S}_\tau)\big)\, dm\Big)\, dm.
\end{aligned}
$$

That is,

$$
\begin{aligned}
y(x) = y(x_0) &+ \int_{x_0}^{x} f\Big(t, y(t), \int_{x_0}^{t} f\big(\tau, y(\tau)\big)\, d\tau\Big)\, dt \\
&+ \int_{x_0}^{x} g\Big(t, y(t), \int_{x_0}^{t} k\big(\tau, y(\tau)\big)\, d\tau\Big)\, dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
y'(x) = f&\Big(x, y(x), \int_{x_0}^{x} h\big(\tau, y(\tau)\big)\, d\tau\Big) \\
&+ g\Big(x, y(x), \int_{x_0}^{x} k\big(\tau, y(\tau)\big)\, d\tau\Big) \quad \text{a.e on } [x_0, x_1).
\end{aligned}
$$

This proves that $y(x)$ is a solution of (2.1) on $[x_0, x_1)$ satisfying

$$y(x_0) = y_0.$$

(b) Conversely, suppose that $y(x)$ is a solution of (2.1) existing on $[x_0, x_1)$. Then, y is absolutely continuous on $[x_0, x_1]$. Now, corresponding to the absolutely continuous function $y(x)$ which is a solution of (2.1) on $[x_0, x_1)$, we can construct a absolutely continuous real Borel measure p on M_{x_1} such that,

$$
\begin{aligned}
&p(E) = 0 \ \text{ for all } \ E \in M_0, \\
&p(\overline{S}_x) = y(x), \ \text{if } \ x \in [x_0, x_1).
\end{aligned}
\tag{2.6}
$$

The details concerning these arguments appear in Rudin [18, pages 163-165]. Since $y(x)$ is a solution of (2.1) we have for $x \in [x_0, x_1)$,

$$
\begin{aligned}
y(x) = y(x_0) &+ \int_{x_0}^{x} f\Big(t, y(t), \int_{x_0}^{t} h\big(\tau, y(\tau)\big)\, d\tau\Big)\, dt \\
&+ \int_{x_0}^{x} g\Big(t, y(t), \int_{x_0}^{t} g\big(\tau, y(\tau)\big)\, d\tau\Big)\, dt.
\end{aligned}
$$

Now, $y(x_0) = p\big(S_{x_0}\big) = 0$ and so, by (2.6) we obtain that

$$\big[p(\overline{S}_x) - p(\overline{S}_{x_0})\big] = \int\limits_{[x_0,x]} f\Big(t, p(\overline{S}_t), \int\limits_{[x_0,t]} h\big(\tau, p(\overline{S}_\tau)\big)\, dm\Big)\, dm + \int\limits_{[x_0,x]} g\Big(t, p(\overline{S}_t), \int\limits_{[x_0,t]} k\big(\tau, p(\overline{S}_\tau)\big)\, dm\Big)\, dm.$$

That is,

$$p([x_0, x]) = \int\limits_{[x_0,x]} f\Big(t, p(\overline{S}_t), \int\limits_{[x_0,t]} h\big(\tau, p(\overline{S}_\tau)\big)\, dm\Big)\, dm + \int\limits_{[x_0,x]} g\Big(t, p(\overline{S}_t), \int\limits_{[x_0,t]} k\big(\tau, p(\overline{S}_\tau)\big)\, dm\Big)\, dm.$$

In general, if $E \in M_{x_1}, E \subset \overline{x_0x_1}$, then

$$p(E) = \int\limits_{[x_0,x]} f\Big(t, p((-\infty, t]), \int\limits_{[x_0,t]} h\big(\tau, p((-\infty, \tau])\big)\, dm\Big)\, dm + \int\limits_{[x_0,x]} g\Big(t, p((-\infty, t]), \int\limits_{[x_0,t]} k\big(\tau, p((-\infty, \tau])\big)\, dm\Big)\, dm.$$

By definition of Radon-Nykodym derivative, we obtain

$$\frac{d}{dm}\Big[p((-\infty, x])\Big] = f\Big(xt, p((-\infty, x]), \int\limits_{[x_0,x]} h\big(\tau, p((-\infty, \tau])\big)\, dm\Big) + g\Big(x, p((-\infty, x]), \int\limits_{[x_0,x]} k\big(\tau, p((-\infty, \tau])\big)\, dm\Big) \;\; a.e.\ [\mu] \;\text{ on }\; \overline{x_0z},$$

$$p(E) = 0 \;\text{ for }\; E \in M_0.$$

This shows that p is a solution of (2.2) on $[x_0, x_1)$ and the proof of (b) is complete.

3. Fixed Point Results

To state the required fixed point techniques that will be used in the proofs of main results, we need the following definitions in what follows.

Definition 3.1 (Dhage [5, 7]). *An upper semi-continuous and nondecreasing function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a $\mathcal{D}$-**function** if $\psi(0) = 0$. The class of all $\mathcal{D}$-functions on $\mathbb{R}+$ is denoted by $\mathfrak{D}$.*

Definition 3.2 (Dhage [5, 7]). *Let $\mathfrak{X}$ be a Banach space with a norm $\|\cdot\|$. An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called $\mathcal{D}$-**Lipschitz** if there exists a $\mathcal{D}$-function $\psi_{\mathcal{T}} \in \mathfrak{D}$ such that*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \psi_{\mathcal{T}}\big(\|x-y\|\big) \tag{3.1}$$

for all elements $x, y \in \mathfrak{X}$.

