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# Some Bilinear and Bilateral Hypergeometric Generating Relations

Chaudhary Wali Mohd.[1], M. I. Qureshi[1], Kaleem A. Quraishi[2] and Ram Pal[3]
[1]Department of Applied Sciences and Humanities,
Faculty of Engineering and Technology, Jamia Millia Islamia
(A Central University), New Delhi-110025 (India)
[2]Mathematics Section, Mewat Engineering College (Wakf),
Palla, Nuh, Mewat-122107, Haryana (India)
[3]Department of Applied Sciences and Humanities, Aryabhat Polytechnic,
G. T. Karnal Road, Delhi-110033 (India)
E-mails: chaudhary.walimohd@gmail.com; miqureshi_delhi@yahoo.co.in;
kaleemspn@yahoo.co.in; rampal1966@rediffmail.com

**Abstract:** The present paper mainly concerns with three theorems involving generating functions expressed in terms of single and double Laplace and Beta integrals. These theorems have been applied to obtain bilinear and bilateral generating functions involving polynomials of Mittag- Leffler, Madhekar- Thakare, Gottlieb, Jacobi, Konhauser, Laguerre and other polynomials hypergeometric in nature. One variable special cases of the hypergeometric polynomials are important in several applied problems.
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## 1. Introduction

Hypergeometric Polynomials occupy the pride of place in the literature on special functions. One variable special functions namely Madhekar- Thakare polynomials, Mittag- Leffler polynomial, Konhauser polynomials, Gottlieb polynomials, Jacobi polynomials, Legendre polynomials, Ultraspherical polynomial and the polynomials hypergeometric in nature, are closely associated with problems of applied nature. For example, Ultraspherical polynomials are deeply connected with axially symmetric potential in $n$ dimensions and contain Legendre and Chebyshev polynomials as special cases. Further Bessel functions used in our work are closely associated with problem possessing circular or cylindrical symmetry. For example, they arise in the study of free vibration of a circular membrane and in finding the temperature distribution in a circular cylinder. They also occur in electromagnetic field theory and numerous other areas of Physics and Engineering.

The Laguerre polynomials used in our work play an important role in finding the wave function associated with the electron in a Hydrogen atom. Further, Laguerre polynomials are encountered in the solution of the problem on the propagation of electromagnetic waves and in the analysis of the motion of electrons in Coulomb field, as well as in certain other problems of theoretical physics.

The present paper mainly aims at applications of the theorems given by Chaudhary [6,pp.261-263] in obtaining bilinear and bilateral generating functions for a class of hypergeometric polynomials. The results obtained in the paper are new and generally not seen in the literature on special functions. The theorems used in our work are as given below:

**Theorem 1:** Let $F(x,t)$ be a function having formal power series expansion in $t$, given by

$$F(x,t) = \sum_{n=0}^{\infty} C_n\, f_n(x)\, t^n \tag{1.1}$$

where $C_n$ is a specified sequence of parameters, independent of $x$ and $t$, and $f_n(x)$; $n = 0, 1, 2, \ldots,$ are polynomials of degree $n$ in $x$. With restrictions on $X_1$, $X_2$, $X_3$ and $t$, such that triple hypergeometric series of Srivastava and $F\left(x, \frac{tz}{z-1}\right)$ remain uniformly convergent for $z \in (0,1)$, then

$$\sum_{n=0}^{\infty} \frac{C_n\,(p)_n}{(1+p-q)_n}\, F^{(3)}\left[\begin{array}{l} p+n,\,(a) :: (b)\,;(b')\,;(b'') : (c)\;;(c')\;;(c'')\;; \\ q,\,(d) \quad\quad :: (e)\,;(e')\,;(e'') : (f)\,;(f')\,;(f'')\,; \end{array} X_1,\, X_2,\, X_3\right]$$

$$f_n(x)\, t^n = \frac{\Gamma(q)}{\Gamma(p)\Gamma(q-p)} \int_0^1 z^{p-1}\,(1-z)^{q-p-1} \times$$

$$\times F^{(3)}\left[\begin{array}{l} (a) :: (b)\,;(b')\,;(b'')\,: (c)\;;(c')\,;(c'')\;; \\ (d) :: (e)\,;(e')\,;(e'') : (f)\,;(f')\,;(f'')\,; \end{array} X_1 z,\, X_2 z,\, X_3 z\right] F\left(x,\, \frac{tz}{z-1}\right) \mathrm{d}z \tag{1.2}$$

$$\left(q-p \notin \mathbb{Z};\; \Re(q) > \Re(p) > 0\right)$$

where $F^{(3)}[X_1,\, X_2,\, X_3]$ is Srivastava's triple hypergeometric function.

