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SPACE-TIME ADMITTING GENERALIZED CONFORMAL CURVATURE TENSOR

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Abstract: The object of the present paper is to study space-time admitting generalized conformal curvature tensor.

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2010 Mathematics Subject Classification: Primary 53C25, 53C50; Secondary 53C80, 53B20.

1. Introduction

The aim of the present work is to study certain investigations in general theory of relativity and cosmology by the coordinate free method of differential geometry. The basic difference between Riemannian and semi-Riemannian geometry is (i) the existence of null vector (i.e. $g(v,v)=0$, for $v\neq 0$, where g is the metric tensor) in semi-Riemannian manifold but not Riemannian manifold, (ii) the signature of metric tensor g in semi-Riemannian manifold is $(-,-,...-,+,+,...,+)$ but in a Riemannian manifold the signature of g is $(+,+,...,+)$. Lorentzian manifold is a spacial case of semi-Riemannian manifold. The signature of metric tensor g in Lorentzian manifold is $(-,+,+,...,+)$. A Lorentzian manifold consists of three types of vectors such as timelike (i.e. $g(v,v)<0$), spacelike (i.e. $g(v,v)>0$) and null vector (i.e. $g(v,v)=0$, for $v\neq 0$). In general, a Lorentzian manifold (M,g) may not have a globally timelike vector field. If (M,g) admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as space-time. The foundation of general relativity is based on a 4−dimensional space-time manifold which is the stage of present modeling of the physical world a torsionless, time-oriented Lorentzian manifold (M, g).

An $n-$dimensoinal generalized Robertson-Walker (GRW) space-time with $n \geq 3$ is a Lorentzian manifold which is a warped product of an open interval I of $\Re$ and an $(n-1)-$dimensional Riemannian manifold ([10], [11], [12]). These Lorentzian manifold broadly extends the classical Robertson-Walker (RW) space-time. RW space-time is regarded as cosmological models since it is spatially homogenous and spatially isotropic whereas GRW space-time serve as inhomogeneous extension of RW space-times that admit an isotropic radiation [18]. A Lorentzian manifold named "Perfect fluid space-time" if its Ricci tensor S has the form

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.1}$$

where α and β are scalars, A is a non-zero one-form such that $g(X, v) = A(X)$ for all v and v is the velocity vector field such that $g(v, v) = -1$. Perfect- fluid space-times in a language of differential geometry are called quasi-Einstein spaces where A is metrically equivalent to a unit space-like vector field. Einstein's field equation without cosmological constant is given by [16]

$$S(X, Y) - \frac{r}{2}g(X, Y) = kT(X, Y). \tag{1.2}$$

The paper is organized as follows: After Preliminaries in Section 3, we deduce the basic algebraic properties of generalized conformal curvature tensor. Next in Section 4, it is proven that a 4−dimensional Ricci simple generalized conformally flat space-time is a perfect fluid space-time. Moreover the space-time is RW space-time. Finally, it is shown that a 4−dimensional Ricci simple conservative generalized conformal curvature tensor with constant ψ is a GRW space-time.

2. Preliminaries

The Weyl conformal curvature tensor is the traceless part of Riemann tensor given as [13]

$$\begin{aligned} \mathcal{C}(U, V, X, Y) =& \mathcal{R}(U, V, X, Y) - \frac{1}{n-2}[S(V, X)g(U, Y) \\ & - S(U, X)g(V, Y) + S(U, Y)g(V, X) - S(V, Y)g(U, X)] \\ & + \frac{r}{(n-1)(n-2)}[g(V, X)g(U, Y) - g(U, X)g(V, Y)], \end{aligned} \tag{2.1}$$

where $\mathcal{R}$ Riemann curvature tensor and r denotes the scalar curvature. The number of algebraically independent components of the Ricci and the Weyl tensors equals that of the Riemann tensor. Since in general relativity only the Ricci tensor is coupled to matter by the Einstein's fields equations, the conformal curvature tensor describes the pure gravity degrees of freedom. Divergence of conformal curvature tensor is given by

$$\begin{aligned}(div\mathcal{C})(U,V)X =& \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X) \\ &- \frac{1}{2(n-1)}\{g(V,X)dr(U) - g(U,X)dr(V)\}].\end{aligned} \tag{2.2}$$

A symmetric $(0,2)$ type tensor field E on a semi-Riemannian manifold (M^n, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_U E)(V,X) = (\nabla_V E)(U,X), \tag{2.3}$$

for arbitrary vector fields U, V and X. The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen [6].

