South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 2 (2020), pp. 271-280

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ON FOUR TUPLE OF DISTINCT INTEGERS SUCH THAT THE SUM OF ANY TWO OF THEM IS CUBE OF A POSITIVE INTEGER

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(**Received:** *Mar. 11, 2020* **Accepted:** *Jul. 29, 2020* **Published:** *Aug. 30, 2020)*

Abstract: In this article we have discussed determination of distinct positive integers a, b, c, d such that $a+b, a+c, b+c, a+d, b+d, c+d$ are cubes of positive integers with
(i) at least three numbers, say a, b, c are positive.
(ii) all four numbers a, b, c, d are positive.
We can obtain infinitely many four tuples from a single four-tuple.

Keywords and Phrases: Perfect squares, cubes, cubefree numbers, taxicab numbers, cubefree taxicab numbers, primes.

2010 Mathematics Subject Classification: 11A67.

1. Introduction

Number theory holds a distinguished position in mathematics for its many results which are profound and yet easy to state. Many of the problems in Number Theory arise from the role of addition and multiplication. One important class of such problems in which numbers can be expressed as sum of some numbers defined multiplicatively. This gives rise to the Pythagorean numbers, triangular numbers,

taxicab numbers, the four-square theorem of Lagrange, Goldbach conjecture and many such numbers and theorems, conjectures.
A remarkable aspect of Number Theory is that there is something in it for every one from puzzles as entertainment for layman to many open problems for scholars and mathematicians. For such problems, one may refer [3] or any standard book on Number Theory.
Perfect cubic numbers are $1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64, \cdots$ and for any integer k, with $m^3 < k < (m+1)^3$; is not a perfect cube number, for any integer m.
$k-$tuple of positive integers $(k \geq 3)$, for any $m \in \mathbb{N}, (4m^3, 4m^3, \cdots, 4m^3)$ is such that the sum of its any two coordinates is $(2m)^3$, cube of the positive integer $2m$. Taking $m = 1, 2, 3, \cdots$, we get such infinitely many $k-$tuples. For any $p, q \in \mathbb{N}$ with $p^3 > 4q^3$, the $k-$tuple of positive integers $(p^3 - 4q^3, 4q^3, 4q^3, \cdots, 4q^3)$ is such that the sum of its any two coordinates is cube of a positive integer, that is p^3 or $(2q)^3$. Such $k-$tuples are infinitely many and first coordinate is different from the other coordinates. Above are trivial examples of $k-$tuples in which sum of any two coordinates is cube of a positive integer.

1.1. Four tuples of distinct positive integers such that sum of any two of its coordinates is a perfect square [1], [2], [5]

Examples of (a, b, c, d) as a four tuple of distinct positive integers such that $a+b, a+c, b+c, a+d, b+d, c+d$ are perfect squares are:
$(18, 882, 2482, 4743)$, $(4190, 10290, 39074, 83426)$, $(7070, 29794, 71330, 172706)$, $(55967,\ 78722,\ 27554, 10082)$, $(15710, 86690, 157346, 27554)$.
Following are four-tuples of integers where one coordinate is negative and sum of any two coordinates is a perfect square:
$(-286, 386, 770, 1730), (-126, 130, 270, 1026)$.

1.2. For determination of distinct $a, b, c, d \in \mathbb{Z}$, such that $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, $c+d$ are cube of positive integers

We consider $a, b, c, d, p, q, r, s, t, u \in \mathbb{Z}$ with $a < b < c, p < q < r$ and $a + b = p^3, a + c = q^3, b + c = r^3, a + d = s^3, b + d = t^3, c + d = u^3$.

Above equations in matrix form is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p^3 \\ q^3 \\ r^3 \\ s^3 \\ t^3 \\ u^3 \end{bmatrix}$

Premultiplying by $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix}$,

and noting $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

we get,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p^3 \\ q^3 \\ r^3 \\ s^3 \\ t^3 \\ u^3 \end{bmatrix}$$

which gives,

$$\begin{aligned} a &= \frac{1}{2}(p^3 + q^3 - r^3) \\ b &= \frac{1}{2}(p^3 - q^3 + r^3) \\ c &= \frac{1}{2}(-p^3 + q^3 + r^3) \\ d &= \frac{1}{2}(-p^3 - q^3 + r^3 + 2s^3) \end{aligned} \tag{1}$$

It is easy to prove that, a, b, c are relatively prime if and only if p, q, r are relatively prime

[$\gcd(a, b, c) = v > 1 \Rightarrow \gcd(a + b, a + c, b + c) \geq v \Rightarrow \gcd(p, q, r) > 1$ etc].

