South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 2 (2020), pp. 255-262

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

PARAMETER ESTIMATION OF NAKAGAMI DISTRIBUTION UNDER PRECAUTIONARY LOSS FUNCTION

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(**Received:** *Oct. 16, 2019* **Accepted:** *Jul. 21, 2020* **Published:** *Aug. 30, 2020)*

Abstract: In this paper Bayes estimators of the scale parameter of Nakagami distribution have been obtained by taking quasi, inverted gamma and uniform prior distribution using the precautionary loss function. These are compared with the corresponding estimators with squared loss function.

Keywords and Phrases: Nakagami Distribution, Bayesian method,Inverted Gamma, Precautionary Loss Function.

2010 Mathematics Subject Classification: 60E05, 62E15, 62H10, 62H12.

1. Introduction

Nakagami distribution can be considered as a flexible lifetime distribution [1]. It is also widely considered in biomedical fields. Shanker et al. [2] and Tsui et al. [3] use the Nakagami distribution to model ultrasound data in medical imaging studies. This distribution is extensively used in reliability theory and reliability engineering and to model the constant hazard rate portion because of its memory less property.

The probability density function of the Nakagami distribution [4] is given by

$$f(x;\theta,k)=\frac{2k^k}{\Gamma(k)\theta^k}x^{2k-1}e^{-\frac{k}{\theta}x^2}\quad ;\quad x>0,\ k>0,\ \theta>0. \tag{1}$$

where θ and k are called scale and shape parameter respectively.
The joint density function of (1) is given by

$$f(\underline{x};\theta,k) = \frac{(2k^k)^n}{(\Gamma(k))^n\theta^{nk}}\prod_{i=1}^{n} x_i^{2k-1} e^{-\frac{k}{\theta}\sum_{i=1}^{n} x_i^2} \tag{2}$$

The maximum likelihood estimator of θ when k is known is given by

$$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i^2 \tag{3}$$

In Bayesian analysis the fundamental problem are that of the choice of prior distribution $g(\theta)$ and a loss function $L(\hat{\theta},\theta)$. The squared error loss function for the scale parameter θ are defined as

$$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2 \tag{4}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_s = E(\theta) \tag{5}$$

This loss function is often used because it does not lead to extensive numerical computations but several authors (Ferguson [5], Berger [6], Zellner [7], Basu and Ebrahimi [8]) have recognized that the inappropriateness of using symmetric loss function. J. G. Norstrom [9] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is given as

$$L(\hat{\theta},\theta) = \frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}. \tag{6}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation.

$$\hat{\theta}_p = \left[E(\theta^2)\right]^{\frac{1}{2}}. \tag{7}$$

Let us consider three prior distributions of θ to obtain the Bayes estimators which are given by

(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d}\ ; \quad \theta > 0,\ d \geq 0, \tag{8}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) Inverted gamma prior: The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and $\beta(> 0)$ with probability density function given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{-(\alpha+1)}e^{-\frac{\beta}{\theta}}\ ; \quad \theta > 0, \tag{9}$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted gamma distribution is conjugate prior for θ.

(iii) Uniform prior: It frequently happens that the life tester known in advance that the probable values of θ lie over a finite range $[\alpha, \beta]$ but he does not have any strong opinion about any subset of values over this range. In such a case a uniform distribution over $[\alpha, \beta]$ may be a good approximation.

$$g_3(\theta) = \frac{1}{\beta - \alpha}; \quad 0 < \alpha \leq \theta \leq \beta. \tag{10}$$

The object of the present paper is to obtain the Bayes estimators of θ using above three prior distributions under precautionary loss function and to study their performance.

2. Bayes Estimators under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{\left(k\sum_{i=1}^{n} x_i^2\right)^{(nk+d-1)}}{\Gamma(nk+d-1)}\theta^{-(nk+d)}e^{-\frac{1}{\theta}k\sum_{i=1}^{n} x_i^2}\ ; \quad \theta > 0. \tag{11}$$

The Bayes estimator under squared error loss function comes out to be

$$\hat{\theta}_s = \frac{k\sum_{i=1}^{n} x_i^2}{nk+d-2}. \tag{12}$$

From equation (7), on using (11), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_p[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}k\sum_{i=1}^{n}x_i^2. \tag{13}$$

