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RESTRICTED MINUS DOMINATION NUMBER OF A GRAPH

B. Chaluvaraju and V. Chaitra*

Department of Mathematics,
Jnana Bharathi Campus,
Bangalore University, Bangalore - 560056 INDIA

E-mail : bchaluvaraju@gmail.com

*Department of Mathematics,
B.M.S. College of Engineering,
Bull Temple Road, Bangalore - 560001, INDIA

E-mail : chaitrav.maths@bmsce.ac.in

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Abstract: A restricted minus dominating function on a graph $G = (V, E)$ is a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 0$ for every vertex $v \in V$ and a vertex assigned 0 is adjacent to at least one vertex assigned 1. The restricted minus domination number $\gamma_r^-(G) = min\{w(f) : f$ is restricted minus dominating function$\}$. In this paper, we initiate the study of $\gamma_r^-(G)$ and its relationship with sign and minus domination are investigated. Many of the known bounds of $\gamma_r^-(G)$ are immediate consequence of our results.

Keywords and Phrases: Graph, domination number, minus domination number, restricted minus domination.

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1. Introduction

All graphs considered in this paper are finite, simple, and undirected. For a general reference on graph theory, the reader is directed to [8]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G, respectively. For any vertex v

of G, let $N(v)$ and $N[v]$ denote its open and closed neighborhoods respectively. $\alpha_0(G)(\alpha_1(G))$, is the minimum number of vertices (edges) in a vertex (edge) cover of G. $\beta_0(G)(\beta_1(G))$, is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G. Let $deg(v)$ be the degree of a vertex v in G, $\triangle(G)$ and $\delta(G)$ be maximum and minimum degree of vertices of G, respectively. The complement G^c of a graph G is the graph having the same set of vertices as G denoted by V^c and in which two vertices are adjacent, if and only if they are not adjacent in G. A tree T is an acyclic connected graph.

A dominating set $D \subseteq V$ for a graph G is such that each $v \in V$ is either in D or adjacent to a vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. For complete review on domination and its related parameters, refer [1], [9] and [10].

For any real valued function $f : V \to R$ the weight of f is denoted and defined as $w(f) = \sum_{v \in V} f(v)$.

A sign dominating function (SDF) of a graph G is a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for all $v \in V$. The sign domination number of a graph G is $\gamma_s(G) = min\{w(f) : f$ is sign dominating function$\}$. For more details on sign domination, we refer [3] and [14].

A minus dominating function (MDF) of a graph G is a function $g : V \to \{-1, 0, 1\}$ such that $g(N[v]) \geq 1$ for all $v \in V$. The minus domination number of a graph G is $\gamma^-(G) = min\{w(g) : g$ is minus dominating function$\}$. For more details on minus domination, we refer [2], [5], [7], [11], [12] and [13].

A restricted minus dominating function (RMDF) on a graph G is a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 0$ for every vertex $v \in V$ and a vertex assigned 0 is adjacent to at least one vertex assigned 1. The restricted minus domination number $\gamma_r^-(G) = min\{w(f) : f$ is restricted minus dominating function$\}$. Let $|V_{-1}|$, $|V_0|$ and $|V_1|$ denote number of vertices assigned -1, 0 and 1 respectively.

2. Existing Result

Theorem 2.1. [4] *For any tree T, $\gamma^-(T) \geq 1$ with equality if and only if $T \cong K_{1,n-1}$.*

Theorem 2.2. [6] *Let G be a graph with n vertices. If $\gamma_s(G) = 0$, then $n \geq 6$.*

Theorem 2.3. [6] *For any graph G, $\gamma_s(G) = n$ if any only if every non isolated vertex is either an endvertex or adjacent to an endvertex.*

3. Results

We start with the couple of observations, which we use in sequel.

Observation 3.1. *A vertex which is assigned -1 is always adjacent to at least one* *vertex assigned 1.*

Proof. Since weight of every vertex of a graph G should not be negative it implies every vertex $v \in V$ which is assigned -1 should be adjacent to at least one vertex assigned 1 such that $f(N[v]) \geq 0$.