Note that $p \in \mathbb{N} \Rightarrow p, q, r \in \mathbb{N}$.

By (1), for $\gcd(a, b, c) = 1$; exactly one from p, q, r is even and remaining two are odd numbers.

Again by (1); a, b, c, d are integers if and only if all integers p, q, r are even or exactly one of them is even.

In above [in equations (1) etc] the role of t^3, u^3 is invisible.

Here clearly $a + b = p^3, a + c = q^3, b + c = r^3$ and $a + d = s^3$.

Now $t^3 = b + d = -q^3 + r^3 + s^3, u^3 = c + d = -p^3 + r^3 + s^3$
$\Rightarrow s^3 + r^3 = t^3 + q^3 = u^3 + p^3$
This gives a taxicab number , expressed as a sum of two positive algebraic cubes in three distinct ways.
By Fermat's last theorem, $a + b + c + d$ never be a cube of any positive integer.

2. Taxicab Number [4], [6], [7]

There is a famous story in the mathematical folklore concerning the brilliant Indian mathematician Ramanujan. When Ramanujan was at Cambridge working with Hardy (1913), he felled ill and had to be admitted to a hospital at Putney. Hardy came to visit him, and remarked that he came in taxicab numbered 1729, which he found to be a dull number. Ramanujan noticed that this was actually a quite interesting number. It is the smallest number which can be expressed as the sum of two positive cubes in two different ways. As a result, the taxicab numbers are defined as those m for which there are solutions in positive integers to the equation, $m = x^3 + y^3 = u^3 + v^3$ for which $\{x, y\} \neq \{u, v\}$.
The nth taxicab number is a positive integer that can be expressed as a sum of two cubes of positive integers in n different ways. The smallest nth taxicab number is denoted by $T_a(n)$.
The concept of second taxicab number was first mentioned in 1657 by Bernard Frenicle de Bersy and was made famous in the early 20th century by a story involving Srinivasa Ramanujan and G. H. Hardy. In 1938, G. H. Hardy and E. M. Wright proved that such numbers exist for all positive integers n, and their proof is easily converted into a program to generate such numbers. However, the proof makes no claim at all about whether these generated numbers are the smallest positive and thus it cannot be used to find the actual value of $T_a(n)$.
Following six taxicab (smallest in size) are known.
$T_a(1) = 2 = 1^3 + 1^3$
$T_a(2) = 1729 = 1^3 + 12^3 = 9^3 + 10^3$
$T_a(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$
$T_a(4) = 6963472309248 = 2421^3 + 19083^3 = 5436^3 + 18948^3 = 10200^3 + 18072^3$
$= 13322^3 + 16630^3$
$T_a(5) = 48988659276962496 = 38787^3 + 365757^3 = 107839^3 + 362753^3 = 205292^3 + 342952^3 = 221424^3 + 336588^3 = 231518^3 + 331954^3$
$T_a(6) = 24153319581254312065344 = 582162^3 + 28906206^3 = 3064173^3 + 28894803^3$
$= 8519281^3 + 28657487^3 = 16218068^3 + 27093208^3 = 17492496^3 + 26590452^3$
$= 18289922^3 + 26224366^3$
$T_a(2)$ is also known as the Hardy-Ramanujan number. The subsequent taxicab numbers were found with the help of supercomputers. John Leech obtained $T_a(3)$

in 1957. E. Resenstiel, J. A. Dardis and C. R. Rosenstiel found $T_a(4)$ in 1991. J. A. Dardis found $T_a(5)$ in 1994 and it was confirmed by David W. Wilson in 1999. $T_a(6)$ was announced by Uwe Hollerbach on March 9, 2008.

Cubefree number means a positive integer that is not divisible by any p^3 where p is a prime. If a cubefree taxicab number T is written as $T = x^3 + y^3$, then x and y are relatively prime. Among the taxicab numbers $T_a(n), 1 \leq n \leq 6$, only $T_a(1)$ and $T_a(2)$ are cubefree taxicab numbers. The smallest cubefree taxicab number with three representations was discovered by Paul Vojta in 1981 while he was a graduate student. It is

$15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3 = 3^2 \times 5 \times 7 \times 31 \times 37 \times 199 \times 211$.

The smallest cubefree taxicab number with four representations was discovered by Staurt GAscoigne and independently by Duncon Moore in 2003. It is

$1801049058342701083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3 = 600259^3 + 1165884^3$.

Positive integers with three representations, not cube free are:

$87539319 = 436^3 + 167^3 = 423^3 + 228^3 = 44^3 + 255^3 = 3^3 \times 7 \times 31 \times 67 \times 223$

$1148834232 = 1044^3 + 222^3 = 920^3 + 718^3 = 846^3 + 816^3 = 2^3 \times 3^3 \times 7 \times 13 \times 211 \times 277$.