If $\psi_{\mathcal{T}}(r) = k\,r$, $k > 0$, $\mathcal{T}$ is called a **Lipschitz** operator on $\mathfrak{X}$ with the Lipschitz constant k. Again, if $0 \leq k < 1$, then $\mathcal{T}$ is called a **contraction** on $\mathfrak{X}$ with contraction constant k. Furthermore, if $\psi_{\mathcal{T}}(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a **nonlinear $\mathcal{D}$-contraction** on $\mathfrak{X}$. The class of all $\mathcal{D}$-functions satisfying the condition of nonlinear $\mathcal{D}$-contraction is denoted by $\mathfrak{D}\mathfrak{N}$.

An operator $\mathcal{T} : \mathfrak{X} \to \mathfrak{X}$ is called compact if $\overline{\mathcal{T}(\mathfrak{X})}$ is a compact subset of $\mathfrak{X}$. $\mathcal{T}$ is called totally bounded if for any bounded subset S of $\mathfrak{X}$, $\mathcal{T}(S)$ is a totally bounded subset of $\mathfrak{X}$. T is called completely continuous if $\mathcal{T}$ is continuous and totally bounded on $\mathfrak{X}$. Every compact operator is totally bounded, but the converse may not be true, however, two notions are equivalent on bounded subsets of $\mathfrak{X}$. The details of different types of nonlinear contraction, compact and completely continuous operators appear in Granas and Dugundji [16].

To prove the main existence results of next section, we need the following hybrid fixed point principle of Dhage [8] involving the sum of two operators in a Banach space $\mathfrak{X}$.

Theorem 3.1 (Dhage [7]). *Let S be a closed convex and bounded subset of a Banach space $\mathfrak{X}$ and let $\mathcal{A} : \mathfrak{X} \to \mathfrak{X}$ and $\mathcal{B} : S \to \mathfrak{X}$ be two operators satisfying the following conditions.*

(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is completely continuous, and

(c) $\mathcal{A}x + \mathcal{B}y = x \implies x \in S$ for all $y \in S$.

Then the operator equation $\mathcal{A}x + \mathcal{B}x = x$ has a solution in S.

4. Existence Theorem

We need the following definition in the sequel.

Definition 4.1. *A function $\beta : S_z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Carathéodory if*

(i) $x \mapsto \beta(x, u, v)$ is μ-measurable for each $u \in \mathbb{R}$, and

(ii) $(u,v) \mapsto \beta(x,u,v)$ *is jointly continuous almost everywhere* $[\mu]$ *on* $\overline{x_0 z}$.

Further a Carathéodory function $\beta(x,u,v)$ *is called* $L^{\mu}_{\mathbb{R}}$*-Carathéodory if*
(iii) there exists a μ*-integrable function* $\gamma : S_z \to \mathbb{R}$ *such that*

$$|\beta(x,u,v)| \leq \gamma(x) \quad a.e.\ [\mu] \ \ for\ x \in \overline{x_0 z},$$

for all $u, v \in \mathbb{R}$.

We consider the following set of assumptions.

(H$_0$) For any $z > x_0$, the σ-algebra M_z is compact with respect to the topology generated by the Pseudo-metric d defined on M_z by

$$d(E_1, E_2) = \mu(E_1 \Delta E_2)$$

for all $E_1, E_2 \in M_z$.

(H$_1$) $\mu(\{x_0\}) = 0$.

(H$_2$) There exists a $\mathcal{D}$-function $\psi_f \in \mathfrak{D}$ such that

$$|f(x,u_1,u_2) - f(x,v_1,v_2)| \leq \psi_f\big(\max\{|u_1 - v_1|, |u_2 - v_2|\}\big) \ \text{a.e.}\ [\mu] \text{ for } x \in \overline{x_0 z},$$

for all $u_1, u_2, v_1, v_2 \in \mathbb{R}$. Moreover, $\psi_f(r) < r$ for each $r > 0$.

(H$_3$) q is continuous on M_0 with respect to the Pseudo-metric d defined in (H_0).

(H$_4$) The function g is $L^{\mu}_{\mathbb{R}}$-Carathéodory on $S_z \times \mathbb{R} \times \mathbb{R}$.

Theorem 4.1. *Suppose that the hypotheses (H$_0$)-(H$_4$) hold. Then the AMIGDE (1.6)-(1.7) has a solution.*

Proof. By expressions (1.2) and (1.3), we have a real number $r(>1)$ such that $r \to 1$ and $S_{rx_0} \supset S_{x_0}$. Then, from hypothesis (H$_1$), it follows that

$$\bigcap_{r \to 1} \left(\overline{S}_{rx_0} - S_{x_0}\right) = \{x_0\}$$

and

$$\mu\Big(\overline{S}_{rx_0} - S_{x_0}\Big) = \mu(\{x_0\}) = 0$$

whenever $r \to 1$.