The risk function of the estimators $\hat{\theta}_s$ and $\hat{\theta}_p$ relative to precautionary loss function, denoted by $R_p(\hat{\theta}_s)$ and $R_p(\hat{\theta}_p)$, respectively, are as follows.

$$R_p(\hat{\theta}_s) = \theta\left[\left(\frac{nk+d-2}{nk-1}\right) + \left(\frac{nk}{nk+d-2}\right) - 2\right] \tag{14}$$

$$R_p(\hat{\theta}_p) = \theta\left[\frac{[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}}{nk-1} + \frac{nk}{[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}} - 2\right] \tag{15}$$

The risk function of the estimators $\hat{\theta}_s$ and $\hat{\theta}_p$ relative to squared error loss function, denoted by $R_s(\hat{\theta}_s)$ and $R_s(\hat{\theta}_p)$, respectively, and are given by

$$R_s(\hat{\theta}_s) = \theta^2\left[\frac{nk(nk+1)}{(nk+d-2)^2} - \frac{2nk}{(nk+d-2)} + 1\right] \tag{16}$$

$$R_s(\hat{\theta}_p) = \theta^2\left[\frac{nk(nk+1)}{[(nk+d-2)(nk+d-3)]} - \frac{2nk}{[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}} + 1\right] \tag{17}$$

In general neither of the estimators uniformly dominates the other. For example, if $k = 1, n = 5,\ d = 1$, then

$$\frac{R_s(\hat{\theta}_s)}{\theta^2} = 0.375 < 0.613 = \frac{R_s(\hat{\theta}_p)}{\theta^2}$$

$$\frac{R_p(\hat{\theta}_s)}{\theta} = 0.25 < 0.31 = \frac{R_p(\hat{\theta}_p)}{\theta}$$

If $k = 1,\ n = 5,\ d = 5$ then

$$\frac{R_s(\hat{\theta}_s)}{\theta^2} = 0.219 < 0.199 = \frac{R_s(\hat{\theta}_p)}{\theta^2}$$

$$\frac{R_p(\hat{\theta}_s)}{\theta} = 0.625 < 0.539 = \frac{R_p(\hat{\theta}_p)}{\theta}$$

3. Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f(\theta/\underline{x}) = \frac{\left(\beta + k\sum_{i=1}^{n} x_i^2\right)^{(nk+\alpha)}}{\Gamma(nk+\alpha)} \theta^{-(nk+\alpha-1)} e^{-\frac{1}{\theta}\left(\beta + k\sum_{i=1}^{n} x_i^2\right)}; \quad \theta > 0. \tag{18}$$

The Bayes estimator under squared error loss function on using (18) comes out to be

$$\hat{\theta}_s^* = \frac{\beta + k\sum_{i=1}^{n} x_i^2}{nk+\alpha-1}. \tag{19}$$

From equation (7), on using (18), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_p^*[(nk+\alpha-1)(nk+\alpha-2)]^{\frac{1}{2}} \left(\beta + k\sum_{i=1}^{n} x_i^2\right). \tag{20}$$

The risk function of the estimators $\hat{\theta}_s^*$ and $\hat{\theta}_p^*$ relative to squared error loss function are given by

$$R_s\left(\hat{\theta}_s^*\right) = \theta^2\left[\frac{nk(nk+1) + 2nk\left(\frac{\beta}{\theta}\right) + \left(\frac{\beta}{\theta}\right)^2}{(nk+\alpha-1)^2} - \frac{2\left(nk+\frac{\beta}{\theta}\right)}{(nk+\alpha-1)} + 1\right] \tag{21}$$

$$R_s\left(\hat{\theta}_p^*\right) = \theta^2\left[C^2\left\{nk(nk+1) + 2nk\left(\frac{\beta}{\theta}\right) + \left(\frac{\beta}{\theta}\right)^2\right\} - 2C\left(nk+\frac{\beta}{\theta}\right) + 1\right] \tag{22}$$

where $C = [(nk+\alpha-1)(nk+\alpha-2)]^{-\frac{1}{2}}$.

The Bayes risk associated with estimators $\hat{\theta}_s^*$ and $\hat{\theta}_p^*$ relative to squared error loss function are given by

$$r_s\left(\hat{\theta}_s^*\right) = \frac{\beta^2}{(\alpha-1)(\alpha-2)(nk+\alpha-1)} \tag{23}$$

$$r_s\left(\hat{\theta}_p^*\right) = \beta^2\left[\frac{nk(nk+1)C^2 - 2nkC + 1}{(\alpha-1)(\alpha-2)} + \frac{2C(nkC-1)}{(\alpha-1)} + C^2\right] \tag{24}$$

In this case the risk functions relative to precautionary loss function and the corresponding Bayes risks can not be obtained in closed forms.