3. Main Results

Consider a (taxicab) number which is expressed as a sum of two cubes of positive integers in three different ways. Let it be $s^3 + r^3 = t^3 + q^3 = u^3 + p^3 \qquad \cdots (*)$
where $p, q, r \in \mathbb{N}$ are all even or exactly one of them is even.

Let $p < q < r$ and $p^3 + q^3 > r^3$.

Take

$$a = \frac{1}{2}(p^3 + q^3 - r^3)$$
$$b = \frac{1}{2}(p^3 - q^3 + r^3)$$
$$c = \frac{1}{2}(-p^3 + q^3 + r^3)$$
$$d = \frac{1}{2}(-p^3 - q^3 + r^3 + 2s^3)$$

Then $a, b, c \in \mathbb{N}$ and $a < b < c, d = s^3 - a$ and $d \in \mathbb{N}$ iff $s^3 > a$.

Clearly $a + b = p^3, a + c = q^3, b + c = r^3, a + d = s^3$ and $b + d = s^3 + r^3 - q^3 = t^3$, $c + d = s^3 + r^3 - p^3 = u^3$ by (*).

Hence (a, b, c, d) is a four tuple of integers where a, b, c are positive and sum of any two of its coordinates is a cube of a positive integer.

Example 3.1. $T_a(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$

Taking $p = 255, q = 423, r = 436, s = 167$; (1) gives

$$a = \frac{1}{2}(255^3 + 423^3 - 436^3) = \frac{1}{2}(16581375 + 75686967 - 82881856) = \frac{9386486}{2} = 4693243,$$
$$b = \frac{1}{2}(255^3 - 423^3 + 436^3) = \frac{1}{2}(16581375 - 75686967 + 82881856) = \frac{23776264}{2} = 118881132,$$
$$c = \frac{1}{2}(-16581375 + 75686967 + 82881856) = \frac{141987448}{2} = 70993724,$$
$$d = \frac{1}{2}(-16581375 - 75686967 + 82881856 + 2 \times 167^3) = \frac{1}{2}(-938648 + 9314926)$$
$$= \frac{-71560}{2} = -35780$$

Clearly, $a + b = 4693243 + 1188813 = 16581375 = 255^3$,
$a + c = 75686967 = 423^3, a + d = 4657463 = 167^3, b + c = 82881856 = 436^3$
$b + d = 11888132 - 35780 = 11852352 = 228^3$,
$c + d = 70993724 - 35780 = 70957944 = 414^3$.
Thus $(4693243, 11888132, 70993724, -35780)$ is a four-tuple of integers such that sum of any two coordinates is cube of a positive integer.
Example 3.2. $15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3$ is the smallest cubefree taxicab number with three representations.
Taking $p = 2152, q = 2456, r = 2468$ and $s = 517$, then by (1),

$$a = \frac{1}{2}(p^3 + q^3 - r^3) = 4873961696,$$
$$b = \frac{1}{2}(p^3 - q^3 + r^3) = 5092174112,$$
$$c = \frac{1}{2}(-p^3 + q^3 + r^3) = 9940473120$$
$$d = s^3 - a = -4735773283$$

where $a+b = 2152^3, a+c = 2456^3, b+c = 2468^3, a+d = 517^3, b+d = 709^3, c+d = 1733^3$.
$\Rightarrow$ Four tuple of integers $(4873961696, 5092174112, 9940473120, -4735773283)$ is such that sum of any two of its coordinates is cube of a positive integer.
Example 3.3. Positive integer, five representations, not cubefree:
$26059452841000 = 29620^3 + 4170^3 = 28810^3 + 12900^3 = 28423^3 + 14577^3$
$= 28423^3 + 14577^3 = 24940^3 + 21930^3$
Dividing by 8, from above we get
$3257431605125 = 14810^3 + 2085^3 = 14405^3 + 6450^3 = 12470^3 + 10965^3$

Taking $p = 10965, q = 14405, r = 14810$ and $s = 2085$, we get

$$\begin{aligned} a &= \frac{1}{2}(p^3 + q^3 - r^3) = 529531610625, \\ b &= \frac{1}{2}(p^3 - q^3 + r^3) = 788803771500, \\ c &= \frac{1}{2}(-p^3 + q^3 + r^3) = 2459563869500 \\ d &= s^3 - a = -520467646500 \end{aligned}$$

$\Rightarrow a+b = 10965^3, a+c = 14405^3, b+c = 14810^3, a+d = 2085^3, b+d = 6450^3, c+d = 12470^3$

$\Rightarrow$ $(529531610625, 788803771500, 2459563869500, -520467646500)$ is a four-tuple where sum of its any two coordinates is a cube of positive integer.