Therefore, we can choose a real number r^* such that $S_{r^*x_0} \supset S_{x_0}$ and satisfying

$$\mu\big(\overline{S}_{r^*x_0} - S_{x_0}\big) < 1 \quad \text{and} \quad \int\limits_{\overline{S}_{r^*x_0} - S_{x_0}} h(x)\, d\mu < 1. \tag{*}$$

Let $z^* = r^*x_0$. Consider the vector measure p_0 on M_{z^*} which is a continuous extension of the measure q on M_0 defined by

$$p_0(E) = \begin{cases} q(E) & \text{if} \quad E \in M_0, \\ 0 & \text{if} \quad E \notin M_0. \end{cases}$$

Now, we define a subset $S(\rho)$ of $ca(S_{z^*}, M_{z^*})$ by

$$S(\rho) = \big\{p \in ca(S_{z^*}, M_{z^*}) \mid \|p - p_0\| \le \rho\big\} \tag{4.1}$$

where $\rho = M_f + 1$. Clearly, $S(\rho)$ is a closed convex ball in $ca(S_{z^*}, M_{z^*})$ centered at p_0 of radius ρ and $q \in S(\rho)$.

Define the two operators $\mathcal{A} : ca(S_{z^*}, M_{z^*}) \to ca(S_{z^*}, M_{z^*})$ and $\mathcal{B} : S(\rho) \to ca(S_{z^*}, M_{z^*})$ by

$$\mathcal{A}p(E) = \begin{cases} \displaystyle\int_E f\Bigg(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Bigg)\, d\mu & \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*}, \\ 0 & \text{if } E \in M_0. \end{cases} \tag{4.2}$$

and

$$\mathcal{B}p(E) = \begin{cases} \displaystyle\int_E g\Bigg(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Bigg)\, d\mu & \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*}, \\ q(E) & \text{if } E \in M_0. \end{cases} \tag{4.3}$$

Then the AMIGDE (1.6)-(1.7) is equivalent to the operator equation

$$\mathcal{A}p(E) + \mathcal{B}p(E) = p(E), \quad E \in M_z. \tag{4.4}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of the hybrid fixed point theorem, Theorem 2.1 on $S(\rho)$. This will be done in a series of following steps.

Step I: Firstly, we show that $\mathcal{A}$ is bounded on $\mathfrak{X} = ca(S_{z^*}, M_{z^*})$. Let $p \in ca(S_{z^*}, M_{z^*})$ be arbitrary element. Then for any $E \in M_{z^*}$, there exist subsets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that $E = F \cup G$ and $F \cap G \neq \emptyset$. Now, by definition of the operator $\mathcal{A}$, we obtain $\mathcal{A}p(F) = 0$. Therefore, we have

$$|\mathcal{A}p(E)| \leq \int_G \left| f\left(x, p(\overline{S}_x, \int\limits_{\overline{S}_x - S_{x_0}} h(x, p(\overline{S}_\tau))\, d\mu\right)\right| d\mu \leq M_f$$

for all $E \in M_{z^*}$. Therefore, by definition of the norm,

$$\|\mathcal{A}p\| = |\mathcal{A}p|(E) = \sup_\sigma \sum_{i=1}^{\infty} |\mathcal{A}p(E_i)| \leq M_f$$

for all $p \in \mathfrak{X}$. As a result, $\mathcal{A}$ is a bounded operator on $ca(S_{z^*}, M_{z^*})$ into itself.

Step II: First we show that $\mathcal{A}$ is a nonlinear $\mathcal{D}$- contraction on $ca(S_{z^*}, M_{z^*})$. Let $p_1, p_2 \in ca(S_{z^*}, M_{z^*})$ be any two elements. Then, by definition of the operator $\mathcal{T}$, we obtain

$$\mathcal{A}p_1(E) - \mathcal{A}p_2(E) = 0 \quad \text{if} \;\; E \in M_0,$$

and

$$\begin{aligned}\mathcal{A}p_1(E) - \mathcal{A}p_2(E) = \int_E \Bigg[& f\Bigg(x, p_1(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p_1(\overline{S}_\tau)\big)\, d\mu\Bigg) \\ & - f\Bigg(x, p_2(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p_2(\overline{S}_\tau)\big)\, d\mu\Bigg)\Bigg] d\mu\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$.

Therefore, by hypotheses (H_4), we obtain

$$\begin{aligned}&|\mathcal{A}p_1(E) - \mathcal{A}p_2(E)| \\ &\quad \leq \int_E \Bigg| f\Bigg(x, p_1(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p_1(\overline{S}_\tau)\big)\, d\mu\Bigg) \\ &\qquad - f\Bigg(x, p_2(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p_2(\overline{S}_\tau)\big)\, d\mu\Bigg)\Bigg| d\mu\end{aligned}$$

$$\leq \int_E \psi_f\bigg(\max\Big\{ \big|p_1(\overline{S}_x) - p_2(\overline{S}_x)\big| \,,\, \int\limits_{\overline{x_0 x}} \psi_k\left(\big|p_1(\overline{S}_\tau) - p_2(\overline{S}_\tau)\big|\right)\, d\mu \Big\}\bigg)\, d\mu$$

$$\leq \int_E \psi_f\bigg(\max\Big\{ \big|p_1 - p_2\big|(\overline{S}_x) \,,\, \int\limits_{\overline{x_0 x}} \psi_k\left(\big|p_1 - p_2\big|(\overline{S}_\tau)\right)\, d\mu \Big\}\bigg)\, d\mu$$