**Theorem 2:** Let

$$G(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n \tag{1.3}$$

where $f_n(x)$ are the polynomials of degree $n$ in $x$, then

$$\frac{1}{\Gamma(h)}\int_0^\infty e^{-p}\,p^{h-1}\,{}_1F_1[c;b;yp]\,{}_1F_1[a;d;zp]\,G(x,\,tp)\,\mathrm{d}p$$

$$=\sum_{n=0}^{\infty}(h)_n\,F_2[h+n;a,c;d,b;z,y]\,f_n(x)\,t^n \tag{1.4}$$

$$(\Re(h)>0)$$

where $F_2$ is Appell's function of second kind.

**Theorem 3:** Let

$$G(x,t)=\sum_{n=0}^{\infty}f_n(x)\,t^n$$

where $f_n(x)$ are the polynomials of degree $n$ in $x$, then

$$\frac{1}{\Gamma(a)\,\Gamma(b)}\int_0^\infty\int_0^\infty e^{-(p+q)}\,p^{a-1}\,q^{b-1}\,{}_0F_1[\text{---};c;ypq]\,{}_0F_1[\text{---};d;zpq]\,G(x,\,tpq)\,\mathrm{d}p\mathrm{d}q$$

$$=\sum_{n=0}^{\infty}(a)_n\,(b)_n\,F_4[a+n,b+n;c,d;y,z]\,f_n(x)\,t^n \tag{1.5}$$

$$\Big(\Re(a)>0;\ \Re(b)>0\Big)$$

where $F_4$ is Appell's function of fourth kind.

**Corollary:** On taking $B=B'=B''=E=E'=E''=C''=F''=0$, theorem 1 with $X_3=0$ gives

$$\sum_{n=0}^{\infty}\frac{C_n\,(p)_n}{(1+p-q)_n}\,F^{(2)}\left[\begin{array}{l} p+n,\,(a):(c);(c');\\ \\ q,\,(d)\qquad :(f);(f'); \end{array}\ X_1,\,X_2\right]f_n(x)\,t^n$$

$$=\frac{\Gamma(q)}{\Gamma(p)\,\Gamma(q-p)}\int_0^1 z^{p-1}(1-z)^{q-p-1}\,F^{(2)}\left[\begin{array}{l}(a):(c)\,;\,(c');\\ \\ (d):(f);(f');\end{array}\ X_1z,\,X_2z\right]\times$$

$$F\left(x,\,\frac{tz}{z-1}\right)\mathrm{d}z \tag{1.6}$$

$$\Big(q-p\notin\mathbb{Z};\ \Re(q)>\Re(p)>0\Big)$$

where $F^{(2)}$ is Kampé de Fériet's double hypergeometric function.

Again on adjusting parameters and variables suitably, the results for Lauricella's functions $F_A^{(3)}$, $F_B^{(3)}$, $F_C^{(3)}$ and $F_D^{(3)}$ [28,p.60] given by Mathur [16,pp.222-223, Equations (2.2)-(2.5)] follow as special cases of Theorem 1.

Further on making applications of these theorem, many more known and new results can be obtained by specializing the parameters or variables or both.

It is worth pointing out that $F^{(3)}$ is a generalization of $F_1$ to $F_{14}$[14, pp. 113-114] series of Lauricella, Kampé de Fériet's double series $F^{(2)}$[1, p. 150(29); see also 5,p.112], $H_A$, $H_B$ and $H_C$ of Srivastava[23, pp. 99-100; see also 25] and $F_K$ of Sharma [22, p. 613(2)].