Example 3.4. We have $143604279 = 522^3 + 111^3 = 460^3 + 359^3 = 423^3 + 408^3$

Taking $p = 408, q = 460, r = 522, s = 111$ we get by (1)

$a = 11508332, b = 56408980, c = 85827668, d = -10140701.$

$\Rightarrow$ $(11508332, 56408980, 85827668, -10140701)$ is a four-tuple where sum of its any two coordinates is a cube of positive integer.

Example 3.5. $T_a(5)$ gives,

$48988659276962496 = 205292^3+342952^3 = 221424^3+336588^3 = 231518^3+331954^3$

Taking $p = 331954, q = 336588, r = 342952, s = 205292$, we get by (1),

$a = 17118752226891364, b = 19391636926447300, c = 20945030988258108,$

$d = -8535530906734276$

$\Rightarrow a + b = 331954^3, a + c = 336588^3, b + c = 342952^3,$

$a + d = 205292^3, b + d = 221424^3, c + d = 231518^3$

$\Rightarrow$ $(17118752226891364, 19391636926447300, 20945030988258108,$
$-8535530906734276)$ is a four-tuple where sum of its any two coordinates is cube of a positive integer.

Example 3.6. We have

$1801049058342701083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3$

Taking $p = 1207602, q = 1216102, r = 1216500, s = 92227$ and using (1), we get

$a = 879641367671452208, b = 881407757105599000, c = 918856835019401000,$

$d = -878856901453751125$

$\Rightarrow a + b = 1207602^3, a + c = 1216102^3, b + c = 1216500^3,$

$a + d = 922273^3, b + d = 136635^3, c + d = 341995^3$

$\Rightarrow$ $(879641367671452208, 881407757105599000, 918856835019401000,$
$-878856901453751125)$ is a four-tuple where sum of its any two coordinates is

cube of a positive integer.

Theorem 3.7. *There exist four-tuple of distinct positive integers such that sum of any two of its coordinates is cube of a positive integer.*

Proof. $T_a(6)$ gives
$24153319581254312065344 = 16218068^3 + 27093208^3 = 17492496^3 + 26590452^3$
$= 18289922^3 + 26224366^3$
Taking $p = 16218068, q = 17492496, r = 18289922, s = 26224366$, we get by (1),

$$a = \frac{1}{2}(16218068^3 + 17492496^3 - 18289922^3) = 1749942657207366722460,$$
$$b = \frac{1}{2}(16218068^3 - 17492496^3 + 18289922^3) = 2515826512048505687972,$$
$$c = \frac{1}{2}(-16218068^3 + 17492496^3 + 18289922^3) = 3602540998645922917476,$$
$$d = 26224366^3 - a = 16285009413352516737436$$

Here $a + b = 16218068^3, a + c = 17492496^3, b + c = 18289922^3$
$a + d = 26224366^3, b + d = 26590452^3, c + d = 27093208^3$
$\Rightarrow$ (1749942657207366722460, 2515826512048505687972, 3602540998645922917476, 16285009413352516737436) is a four-tuple of distinct positive integers where sum of any two of its coordinates is cube of a positive integer.

4. Conclusion

From above theorem, there is a four tuple of distinct positive integers such that sum of its any two coordinates is cube of a positive integer. If (a, b, c, d) be such a four tuple of distinct positive integers, then for any $n \in \mathbb{N}, (4n^3a, 4n^3b, 4n^3c, 4n^3d)$ is a four tuple of distinct positive integers where sum of any two of its coordinates is cube of a positive integer. Thus there are infinitely many such four tuples of distinct positive integers.

For any integer $n \geq 4$, if we have $p, q, r, s, t, u \in \mathbb{N}$ such that $s^n + r^n = t^n + q^n = u^n + p^n$ with $p < q < r$ and $s^n > p^n + q^n - r^n > 0$, then taking

$$a = \frac{1}{2}(p^n + q^n - r^n)$$
$$b = \frac{1}{2}(p^n - q^n + r^n)$$
$$c = \frac{1}{2}(-p^n + q^n + r^n)$$
$$d = s^n - a$$

(where all p, q, r are even or exactly one of them is even) we get a four-tuple of positive integers (a, b, c, d) such that sum of its any two coordinates is nth power of a positive integer, i.e. $a+b=p^n, a+c=q^n, b+c=r^n, a+d=s^n, b+d=t^n, c+d=u^n$.

5. Future Scope

For budding researchers, there are open problems to determine four-tuple (a, b, c, d) of positive integers, such that sum of any two of them is fourth, fifth, ... power of a positive integer.

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