$$\leq \int_E \psi_f\bigg(\max\Big\{ \big\|p_1 - p_2\big\| \,,\, \int\limits_{\overline{x_0 z^*}} \psi_k\left(\big\|p_1 - p_2\big\|\right)\, d\mu \Big\}\bigg)\, d\mu$$

$$\leq \int_E \psi_f\Big(\big\|p_1 - p_2\big\|\Big)\, d\mu$$

$$\leq \int\limits_{\overline{x_0 z^*}} \psi_f\Big(\big\|p_1 - p_2\big\|\Big)\, d\mu$$

$$\leq \psi_f\Big(\big\|p_1 - p_2\big\|\Big)$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. This further in view of definition of the norm in $ca(S_{z^*}, M_{z^*})$ implies that

$$\|\mathcal{A}p_1 - \mathcal{A}p_2\| \leq \psi_f\Big(\big\|p_1 - p_2\big\|\Big)$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. Hence, $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction on $ca(S_{z^*}, M_{z^*})$.
Step III : Thirdly, we show that $\mathcal{B}$ is continuous on $S(\rho)$. Let $\{p_n\}$ be a sequence of vector measures in $S(\rho)$ converging to a vector measure p. Then by dominated convergence theorem,

$$\begin{aligned}
\lim_{n\to\infty} \mathcal{B}p_n(E) &= \lim_{n\to\infty} \int_E g\bigg(x, p_n(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu\bigg)\, d\mu \\
&= \int_E g\bigg(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} \Big[\lim_{n\to\infty} k\big(\tau, p_n(\overline{S}_\tau)\big)\Big]\, d\mu\bigg)\, d\mu \\
&= \int_E g\bigg(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu\bigg)\, d\mu \\
&= \mathcal{B}p(E)
\end{aligned}$$

for all $E \in M_{z^*}$, $E \subset \overline{x_0 z^*}$. Similarly, if $E \in M_0$, then

$$\lim_{n\to\infty} \mathcal{B}p_n(E) = q(E) = \mathcal{B}p(E),$$

and so $\mathcal{B}$ is a pointwise continuous operator on $S(\rho)$.

Next we show that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous sequence in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \quad \text{with} \quad F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \quad \text{with} \quad F_2 \cap G_2 = \emptyset.$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{4.5}$$

and

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2). \tag{4.6}$$

Therefore, we have

$$\begin{aligned} \mathcal{B}p_n(E_1) &- \mathcal{B}p_n(E_2) \\ &\leq q(F_1) - q(F_2) + \int_{G_1 - G_2} g\Big(x, p_n(\overline{S}_x), \int_{\overline{S}_x - S_{x_0}} k\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu \\ &\quad + \int_{G_2 - G_1} g\Big(x, p_n(\overline{S}_x), \int_{\overline{S}_x - S_{x_0}} k\big(\tau, p_n(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu. \end{aligned}$$

Since $f(x, y)$ is $L^{\mu}_{\mathbb{R}}$- Carathéodory, we have that

$$\begin{aligned} |\mathcal{B}p_n(E_1) &- \mathcal{B}p_n(E_2)| \\ &\leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} \Big| g\Big(x, p_n(\overline{S}_x), \int_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\Big|\, d\mu \\ &\leq |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} \gamma(x)\, d\mu. \end{aligned}$$

Assume that

$$d(E_1, E_2) = \mu(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $\mu(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_{z^*} , it is uniformly continuous and so

$$|\mathcal{B}p_n(E_1) - \mathcal{B}p_n(E_2)| \;\; \leq \;\; |q(F_1) - q(F_2)| + \int_{G_1 \Delta G_2} \gamma(x)\, d\mu$$

$$\to \quad 0 \text{ as } E_1 \to E_2$$

uniformly for all $n \in \mathbb{N}$. This shows that $\{\mathcal{B}p_n : n \in \mathbb{N}\}$ is a equi-continuous set in $ca(S_{z^*}, M_{z^*})$. As a result, $\{\mathcal{B}p_n\}$ converges to $\mathcal{B}p$ uniformly on M_{z^*} and a fortiori $\mathcal{B}$ is a continuous operator on $S(\rho)$ into $ca(S_{z^*}, M_{z^*})$.

Step IV: Next we show that $\mathcal{T}(S(\rho))$ is a totally bounded set in $ca(S_{z^*}, M_{z^*})$. We shall show that the set is uniformly bounded and equi-continuous set in $ca(S_{z^*}, M_{z^*})$. Firstly, we show that $\mathcal{T}(S(\rho))$ is a uniformly bounded set in $ca(S_{z^*}, M_{z^*})$.

Let $\lambda \in \mathcal{T}(S)$ be an arbitrary element. Then, there is a member $p \in S$ such that $\lambda(E) = \mathcal{B}p(E)$ for all $E \in M_{z^*}$. Let $E \in M_{z^*}$. Then there exists two subsets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that

$$E = F \cup G \qquad \text{and} \qquad F \cap G = \phi.$$

Hence by definition of $\mathcal{B}$,

$$\begin{aligned}
|\lambda(E)| &= |\mathcal{B}p(E)| \\
&\leq |q(F)| + \int_G \left| g\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big) \right| d\mu \\
&\leq \|q\| + \int_G \gamma(x)\, d\mu \\
&\leq \|q\| + \int_E \gamma(x)\, d\mu \\
&< \|q\| + 1
\end{aligned}$$

for all $E \in M_{z^*}$. From the above inequality it follows that

$$\|\lambda\| = \|\mathcal{B}p\| = |\mathcal{B}p|(E) = \sup_\sigma \sum_{i=1}^{\infty} |Tp(E_i)| \leq \|q\| + 1$$

for all $\lambda \in \mathcal{B}(S(\rho))$. As a result $\mathcal{B}$ defines a mapping $\mathcal{B} : S(\rho) \to S(\rho)$. Moreover, $\mathcal{B}(S(\rho))$ is a uniformly bounded set in $ca(S_{z^*}, M_{z^*})$.