## 2. Definitions

The hypergeometric functions and polynomials needed in our subsequent work are as defined below:

The Kampé de Fériet's double hypergeometric function $F^{(2)}$[1,p.150] is defined as

$$F^{(2)}\left[\begin{array}{ll}(a):(b);(d); & \\ & x,\, y \\ (f):(g);(h); & \end{array}\right] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{[(a)]_{m+n}\,[(b)]_m\,[(d)]_n\,x^m\,y^n}{[(f)]_{m+n}\,[(g)]_m\,[(h)]_n\,m!\,n!} \tag{2.1}$$

where $(a)$ and $[(a)]_{m+n}$ will mean the sequence of $A$ parameters $a_1, a_2, \ldots, a_A$ and the product $(a_1)_{m+n}(a_2)_{m+n}\cdots(a_A)_{m+n}$ respectively. Thus $[(a)]_m$ is to be interpreted as

$$[(a)]_m = \prod_{i=1}^{A}(a_i)_m = (a_1)_m(a_2)_m\cdots(a_A)_m = \prod_{i=1}^{A}\frac{\Gamma(a_i+m)}{\Gamma(a_i)} \tag{2.2}$$

with similar interpretations for $[(b)]$, $[(d)]$ etc.
$F^{(2)}$ has Appell's functions $F_1$, $F_2$ and $F_4$ as special cases, that is,

$$F_1[\alpha,\, \beta,\, \gamma;\, \delta;\, x,\, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_m\,(\gamma)_n\,x^m\,y^n}{(\delta)_{m+n}\,m!\,n!},\quad \max\{|x|,\,|y|\}<1 \tag{2.3}$$

$$F_2[\alpha,\, \beta,\, \gamma;\, \delta,\, \xi;\, x,\, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_m\,(\gamma)_n\,x^m\,y^n}{(\delta)_m\,(\xi)_n\,m!\,n!},\quad |x|+|y|<1 \tag{2.4}$$

$$F_4[\alpha,\, \beta;\, \gamma;\, \delta;\, x,\, y] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\alpha)_{m+n}\,(\beta)_{m+n}\,x^m\,y^n}{(\gamma)_m\,(\delta)_n\,m!\,n!},\quad \sqrt{|x|}+\sqrt{|y|}<1 \tag{2.5}$$

Srivastava's generalized hypergeometric function $F^{(3)}$[24,p.428] of three variables is defined as

$$F^{(3)}\left[\begin{array}{l} (a) :: (b) ; (b') ; (b'') : (c) ; (c') ; (c'') ; \\ \\ (d) :: (e) ; (e') ; (e'') : (f) ; (f') ; (f'') ; \end{array} x,\, y,\, z \right]$$

$$= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{[(a)]_{m+n+p}\,[(b)]_{m+n}\,[(b')]_{n+p}\,[(b'')]_{p+m}\,[(c)]_m\,[(c')]_n\,[(c'')]_p\,x^m\,y^n\,z^p}{[(d)]_{m+n+p}\,[(e)]_{m+n}\,[(e')]_{n+p}\,[(e'')]_{p+m}\,[(f)]_m\,[(f')]_n\,[(f'')]_p\,m!\,n!\,p!} \tag{2.6}$$

It will be assumed throughout the paper that the absence of parameters shown by horizontal dashes mean that there exist no parameters and in that case, from (2.2), the conventional value of an empty product will be unity, that is, $\prod_{i=1}^{0}(a_i)_m = 1$. Further numerator parameters like $(a)$, $(b)$, $(b')$ etc. may be zero or negative integers, but the denominator parameters like $(d)$, $(e)$, $(e')$ etc. are not allowed to be zero or negative integers.

The region of convergence of above triple series (2.6) is given in the literature [8,p.156; see also 9,p.40].

The function $F_G$ in the notation of Saran[19] indicating also the numbering of Lauricella[14] on the left is as given below:

$$F_8 : F_G[a, a, a;\, b, c, d;\, e, f, f;\, x, y, z] = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{p=0}^{\infty} \frac{(a)_{m+n+p}\,(b)_m\,(c)_n\,(d)_p\,x^m\,y^n\,z^p}{(e)_m\,(f)_{n+p}\,m!\,n!\,p!} \tag{2.7}$$

$$\Big(|x| < r,\ |y| < s,\ |z| < t,\ r + s = 1 = r + t\Big)$$

The Jacobi polynomials [18,p.254] has Ultraspherical (or Gegenbauer) polynomials $C_n^{\alpha}(x)$, Legendre polynomial $C_n^{1/2}(x)$ and the generalized Laguerre polynomials $L_n^{(\alpha)}(x)$ [18,p.200] as special cases and is defined by:

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\,{}_2F_1\left[\begin{array}{ll} -n,\, 1+\alpha+\beta+n & ; \\ \\ 1+\alpha & ; \end{array} \tfrac{1-x}{2}\right] = H_n^{(\alpha,\beta)}\left(\xi,\, \xi,\, \frac{1-x}{2}\right) \tag{2.8}$$

where $H_n^{(\alpha,\beta)}(\xi,\, \xi,\, x)$ [12,p.158] is the generalized Rice polynomial of Khandekar[12].