In 2012, Mantica and Suh [9] introduced a new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ and studied various geometric properties of it an Riemannian manifold. A new tensor $\mathcal{Z}$ is defined as:

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{2.4}$$

where ψ is an arbitrary scalar function and name is generalized $\mathcal{Z}-$tensor.

Definition 2.1. *A Riemannian manifold (M^n, g) of dimension n $(n > 3)$ is said to be Ricci simple tensor [5] if its Ricci tensor $S(X,Y)$ satisfies the condition*

$$S(X,Y) = -rA(X)A(Y), \tag{2.5}$$

where r and A is scalar curvature and unit time-like vector field respectively. This condition has a geometric meaning that a unit time like vector A becomes a principle vector of Ricci operator.

3. Generalized Conformal Curvature Tensor

In view of equation (2.4), equation (2.1) takes the form

$$\begin{aligned}\mathcal{C}(U,V,X,Y) =& \mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[\mathcal{Z}(V,X)g(U,Y) \\ &- \mathcal{Z}(U,X)g(V,Y) + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)] \\ &+ \frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ &+ \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{3.1}$$

Define

$$\begin{aligned}\mathcal{C}^*(U,V,X,Y) = \mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[&\mathcal{Z}(V,X)g(U,Y)\\ &- \mathcal{Z}(U,X)g(V,Y) + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)]\\ &+ \frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{3.2}$$

Thus from above equation, equation (3.1) reduces to

$$\mathcal{C}(U,V,X,Y) = \mathcal{C}^*(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],$$

which gives

$$\mathcal{C}^*(U,V,X,Y) = \mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \tag{3.3}$$

where $\mathcal{C}^*(U,V,X,Y)$ is called generalized conformal curvature tensor.

If $\psi = 0$, then from equation (3.3), we obtain

$$\mathcal{C}^*(U,V,X,Y) = \mathcal{C}(U,V,X,Y). \tag{3.4}$$

Thus we can state as follows-

Theorem 3.1. *A generalized conformal curvature tensor reduces to conformal curvature tensor provided that the scalar function ψ vanishes.*

Now, interchanging the places of U and V in equation (3.3), we obtain

$$\mathcal{C}^*(V,U,X,Y) = \mathcal{C}(V,U,X,Y) - \frac{2\psi}{(n-2)}[g(U,X)g(V,Y) - g(V,X)g(U,Y)]$$

i.e.

$$\mathcal{C}^*(V,U,X,Y) = -\mathcal{C}(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] = -\mathcal{C}^*(V,U,X,Y)$$

i.e.

$$\mathcal{C}^*(U,V,X,Y) = -\mathcal{C}^*(V,U,X,Y),$$

which gives

$$\mathcal{C}^*(U,V,X,Y)+\mathcal{C}^*(V,U,X,Y)=0,$$

which shows that generalized conformal curvature tensor is skew-symmetric with respect to first two slots.

Interchanging the places of X and Y in equation (3.3), we obtain

$$\mathcal{C}^*(U,V,Y,X)=\mathcal{C}(U,V,Y,X)-\frac{2\psi}{(n-2)}[g(V,Y)g(U,X)-g(U,Y)g(V,X)]$$

i.e.

$$\mathcal{C}^*(U,V,Y,X)=-\mathcal{C}(U,V,X,Y)+\frac{2\psi}{(n-2)}[g(V,X)g(U,Y)-g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{C}(U,V,X,Y)-\frac{2\psi}{(n-2)}[g(V,X)g(U,Y)-g(U,X)g(V,Y)]=-\mathcal{C}^*(U,V,Y,X)$$

i.e.

$$\mathcal{C}^*(U,V,X,Y)=-\mathcal{C}^*(U,V,Y,X),$$

which gives

$$\mathcal{C}^*(U,V,X,Y)+\mathcal{C}^*(U,V,Y,X)=0,$$

which shows that generalized conformal curvature tensor is skew-symmetric with respect to last two slots.