Observation 3.2. *By the definitions of* $\alpha_o(G), \alpha_1(G), \beta_o(G)$ *and* $\beta_1(G)$*, Clearly,* $\gamma_r^-(G) < min\{\alpha_o(G), \alpha_1(G), \beta_o(G), \beta_1(G)\}$.

Theorem 3.1. *For any path* P_n *with* $n \geq 1$ *and Cycle* C_n *with* $n \geq 3$ *vertices,*

$$\gamma_r^-(P_n) = \gamma_r^-(C_n) = \begin{cases} 1 & \textit{if } \; n{=}3k{+}1, \\ 0 & \textit{otherwise}, \end{cases}$$

where k *is a positive integer.*

Proof. The result can be easily checked for $n = 1$ and 2. We shall prove the result for $n \geq 3$ vertices. For any positive integer k, if there are $3k$-vertices, then -1, 1, 0 is assigned k-times. Hence $\gamma_r^-(G) = 0$. If there are $(3k+1)$-vertices, then as usual $3k$-vertices are assigned -1, 1, 0 in order. Since the last vertex among $3k$-vertices is assigned 0 and $(3k+1)^{th}$ vertex say v can neither be assigned 0 as it will not be adjacent to 1 nor -1 as $f(N[v]) = -1$. Hence it should be assigned 1. For such assignment $\gamma_r^-(G) = 1$. If there are $(3k+2)$-vertices, then -1, 1, 0 are assigned to $3k$-vertices in order. $(3k+1)^{th}$ vertex is assigned 1 and $(3k+2)^{nd}$ vertex can be assigned either 0 or -1. Since the restricted minus domination number of G is minimum of such assignments, we assign -1 to the last vertex. Hence $\gamma_r^-(G) = |V_1| - |V_{-1}| = 0$.

Theorem 3.2. *For any complete bipartite graph* $K_{p,q}$ *with bipartitions* $|P_1| = p$ *and* $|P_2| = q$*,*

$$\gamma_r^-(K_{p,q}) = 1.$$

Proof. Let $f : V \to \{-1, 0, 1\}$ be a restricted minus dominating function.

Case 1. If the number of vertices assigned 1 is equal to number of vertices assigned -1, then for any vertex $v \in V_{-1}$, $f(N[v]) < 0$.

Case 2. If the number of vertices assigned 1 is less than the number of vertices assigned -1 then there is at least one vertex $v \in V$ such that $f(N[v]) < 0$.

From the above two cases $\gamma_r^-(K_{p,q}) > 0$ and $|V_1| > |V_{-1}|$.

If $|V_1| = |V_{-1}| + 1$ then $f(N[v]) > 0$ for all $v \in V$. Hence $\gamma_r^-(K_{p,q}) = 1$. Thus $\gamma_r^-(G) = 1$.

To prove our next result, we make use of the following definition:

A graph G is outerplanar if it has a crossing-free embedding in the plane such that all vertices are on the same face.

Theorem 3.3. *For any positive integer k, there exist an outerplanar graph G with $\gamma_r^-(G_k) \leq -k$.*
Proof. Consider the outerplanar graph G_k which can be constructed as in Figure 1. Then there are $(3k+3)$-vertices out of which $(2k+2)$ vertices are of degree 1. By assigning -1 to $2k$ vertices of degree 1, 1 to k vertices of degree 5 and 0 to remaining vertices produces RMDF f of G_k of weight $k - 2k = -k$ as illustrated. This implies that the restricted minus domination number $\gamma_r^-(G) \leq -k$.