Szegö[29,p.64; see also 17,p.190] established the formulae:

$$P_n^{(\alpha-n,\beta-n)}(x) = \frac{\Gamma(1+\alpha)}{\Gamma(1+\alpha-n)\,n!}\left(\frac{1+x}{2}\right)^n {}_2F_1\left[\begin{array}{lc} -n,\ -\beta & ; \\ & \frac{x-1}{x+1} \\ 1+\alpha-n & ; \end{array}\right]$$

$$= \left(\frac{1-x}{2}\right)^n P_n^{(-\alpha-\beta-1,\beta-n)}\left(\frac{x+3}{x-1}\right), \tag{2.9}$$

$$P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$$

$$P_n^{(\alpha,\alpha)}(x) = \frac{(1+\alpha)_n}{(1+2\alpha)_n}\, C_n^{\alpha+\frac{1}{2}}(x) \tag{2.10}$$

$$P_n(x) = P_n^{(0,0)}(x) = C_n^{1/2}(x) = (-1)^n\, P_n(-x) \tag{2.11}$$

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!}\, {}_1F_1\left[\begin{array}{lc} -n & ; \\ & x \\ 1+\alpha & ; \end{array}\right] = \lim_{|\beta|\to\infty}\left\{P_n^{(\alpha,\beta)}\left(1-\frac{2x}{\beta}\right)\right\} \tag{2.12}$$

$$x^{-\frac{\alpha}{2}}\, J_\alpha(2\sqrt{x}) = \lim_{n\to\infty}\left\{n^{-\alpha}\, L_n^{(\alpha)}\left(\frac{x}{n}\right)\right\} \tag{2.13}$$

where in each of the equations (2.8) to (2.13), $\Re(\alpha) > -1$, $\Re(\beta) > -1$ and $n$ is a non-negative integer.

The Madhekar-Thakare polynomial $J_n^{(\alpha,\beta)}(x;k)$ [15,p.421;28,p.197(Q.No.64)] is defined as

$$J_n^{(\alpha,\beta)}(x;k) = \frac{(1+\alpha)_{kn}}{n!}\,{}_{k+1}F_k\left[\begin{array}{lc} -n,\ \Delta(k;\alpha+\beta+n+1) & ; \\ & \left(\frac{1-x}{2}\right)^k \\ \Delta(k;1+\alpha) & ; \end{array}\right] \tag{2.14}$$

where $k$ is a positive integer, and

$$J_n^{(\alpha,\beta)}(x;1) = P_n^{(\alpha,\beta)}(x)$$

The Konhauser polynomial $Z_n^\alpha(x;k)$ [13; p. 304,(Eq.5); 28, p. 197 (Q.No. 65)] is defined as

$$Z_n^\alpha(x;k) = \frac{(1+\alpha)_{kn}}{n!}\,{}_1F_k\left[\begin{array}{lc} -n & ; \\ & \left(\frac{x}{k}\right)^k \\ \Delta(k;1+\alpha) & ; \end{array}\right], \quad k\in\mathbb{N} \tag{2.15}$$

$$Z_n^{\alpha}(x;k) = \lim_{|\beta|\to\infty} \left\{ J_n^{(\alpha,\beta)}\left(1 - \frac{2x}{\beta}\right);\, k \right\}$$

$$Z_n^{\alpha}(x;1) = L_n^{(\alpha)}(x)$$

The Gottlieb Polynomials [11,p.454,eq.(2.3)] is defined as

$$\ell_n(x;\lambda) = e^{-n\lambda}\, {}_2F_1 \left[ \begin{array}{lll} -n,\, -x & ; & \\ & & 1 - e^{\lambda} \\ 1 & ; & \end{array} \right] \tag{2.16}$$

$$e^{n\lambda}\,\ell_n(x;\,\lambda) = P_n^{(0,\,-x-n-1)}(2e^{\lambda} - 1) \tag{2.17}$$

The Mittag-Leffler Polynomial $g_n(x)$[28,p.185(Q.No.45); see also 2] is defined as

$$g_n(x) = 2x\ {}_2F_1 \left[ \begin{array}{lll} 1-n,\, 1-x & ; & \\ & & 2 \\ 2 & ; & \end{array} \right] \tag{2.18}$$