Again interchanging pair of slots in equation (3.3), we obtain

$$\mathcal{C}^*(X,Y,U,V)=\mathcal{C}(X,Y,U,V)-\frac{2\psi}{(n-2)}[g(Y,U)g(X,V)-g(X,U)g(Y,V)]$$

i.e.

$$\mathcal{C}^*(X,Y,U,V)=\mathcal{C}(U,V,X,Y)-\frac{2\psi}{(n-2)}[g(V,X)g(U,Y)-g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{C}^*(U,V,X,Y)=\mathcal{C}^*(X,Y,U,V),$$

which gives

$$\mathcal{C}^*(U,V,X,Y)-\mathcal{C}^*(X,Y,U,V)=0,$$

which shows that generalized conformal curvature tensor is symmetric on pair slots. Thus we can state as follows-

Theorem 3.2. *A generalized conformal curvature tensor on (M^n, g) is*
(1) *skew-symmetric in first two slots,*
(2) *skew-symmetric in lost two slots,*
(3) *symmetric in pair of slots.*
Now, writing two more equations by the cyclic permutations of U, V and X of equation (3.3), we obtain

$$\mathcal{C}^*(V,X,U,Y) = \mathcal{C}(V,X,U,Y) - \frac{2\psi}{(n-2)}[g(X,U)g(V,Y) - g(V,U)g(X,Y)], \quad (3.5)$$

and

$$\mathcal{C}^*(X,U,V,Y) = \mathcal{C}(X,U,V,Y) - \frac{2\psi}{(n-2)}[g(U,V)g(X,Y) - g(X,V)g(U,Y)], \quad (3.6)$$

Adding equations (3.3), (3.5) and (3.6), we obtain

$$\mathcal{C}^*(U,V,X,Y) + \mathcal{C}^*(V,X,U,Y) + \mathcal{C}^*(X,U,V,Y) = 0, \quad (3.7)$$

which shows that generalized conformal curvature tensor satisfied Bianchi's first identity. Thus we can state as follows-

Theorem 3.3. *A generalized conformal curvature tensor on (M^n, g) satisfies Bianchi's first identity.*
Now, taking the covariant derivative of equation (3.2), with respect to U, we obtain

$$\begin{aligned}(\nabla_U\mathcal{C}^*)(V,X,Y,W) &= (\nabla_U\mathcal{R})(V,X,Y,W) - \frac{1}{n-2}[g(V,W)(\nabla_U\mathcal{Z})(X,Y)\\ &- g(X,W)(\nabla_U\mathcal{Z})(V,Y) + g(X,Y)(\nabla_U\mathcal{Z})(V,W) - g(V,Y)(\nabla_U\mathcal{Z})(X,W)]\\ &+ \frac{dr(U)}{(n-1)(n-2)}[g(X,Y)g(V,W) - g(V,Y)g(X,W)].\end{aligned} \quad (3.8)$$

Writing two more equations by the cyclic permutations of U, V and X from equation (3.8), we obtain

$$\begin{aligned}(\nabla_V\mathcal{C}^*)(X,U,Y,W) &= (\nabla_V\mathcal{R})(X,U,Y,W) - \frac{1}{n-2}[g(X,W)(\nabla_V\mathcal{Z})(U,Y)\\ &- g(U,W)(\nabla_V\mathcal{Z})(X,Y) + g(U,Y)(\nabla_V\mathcal{Z})(X,W) - g(X,Y)(\nabla_V\mathcal{Z})(U,W)]\\ &+ \frac{dr(V)}{(n-1)(n-2)}[g(U,Y)g(X,W) - g(X,Y)g(U,W)],\end{aligned} \quad (3.9)$$

and

$$(\nabla_X \mathcal{C}^*)(U,V,Y,W) = (\nabla_X \mathcal{R})(U,V,Y,W) - \frac{1}{n-2}[g(U,W)(\nabla_X \mathcal{Z})(V,Y) - g(V,W)(\nabla_X \mathcal{Z})(U,Y) + g(V,Y)(\nabla_X \mathcal{Z})(U,W) - g(U,Y)(\nabla_X \mathcal{Z})(V,W)] + \frac{dr(X)}{(n-1)(n-2)}[g(V,Y)g(U,W) - g(U,Y)g(V,W)]. \tag{3.10}$$