Next we show that $\mathcal{B}(S(\rho))$ is a equi-continuous set of measures in $ca(S_{z^*}, M_{z^*})$. Let $E_1, E_2 \in M_{z^*}$. Then there exist subsets $F_1, F_2 \in M_0$ and $G_1, G_2 \in M_{z^*}$, $G_1 \subset \overline{x_0 z^*}$, $G_2 \subset \overline{x_0 z^*}$ such that

$$E_1 = F_1 \cup G_1 \qquad \text{with} \qquad F_1 \cap G_1 = \emptyset$$

and

$$E_2 = F_2 \cup G_2 \quad \text{with} \quad F_2 \cap G_2 = \emptyset.$$

We know the identities

$$G_1 = (G_1 - G_2) \cup (G_2 \cap G_1), \tag{4.7}$$

and

$$G_2 = (G_2 - G_1) \cup (G_1 \cap G_2). \tag{4.8}$$

Therefore, we have

$$\begin{aligned} |\lambda(E_1) &- \lambda(E_2)| \\ &= |\mathcal{B}p(E_1) - \mathcal{B}p(E_2)| \\ &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 - G_2} \left| g\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \right) \right| d\mu \\ &\quad + \int\limits_{G_2 - G_1} \left| g\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \right) \right| d\mu. \end{aligned}$$

Since $g(x, y)$ is $L^{\mu}_{\mathbb{R}}$- Carathéodory, we have that

$$\begin{aligned} |\lambda(E_1) - \lambda(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \left| g\left(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} g\big(\tau, p(\overline{S}_\tau)\big)\, d\mu \right) \right| d\mu \\ &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} \gamma(x)\, d\mu. \end{aligned}$$

Assume that

$$d(E_1, E_2) = \mu(E_1 \Delta E_2) \to 0.$$

Then we have that $E_1 \to E_2$. As a result $F_1 \to F_2$ and $\mu(G_1 \Delta G_2) \to 0$. As q is continuous on compact M_0, it is uniformly continuous and so

$$\begin{aligned} |\lambda(E_1) - \lambda(E_2)| &\leq |q(F_1) - q(F_2)| + \int\limits_{G_1 \Delta G_2} h(x)\, d\mu \\ &\to 0 \text{ as } E_1 \to E_2 \end{aligned}$$

uniformly for all $\lambda \in \mathcal{B}(S)$. This shows that $\mathcal{T}(S(\rho))$ is a equi-continuous set in $ca(S_{z^*}, M_{z^*})$. Now an application of the Arzelà-Ascoli theorem yields that $\mathcal{B}$ is a totally bounded operator on $S(\rho)$. Now, $\mathcal{B}$ is continuous and totally bounded, it is completely continuous operator on $S(\rho)$ into itself.

Step V: Finally, we show that the hypothesis (c) of Theorem 3.1 is satisfied. Let $p \in S(\rho)$ be arbitrary and let there is an element $u \in ca(S_{z^*}, M_{z^*})$ such that $\mathcal{A}u + \mathcal{B}p = u$. We show that $u \in S$. Now, by definitions of the operators $\mathcal{A}$ and $\mathcal{B}$,

$$u(E) = \begin{cases} \int\limits_E f\Big(x, p_n(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu \\ + \int\limits_E g\Big(x, p_n(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu, & \text{if } E \in M_z,\ E \subset \overline{x_0 z^*}, \\ q(E), & \text{if } E \in M_0. \end{cases}$$

for all $E \in M_z$.

If $E \in M_{z^*}$, then there exist sets $F \in M_0$ and $G \in M_{z^*}$, $G \subset \overline{x_0 z^*}$ such that $E = F \cup G$ and $F \cap G = \emptyset$. Then we have

$$\begin{aligned} u(E) = q(F) &+ \int\limits_E f\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu \\ &+ \int\limits_E g\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu. \end{aligned}$$

which further yields

$$\begin{aligned} u(E) - p_0(E) &= \int\limits_E f\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu \\ &\quad + \int\limits_E g\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, p(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu. \end{aligned}$$

Hence,

$$\begin{aligned} |u(E) - p_0(E)| \ &\leq \ \int\limits_E \Big| f\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} \big|k\big(\tau, p(\overline{S}_\tau)\big)\big|\, d\mu\Big)\Big|\, d\mu \\ &\quad + \int\limits_E \Big| g\Big(x, p(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} \big|k\big(\tau, p(\overline{S}_\tau)\big)\big|\, d\mu\Big)\Big|\, d\mu \end{aligned}$$

$$\begin{aligned} &\le M_f + \int\limits_{\overline{x_0 z^*}} \gamma(x)\, d\mu \\ &< M_f + 1 \end{aligned}$$

which further implies that

$$\|u - p_0\| \le M_f + 1 = \rho.$$

As a result, we have $u \in S$ and so hypothesis (c) of Theorem 3.1 is satisfied. In consequence, the operator equation $\mathcal{A}p(E) + \mathcal{B}p(E) = p(E)$ has a solution $p(S_{x_0}, q)$ in $ca(S_{z^*}, M_{z^*})$. This further implies that the AMIGDE (1.6)-(1.7) has a solution on $\overline{x_0 z^*}$. This completes the proof.