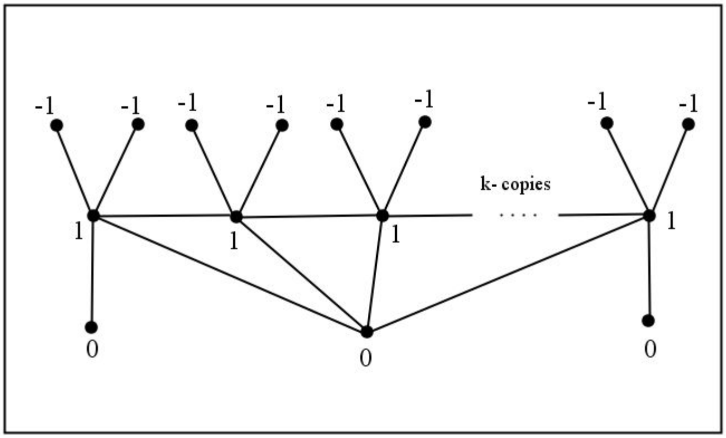

Figure-1: An outerplanar graph G_k with $\gamma_r^-(G_k) \leq -k$

Theorem 3.4. *For any connected graph G, $\gamma_r^-(G) = 0$ if and only if $|V_1| = |V_{-1}|$.*
Proof. As vertices assigned 0 is adjacent to at least one vertex assigned 1, implies that V_1 dominates vertices of V_0. Hence $\gamma_r^-(G) = |V_1| - |V_{-1}|$. Suppose $|V_1| = |V_{-1}|$. This implies that $\gamma_r^-(G) = 0$. On the other hand, if $\gamma_r^-(G) = 0$, then $|V_1| - |V_{-1}| = 0$.

Theorem 3.5. *Let G be a nontrivial graph with $\triangle(G) = n - 1$. Then*

(i) $\gamma_r^-(G) = 0$.

(ii) $\gamma_r^-(G) \leq \gamma_r^-(G^c)$.

Proof. Let G be a nontrivial graph with n-vertices.
(i) Let v be a vertex of degree $n - 1$. If we assign 1 to vertex v, assign -1 to a vertex adjacent to v and remaining $(n-2)$-vertices are assigned 0, then such an assignment satisfies both the conditions RMDF. Hence $\gamma_r^-(G) = 0$. (ii) If G is a graph with $\triangle(G) = n - 1$, then by (i), $\gamma_r^-(G) = 0$. Also, the graph G^c is a disconnected graph, this implies that $\gamma_r^-(G) \leq \gamma_r^-(G^c)$.

Theorem 3.6. *For any connected graph G,*

$$\gamma_r^-(G) \leq \gamma(G).$$

Proof. Let $f : V \to \{0, 1\}$ be a dominating function and $g : V \to \{-1, 0, 1\}$ be RMDF on a graph G. Then $f(N[v]) \geq 1$ and $g(N[v]) \geq 0$ for every $v \in V$. As $\gamma(G) \geq 1$ and due to the fact of the Theorem 3.3, the result follows.

Theorem 3.7. *For any nontrivial graph G, $\gamma_r^-(G) \leq n - \Delta(G)$. Further, the bound is attained if the graph G is totally disconnected.*

Proof. Let G be a graph with n-vertices. Then, we consider the following cases:

Case 1. If $\Delta(G) = 0$, then $G \cong K_n^c$ and $n - \Delta(G) = n$. We have $\gamma_r^-(G) = n$.

Case 2. If $\Delta(G) = 1$, then $G \cong K_2$ and $n - \Delta(G) = 1$. Here, one vertex is assigned 1 and other vertex is assigned -1. Then $\gamma_r^-(G) = 0$.

Case 3. If $\Delta(G) = n - 1$, then $n - \Delta(G) = 1$. Then by Theorem 3.5, $\gamma_r^-(G) = 0$.

Case 4. If $\Delta(G) = k$ other than above considerations, then $n - \Delta(G) = n - k > 1$. We have $\gamma_r^-(G) < n - k$.

Hence, from all the above cases the result is proven.

Theorem 3.8. *For any tree T,*

$$\gamma_r^-(T) \leq \gamma^-(T).$$

Proof. Let T be a tree. Then by Theorem 2.1, $\gamma^-(T) \geq 1$ and by Theorem 3.3, we have $\gamma_r^-(T) \leq \gamma^-(T)$.

There is no good relation between $\gamma_r^-(G)$ and $\gamma^-(G)$ except for tree. For illustration, consider the graphs G_1, G_2 and G_3.