Adding equations (3.8), (3.9) and (3.10) with the fact that $(\nabla_U \mathcal{R})(V,X,Y,W) + (\nabla_V \mathcal{R})(X,U,Y,W) + (\nabla_X \mathcal{R})(U,V,Y,W) = 0$, we get

$$\begin{aligned}(\nabla_U \mathcal{C}^*)(V,X,Y,W) &+ (\nabla_V \mathcal{C}^*)(X,U,Y,W) + (\nabla_X \mathcal{C}^*)(U,V,Y,W) \\ &= -\frac{1}{n-2}[g(V,W)\{(\nabla_U \mathcal{Z})(X,Y) - (\nabla_X \mathcal{Z})(U,Y)\} \\ &- g(X,W)\{(\nabla_U \mathcal{Z})(V,Y) - (\nabla_V \mathcal{Z})(U,Y)\} \\ &+ g(X,Y)\{(\nabla_U \mathcal{Z})(V,W) - (\nabla_V \mathcal{Z})(U,W)\} \\ &- g(V,Y)\{(\nabla_U \mathcal{Z})(X,W) - (\nabla_X \mathcal{Z})(U,W)\} \\ &- g(U,W)\{(\nabla_V \mathcal{Z})(X,Y) - (\nabla_X \mathcal{Z})(V,Y)\} \\ &+ g(U,Y)\{(\nabla_V \mathcal{Z})(X,W) - (\nabla_X \mathcal{Z})(V,W)\}] \\ &+ \frac{dr(U)}{(n-1)(n-2)}[g(X,Y)g(V,W) - g(V,Y)g(X,W)] \\ &+ \frac{dr(V)}{(n-1)(n-2)}[g(U,Y)g(X,W) - g(X,Y)g(U,W)] \\ &+ \frac{dr(X)}{(n-1)(n-2)}[g(V,Y)g(U,W) - g(U,Y)g(V,W)].\end{aligned} \tag{3.11}$$

Assuming that $\mathcal{Z}-$tensor is Codazzi tensor, then in view of equation (2.4), Ricci tensor $S(X,Y)$ is also Codazzi tensor, i.e. $(\nabla_U S)(X,Y) - (\nabla_X S)(U,Y) = 0$, which gives that the scalar curvature tensor r is constant. Thus above equation (3.11), reduces to

$$(\nabla_U \mathcal{C}^*)(V,X,Y,W) + (\nabla_V \mathcal{C}^*)(X,U,Y,W) + (\nabla_X \mathcal{C}^*)(U,V,Y,W) = 0. \tag{3.12}$$

Thus we can state as follows-

Theorem 3.4. *A generalized conformal curvature tensor on (M^n, g) satisfies Bianchi's second identity, if the $\mathcal{Z}-$tensor is Codazzi tensor.*

4. Generalized Conformally Flat Space-time

We consider a 4−dimensinal generalized conformally flat manifold (M, g) with Lorentzian metric g. From equation (3.3), we have

$$\mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] = 0.$$

i.e.

$$\mathcal{C}(U,V,X,Y) = \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]. \tag{4.1}$$

Using equation (2.1) in equation (4.1), we have

$$\begin{aligned}\mathcal{R}(U,V,X,Y) &- \frac{1}{n-2}[S(V,X)g(U,Y) - S(U,X)g(V,Y) + S(U,Y)g(V,X) \\ &- S(V,Y)g(U,X)] + \frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ &= \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{4.2}$$

For $n = 4$, above equation (4.2) takes the form

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& \frac{1}{2}[S(V,X)g(U,Y) - S(U,X)g(V,Y) \\ &+ S(U,Y)g(V,X) - S(V,Y)g(U,X)] \\ &+ \frac{6\psi - r}{6}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{4.3}$$

Assuming space-time is Ricci simple, then in view of equation (2.5) above equation reduces to