Remark 4.1. *We note that Theorem 4.1 is a local existence theorem for the AMIGDE (1.6)-(1.7) and is obtained in a closed ball centered at the given initial vector q on M_0. Therefore, Theorem 4.1 includes the local existence results for the abstract measure differential and abstract measure integrodifferential equations considered in Dhage [1], Dhage and Bellale [11] and Sharma [19, 20] which are also new to the literature.*

In the following we give a numerical example to illustrate the hypotheses and abstract existence result proved above for the AMIGDE (1.6)-(1.7).

Example 4.1. Given a vector measure $p \in ca(X, M)$ with $p << \mu$, consider the AMIGDE with a linear perturbation of second type of the form

$$\frac{dp}{d\mu} = \frac{|p(\overline{S}_\tau)|}{1 + |p(\overline{S}_\tau)|} + \int\limits_{\overline{S}_x - S_{x_0}} \frac{1 + |p(\overline{S}_\tau)|}{2 + p^2(\overline{S}_\tau)}\, d\mu \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}. \tag{4.9}$$

and

$$p(E)) = 0, \tag{4.10}$$

where $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ.

Choose a point $z^* \in \overline{x_0 z}$ such that $\mu(\overline{x_0 z^*}) < 1$. Here, $f(x, u, v) = \dfrac{|u|)}{1 + |u|}$ and $g(x, u, v) = v$, where $k(x, u) = \dfrac{1 + |u|}{2 + u^2}$ for all $x \in \overline{x_0 z}$ and $u, v \in \mathbb{R}$. Clearly, f is a continuous and bounded function on $S_z \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = 1$. Again, the function f satisfies the hypothesis (H_3) on $S_z \times \mathbb{R} \times \mathbb{R}$ with the $\mathcal{D}$-function $\psi_f(r) = \dfrac{r}{1 + r}$. Furthermore, g is a continuous and bounded function on $S_z \times \mathbb{R} \times \mathbb{R}$ with the growth or comparison function $\gamma(x) = 1$ for all $x \in S_z$ and so, the hypotheses (H_3) and (H_4) are satisfied. Therefore, if the assumptions (H_0)-(H_1) hold, then the AMIGDE (4.9) - (4.10) has a solution $p(\overline{S}_{x_0}, q)$ defined on $\overline{x_0 z^*}$ provided $\mu(\overline{x_0 z^*}) < 1$.

5. Existence of Extremal Solutions

In this section we prove the existence of the extremal solutions for the AMIGDE (1.6)-(1.7) on $\overline{x_0 z}$ under certain monotonicity conditions. We define an order relation $\preceq$ in $ca(S_z, M_z)$ with the help of the cone K in $ca(S_z, M_z)$ given by

$$K = \{p \in ca(S_z, M_z) \mid p(E) \geq 0 \text{ for all } E \in M_z\}. \tag{5.1}$$

Thus for any $p_1, p_2 \in ca(S_z, M_z)$, one has

$$p_1 \preceq p_2 \iff p_2 - p_1 \in K \tag{5.2}$$

or, equivalently,

$$p_1 \preceq p_2 \iff p_1(E) \leq p_2(E)$$

for all $E \in M_z$. A cone K in $ca(S_z, M_z)$ is called normal if the norm is semi-monotone on K. The details of different properties of cones in Banach spaces appear in Heikkilä and Lakshmikantham [17].

The following lemma follows immediately from the definition of the order cone K in in the Banach space $ca(S_z, M_z)$.

Lemma 5.1. *The cone K is normal in the Banach space* $ca(S_z, M_z)$.

Proof. To finish, it is enough to prove that the norm $\|\cdot\|$ is semi-monotone on K. Let $p_1, p_2 \in K$ be such that $p_1 \preceq p_2$ on M_z . Then, we have

$$0 \leq p_1(E) \leq p_2(E)$$

for all $E \in M_z$. Now, for a countable partition $\sigma = \{E_n : n \in \mathbb{N}\}$ of measurable subsets of S_z, by definition of the norm in $ca(S_z, M_z)$, one has

$$\begin{aligned} \|p_1\| = |p_1|(S_z) &= \sup_\sigma \sum_{i=1}^{\infty} |p_1(E_i)| = \sup_\sigma \sum_{i=1}^{\infty} p_1(E_i) \\ &\leq \sup_\sigma \sum_{i=1}^{\infty} p_2(E_i) = \sup_\sigma \sum_{i=1}^{\infty} |p_2(E_i)| = |p_2|(S_z) = \|p_2\|. \end{aligned}$$

This shows that $\|\cdot\|$ is semi-monotone on K and consequently the order cone K is normal in $ca(S_z, M_z)$. The proof of the lemma is complete.

We need the following fixed point theorem for monotonic operators in an ordered Banach space (see Dhage [3, 5, 7] and references therein).