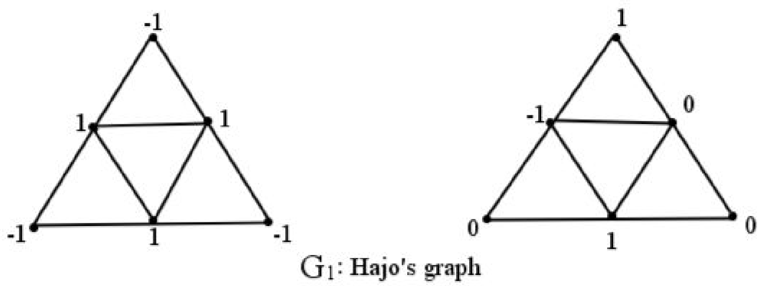

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Figure-2 Graphs with $\gamma^-(G_1)$ and $\gamma_r^-(G_1)$.

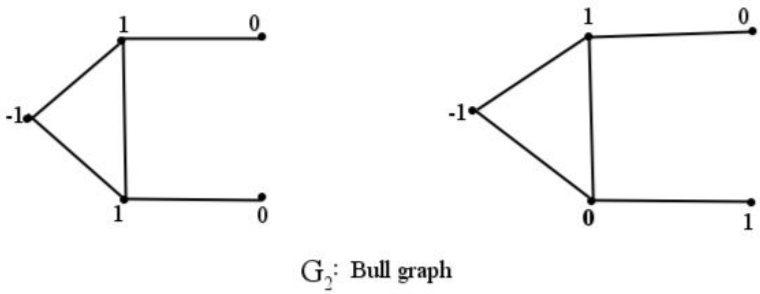

Figure-3 Graphs with $\gamma^-(G_2)$ and $\gamma_r^-(G_2)$.

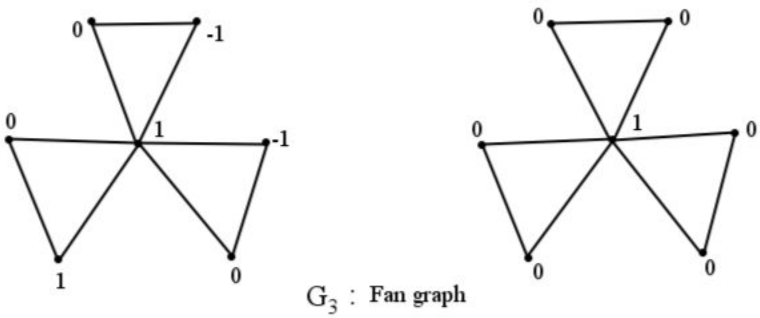

Figure-4 Graphs with $\gamma_r^-(G_3)$ and $\gamma^-(G_3)$.

In Figure 2, we have $\gamma^-(G_1) < \gamma_r^-(G_1)$.
In Figure 3, we have $\gamma_r^-(G_2) = \gamma^-(G_2)$.
In Figure.4, we have $\gamma^-(G_3) > \gamma_r^-(G_3)$.

Theorem 3.9. *Let G be a graph with $n \geq 1$ vertices. Then $\gamma_s(G) = \gamma_r^-(G) = n$ if and only if $G \cong K_n^c$.*
Proof. Let $G \cong K_n^c$. Then, under RMDF and SDF every vertex of G is assigned 1. Hence $\gamma_s(G) = \gamma_r^-(G) = n$. On the other hand, let $\gamma_s(G) = \gamma_r^-(G) = n$. By Theorem 2.3, $\gamma_s(G) = n$ if and only if every vertex of G is either endvertex or support vertex. For any graph G other than K_n^c, where every vertex of G is either endvertex or support vertex, we get a contradiction. Hence the result.

Theorem 3.10. *Let G be a graph with $n \geq 6$ vertices. If $| V_1 |=| V_{-1} |$, then*

$$\gamma_r^-(G) = \gamma_s(G).$$

Proof. By Theorem 3.4 and Theorem 2.2, the desired result follows.

Open Problem: For which class of graphs G is

1. $\gamma_r^-(G) = \gamma(G)$.
2. $\gamma_r^-(G) = \gamma^-(G)$.
3. $\gamma_r^-(G) = \gamma_s(G)$.

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