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& \frac{1}{2}[-rA(V)A(X)g(U,Y) + rA(U)A(X)g(V,Y) \\ &- rA(U)A(Y)g(V,X) + rA(V)A(Y)g(U,X)] \\ &+ \frac{6\psi - r}{6}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],\end{aligned} \tag{4.4}$$

which gives

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& \frac{6\psi - r}{6}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ &- \frac{r}{2}[g(U,Y)A(V)A(X) + g(V,X)A(U)A(Y) \\ &- g(V,Y)A(U)A(X) - g(U,X)A(V)A(Y)],\end{aligned} \tag{4.5}$$

which is of the form of quasi-constant curvature tensor.
Contracting equation (4.5), we obtain

$$\begin{aligned} S(V,X) = &\frac{6\psi - r}{6}[4g(V,X) - g(V,X)] \\ &- \frac{r}{2}[4A(V)A(X) - g(V,X) - A(V)A(X) - A(V)A(X)], \end{aligned} \tag{4.6}$$

i.e.

$$S(V,X) = \frac{6\psi - r}{2}g(V,X) + \frac{r}{2}g(V,X) - rA(V)A(X),$$

which in view of equation (2.5) becomes

$$S(V,X) = 3\psi g(V,X) + S(V,X),$$

which yields to

$$\psi = 0. \tag{4.7}$$

In view of equation (4.7), equation (4.1) gives

$$\mathcal{C}(U,V,X,Y) = 0.$$

This shows that generalized conformally Ricci simple flat space-time is conformally flat. Thus we can state as follows-

Theorem 4.1. *A* $4-$*dimensional Ricci simple generalized conformally flat space-time* M *is conformally flat perfect fluid space-time.*

In [11], the authors have shown that an $n-$dimensional ($n \geq 4$) Ricci simple conformally flat perfect fluid space-time is RW space-time. Thus we can state as follows-

Corollary 4.2. *A* $4-$*dimensional Ricci simple generalized conformally flat space-time* M *is RW space-time.*

5. Conservative Generalized Conformal Space-time

From equation (3.3), generalized conformal curvature tensor is given by

$$\mathcal{C}^*(U,V)X = \mathcal{C}(U,V)X - \frac{2\psi}{(n-2)}[g(V,X)U - g(U,X)V]. \tag{5.1}$$

The divergence of $C^*(U,V)X$ is defined as

$$(div\mathcal{C}^*)(U,V)X = g((\nabla_{e_i}\mathcal{C}^*)(U,V)X, e_i)$$

i.e.

$$(div\mathcal{C}^*)(U,V)X = g((\nabla_{e_i}\mathcal{C})(U,V)X, e_i) - \frac{2}{n-2}[g((\nabla_{e_i}\psi)\{g(V,X)U - g(U,X)V\}, e_i)],$$

which gives

$$(div\mathcal{C}^*)(U,V)X = (div\mathcal{C})(U,V)X - \frac{2}{(n-2)}[(U\psi)g(V,X) - (V\psi)g(U,X)]. \quad (5.2)$$

From equations (2.2) and (5.2), we obtain

$$\begin{aligned}(div\mathcal{C}^*)(U,V)X =& \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X) \\ &- \frac{1}{2(n-1)}\{g(V,X)dr(U) - g(U,X)dr(V)\}] \\ &- \frac{2}{(n-2)}[(U\psi)g(V,X) - (V\psi)g(U,X)].\end{aligned} \quad (5.3)$$

If scalar function ψ is constant then from equation (5.2), we obtain

$$(div\mathcal{C}^*)(U,V)X = (div\mathcal{C})(U,V)X. \quad (5.4)$$

If $(div\mathcal{C}^*)(U,V)X = 0$ then from equation (5.4), we obtain

$$(div\mathcal{C})(U,V)X = 0.$$

Thus we can state as follows-

Theorem 5.1. *A 4−dimensional relativistic conservative generalized conformal curvature space-time M admitting constant scalar function ψ is conservative conformal curvature tensor.*

In [11], the authors have shown that an $n-$dimensional ($n \geq 4$) Ricci simple conservative conformal curvature space-time is a GRW space-time with Einstein fibers. Thus we can state as follows-

Corollary 5.2. *A 4−dimensional Ricci simple conservative generalized conformal curvature space-time M admitting constant scalar function ψ a GRW space-time with Einstein fibers.*

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