Theorem 5.1 (Dhage [3, 5, 7]). *Let K be an order cone in an ordered real Banach space $\mathfrak{X}$ and let $\mathcal{A}, \mathcal{B} : \mathfrak{X} \to \mathfrak{X}$ be a nondecreasing operators such that*

(a) $\mathcal{A}$ is nonlinear $\mathcal{D}$-contraction,

(b) $\mathcal{B}$ is completely continuous, and

(c) there exist elements $u, v \in \mathfrak{X}$ such that $u \preceq v$ satisfying $u \preceq \mathcal{A}u + \mathcal{B}u$ and $\mathcal{A}v + \mathcal{B}v \preceq v$.

Furthermore, if the order cone K is normal, then the operator equation $\mathcal{A}p+\mathcal{B}p = p$ has a minimal and a maximal solution in $[u, v]$.

We need the following definitions in the sequel.

Definition 5.1. *A vector measure $u \in \mathrm{ca}(S_z, M_z)$ is called a lower solution of AMIGDE (1.6)-(1.7) if*

$$\frac{du}{d\mu} \leq f\bigg(x, u(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, u(\overline{S}_\tau)\big)\, d\mu\bigg) + g\bigg(x, u(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, u(\overline{S}_\tau)\big)\, d\mu\bigg) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z},$$

and

$$u(E) \leq q(E), \quad E \in M_0.$$

Similarly, a vector measure $v \in \mathrm{ca}(S_z, M_z)$ is called an upper solution to AMIGDE (1.6)-(1.7) if

$$\frac{dv}{d\mu} \geq f\bigg(x, v(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, v(\overline{S}_\tau)\big)\, d\mu\bigg) + g\bigg(x, v(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} k\big(\tau, v(\overline{S}_\tau)\big)\, d\mu\bigg) \text{ a.e. } [\mu] \text{ on } \overline{x_0 z},$$

and

$$v(E) \geq q(E), \quad E \in M_0.$$

A vector measure $p \in \mathrm{ca}(S_z, M_z)$ is a solution to AMIGDE (1.6)-(1.7) if it is upper as well as lower solution to the AMIGDE (1.6)-(1.7) on $\overline{x_0 z}$.

Definition 5.2. *A solution p_M is called a maximal solution for the AMIGDE (1.6)-(1.7) if for any other solution $p(\overline{S}_{x_0}, q)$ of the AMIGDE (1.6)-(1.7) we have that*

$$p(E) \leq p_M(E) \quad \forall E \in M_z.$$

Similarly, a minimal solution $p_m(\overline{S}_{x_0}, q)$ for the AMIGDE (1.6)-(1.7) is defined on $\overline{x_0 z}$.

Definition 5.3. *A function $\beta : S_z \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is called Chandrabhan if*

(i) *$x \to \beta(x, u, v)$ is μ-measurable for each $u, v \in \mathbb{R}$,*

(ii) *$(u, v) \to \beta(x, u, v)$ is jointly continuous almost everywhere $[\mu]$ on $\overline{x_0 z}$, and*

(iii) *The function $\beta(x, u, v)$ is nondecreasing in u and v for each $x \in S_z$.*

Further a Chandrabhan function $\beta(x, u, v)$ is called $L^{\mu}_{\mathbb{R}}$-Chandrabhan if

(iii) *there exists a μ-integrable function $\gamma : S_z \to \mathbb{R}$ such that*

$$|\beta(x, u, v)| \leq \gamma(x) \quad a.e. \ [\mu] \ \text{for} \ x \in \overline{x_0 z},$$

for all $u, v \in \mathbb{R}$.

We consider the following assumptions:

(H_6) The function $f(x, u, v)$ is nondecreasing in u and v for each $x \in S_z$.

(H_7) The function $h(x, u)$ is nondecreasing in u for each $x \in S_z$.

(H_8) The function g is $L^{\mu}_{\mathbb{R}}$-Chandrabhan on $S_z \times \mathbb{R} \times \mathbb{R}$.

(H_9) The function $k(x, u)$ is nondecreasing in u for each $x \in S_z$.

(H_{10}) The AMIGDE (1.6)-(1.7) has a lower solution u and an upper solution v satisfying $u \preceq v$ on M_z.

Theorem 5.2. *Suppose that the assumptions (H_0) - (H_3) and (H_6)-(H_{10}) hold. Then the AMIGDE (1.6)-(1.7) has a minimal and a maximal solution in the vector segment $[u, v]$ defined on $\overline{x_0 z}$.*

Proof. Now, AMIGDE (1.6)-(1.7) is equivalent to the abstract measure integral equation (in short AMIE) (1.8)-(1.9). Given the lower solution u and upper solution

v in hypothesis (H7), we consider the order interval $[u, v]$ in the ordered Banach space $ca(S_z, M_z)$ defined by

$$[u, v] = \{p \in ca(S_z, M_z) \mid u \preceq p \preceq v\}.$$

Define two operators $\mathcal{A}, \mathcal{B} : [u, v] \to ca(S_z, M_z)$ by the expressions (4.2) and (4.3) respectively. Then the AMIE (1.8)-(1.9) is equivalent to the operator equation