4. Bayes Estimators under $g_3(\theta)$

Under $g_3(\theta)$, using (2), the posterior density of θ, is given by

$$f(\theta/\underline{x}) = \frac{\left(k\sum_{i=1}^{n}x_i^2\right)^{(nk-1)}\theta^{-nk}e^{-\frac{1}{\theta}\left(k\sum_{i=1}^{n}x_i^2\right)}}{Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\alpha}, nk-1\right) - Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\beta}, nk-1\right)}; 0 < \alpha \leq \theta \leq \beta. \quad (25)$$

where, $Ig(x, n) = \int_0^x e^{-t}t^{n-1}dt.$

The Bayes estimator under squared error loss function is given by

$$\hat{\theta_s^{**}} = \left[\frac{Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\alpha}, nk-2\right) - Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\beta}, nk-2\right)}{Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\alpha}, nk-1\right) - Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\beta}, nk-1\right)}\right] k\sum_{i=1}^{n}x_i^2 \quad (26)$$

From equation (7), using (26), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta_p^{**}} = \left[\frac{Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\alpha}, nk-3\right) - Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\beta}, nk-3\right)}{Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\alpha}, nk-1\right) - Ig\left(\frac{k\sum_{i=1}^{n}x_i^2}{\beta}, nk-1\right)}\right] k\sum_{i=1}^{n}x_i^2 \quad (27)$$

The equations (26) and (27), can be solved numerically. In this case the risk function and the corresponding Bayes risks can not be obtained in a closed form.

5. Conclusions

From the given example in section (2), it is clear that neither of the estimators uniformly dominates the other. We therefore recommend that the estimator's choice lies according to the value of 'd' in the quasi density used as the prior distribution which in turn depends on the situation at hand.

The risk function and Bayes risks under the natural conjugate are dependent on the population parameter θ and θ is not reparable, therefore, comparison could only be done by using numerical techniques.

Also, it is clear that from the equations (26) and (27) that only numerical solutions exist for the estimators $\hat{\theta}_s^{**}$ and $\hat{\theta}_p^{**}$. In this case the risk functions and Bayes risk cannot be obtained in closed forms. Thus, the comparison could only be done after obtaining the results numerically, which depends on the value of the parameter itself.

References

[1] M. Nakagami, The m-distribution – a general formula of intensity distribution of rapid fading, in Statistical Methods in Radio Wave Propagation: Proceedings of a Symposium Held at the University of California, Los Angeles, June 18-20, 1958, W. C. Hoffman, Ed., pp. 3-36, Pergamon Press, Oxford, UK, 1960.

[2] A. K. Shanker, C. Cervantes, H. Loza - Tavera and S. Avudainayagam, Chromium toxicity in plants, Environment International, vol. 31, no. 5(2005), pp. 739-753.

[3] P. H. Tsui, C. C. Huang, and S. H. Wang, Use of Nakagami distribution and logarithmic compression in ultrasonic tissue characterization, Journal of Medical and Biological Engineering, vol. 26, no. 2(2006), pp. 69-73.

[4] Kaisar Ahmad, S. P. Ahmad, and A. Ahmad, Classical and Bayesian Approach in Estimation of Scale Parameter of Nakagami distribution, Journal of Probability and Statistics, Hindawi Publishing Corporation, Article ID 7581918, 8 pages, Volume 2016.

[5] Ferguson, T. S., Mathematical Statistics: A Decision Theoretic Approach, Academic Press, New York, 1967.

[6] Berger, J. O., Statistical Decision Theory – Foundation Concepts and Methods, Springer – Verlag, 1980.

[7] Zellner, A., Bayesian estimation and prediction using asymmetric loss functions, Jour. Amer. Stat. Assoc., 91(1986), 446-451.

[8] Basu, A. P. and Ebrahimi, N., Bayesian approach to life testing and reliability estimation using asymmetric loss function," Jour. Stat. Plann. Infer., 29(1991), 21-31.

[9] Norstrom, J. G., The use of precautionary loss functions in Risk Analysis, IEEE Trans. Reliab., 45(3)(1996), 400-403.