$$\mathcal{A}p(E) + \mathcal{B}p(E) = p(E), \quad E \in M_z. \tag{5.3}$$

We shall show that the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 5.1 on $ca(S_z, M_z)$. Let $p_1, p_2 \in ca(S_z, M_z)$ be such that $p_1 \preceq p_2$ on M_z. Since μ is a positive measure, by hypotheses (H_5) and (H_5), we obtain

$$\begin{aligned}
\mathcal{A}p_1(E) &= \begin{cases} \displaystyle\int_E f\Big(x, p_1(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, p_1(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu & \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*}, \\ 0 & \text{if } E \in M_0. \end{cases} \\
&\leq \begin{cases} \displaystyle\int_E f\Big(x, p_2(\overline{S}_x), \int\limits_{\overline{S}_x - S_{x_0}} h\big(\tau, p_2(\overline{S}_\tau)\big)\, d\mu\Big)\, d\mu & \text{if } E \in M_{z^*},\ E \subset \overline{x_0 z^*}, \\ 0 & \text{if } E \in M_0. \end{cases} \\
&= \mathcal{A}p_2(E)
\end{aligned}$$

for all $E \in M_z$, $E \subset \overline{x_0 z}$ and

$$\mathcal{A}p_1(E) = q(E) = \mathcal{A}p_2(E)$$

for $E \in M_0$. Hence $\mathcal{A}$ is nondecreasing on $ca(S_z, M_z)$. Similarly, it is shown that the operator $\mathcal{B}$ is also nondecreasing on $ca(S_z, M_z)$.

Since the measures u and v are respectively lower and upper solutions of the AMIGDE (1.6) and (1.7), one has $u \preceq \mathcal{A}u + \mathcal{B}u$ and $\mathcal{A}v + \mathcal{B}u \preceq v$ and consequently from nondecresing nature of $\mathcal{A}$ and $\mathcal{B}$ it follows that $u \preceq \mathcal{A}u + \mathcal{B}u \preceq \mathcal{A}v + \mathcal{B}v \preceq v$. As a result, $\mathcal{A} + \mathcal{B}$ defines a mapping $\mathcal{A} + \mathcal{B} : [u, v] \to [u, v]$. Next, as the cone K in the Banach space $ca(S_z, M_z)$ is normal $[u, v]$ is a norm-bounded subset of $ca(S_z, M_z)$. Therefore, proceeding with the argument in the proof of Theorem 4.1 it can be shown that the operator $\mathcal{A}$ is a nonlinear $\mathcal{D}$-contraction and the operator $\mathcal{B}$ is a completely continuous operator on $ca(S_z, M_z)$ into itself.

Thus, the operators $\mathcal{A}$ and $\mathcal{B}$ satisfy all the conditions of Theorem 5.1 and so the operator equation $\mathcal{A}p + \mathcal{B}p = p$ has a maximal and a minimal solution in the vector segment $[u, v]$ of $ca(S_z, M_z)$. This further implies that AMIGDE (1.6)-(1.7) has a maximal and a minimal solution in the vector segment $[u, v]$ defined on $\overline{x_0 z}$. This completes the proof.

Example 5.1. Given a vector measure $p \in ca(X, M)$ with $p << \mu$, consider the AMIGDE with a linear perturbation of second type of the form

$$\frac{dp}{d\mu} = \tan^{-1} p(\overline{S}_x) + \int\limits_{\overline{S}_x - S_{x_0}} \coth p(\overline{S}_\tau)\, d\mu \quad \text{a.e. } [\mu] \text{ on } \overline{x_0 z}. \tag{5.4}$$

and

$$p(E)) = 0, \tag{5.5}$$

where $\dfrac{dp}{d\mu}$ is a Radon-Nikodym derivative of p with respect to μ.

Choose a point $z^* \in \overline{x_0 z}$ such that $\mu(\overline{x_0 z^*}) < 1$. Here, $f(x, u, v) = \tan^{-1} u$ and $g(x, u, v) = v$, where $k(x, u) = \coth u$ for all $x \in \overline{x_0 z}$ and $u, v \in \mathbb{R}$. Clearly, f is a continuous and bounded function on $S_z \times \mathbb{R} \times \mathbb{R}$ with bound $M_f = \dfrac{\pi}{2}$. Again, the function f satisfies the hypothesis (H_3) on $S_z \times \mathbb{R} \times \mathbb{R}$ with the $\mathcal{D}$-function $\psi_f(r) = \dfrac{r}{1+r}$. Furthermore, g is a continuous and bounded function on $S_z \times \mathbb{R} \times \mathbb{R}$ with the growth or comparison function $\gamma(x) = 1$ for all $x \in S_z$ which μ-integrable on $\overline{x_0 z}$. Also the functions $f(x, u, v)$, $g(x, u, v)$ and $k(x, u)$ are nondecreasing in u and v for each $x \in S_z$. Hence, the hypotheses (H_6) through (H_9) are satisfied. Furthermore, the AMIGDE (5.4) - (5.5) has a lower solution $u(\overline{S}_x) = -1$ and an upper solution $v(\overline{S}_x) = -2$ with $u \preceq v$ on M_z, and so the hypotheses (H_{10}) is satisfied. Therefore, if the assumptions (H_0)-(H_1) hold, then the AMIGDE (5.4) - (5.5) has a solution $p(\overline{S}_{x_0}, q)$ defined on $\overline{x_0 z^*}$ provided $\mu(\overline{x_0 z^*}) < 1$.

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