The Bessel function $J_\nu(z)$[28,p.209(Q.No.1)] of order $\nu$ is defined as

$$J_\nu(z) = \frac{(\frac{1}{2}z)^\nu}{\Gamma(\nu+1)}\, {}_0F_1 \left[ \begin{array}{lll} \text{———} & ; & \\ & & -\frac{z^2}{4} \\ \nu+1 & ; & \end{array} \right];\ z \in \mathbb{R} \tag{2.19}$$

## 3. Applications

(i) Consider the generating function [28,p.209(Eq.10); see also 7,p.62(Eq.27)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{n!}\, {}_{G+1}F_{H+1} \left[ \begin{array}{lll} -n,\, (g) & ; & \\ & & x \\ 1-\lambda-n,\, (h) & ; & \end{array} \right] t^n$$

$$= (1-t)^{-\lambda}\, {}_GF_H \left[ \begin{array}{lll} (g) & ; & \\ & & xt \\ (h) & ; & \end{array} \right] \tag{3.1}$$

where $(g)$ stands for $G$ number of parameters $g_1,\, g_2,\, \ldots,\, g_G$ with similar interpretation for $(h)$ and $|t| < 1$.

In (3.1), we take $F(x,\,t) = (1-t)^{-\lambda}\, {}_GF_H \left[ \begin{array}{lll} (g) & ; & \\ & & xt \\ (h) & ; & \end{array} \right]$

Combining (3.1) with (1.1) and using corollary of Theorem 1 we get

$$\sum_{n=0}^{\infty} \frac{(p)_n\,(\lambda)_n}{(1+p-q)_n\, n!}\, {}_{1+A+C}F_{1+D+F} \left[ \begin{array}{lll} p+n,\, (a),\, (c) & ; & \\ & & y \\ q,\, (d),\, (f) & ; & \end{array} \right] \times$$

$$
{}_{G+1}F_{H+1}\left[\begin{array}{lc} -n,\,(g) & ; \\ & x \\ 1-\lambda-n,\,(h) & ; \end{array}\right] t^n
$$

$$
= F^{(3)}\left[\begin{array}{l} p\ ::\text{—};\text{———};\text{—}:\ (a),\,(c)\ ;(g);\lambda\ ; \\ \text{—}::\text{—};1+p-q;\text{—}:q,\,(d),\,(f);(h);\text{—}; \end{array}\ y,\,xt,\,t\right] \quad (3.2)
$$

which for $G = H = 0$, $\lambda = -\alpha$ and with $t$ replaced by $-t$, gives

$$
\sum_{n=0}^{\infty}\frac{(p)_n}{(1+p-q)_n}\,{}_{1+A+C}F_{1+D+F}\left[\begin{array}{lc} p+n,\,(a),\,(c) & ; \\ & y \\ q,\,(d),\,(f) & ; \end{array}\right] L_n^{(\alpha-n)}(x)\,t^n
$$

$$
= F^{(3)}\left[\begin{array}{l} p\ ::\text{—};\text{———};\text{—}:\ (a),\,(c)\ ;\text{—};-\alpha; \\ \text{—}::\text{—};1+p-q;\text{—}:q,\,(d),\,(f);\text{—};\text{——}; \end{array}\ y,\,-xt,\,-t\right] \quad (3.3)
$$

where $L_n^{(\alpha-n)}(x)$ is the restricted Laguerre polynomial given by (2.12).

Further in (3.2), taking $C = D = F = H = 0$, $A = G = 1$ and then replacing $\lambda$ by $1+p-q$, we get

$$
\sum_{n=0}^{\infty}\frac{(p)_n}{n!}\,{}_2F_1\left[\begin{array}{lc} -n,\,\beta & ; \\ & x \\ q-p-n & ; \end{array}\right]{}_2F_1\left[\begin{array}{lc} p+n,\,\alpha & ; \\ & y \\ q & ; \end{array}\right] t^n
$$

$$
= F_G[p,p,p,\alpha,\beta,1+p-q;\ q,1+p-q,1+p-q;\ y,xt,t] \quad (3.4)
$$

where Saran's function $F_G$ is given by (2.7).

Equation (3.4) for $y = 0$ gives

$$
\sum_{n=0}^{\infty}\frac{(p)_n}{n!}\,{}_2F_1\left[\begin{array}{lc} -n,\,\beta & ; \\ & x \\ q-p-n & ; \end{array}\right] t^n = F_1[p,\beta,1+p-q;\ 1+p-q;\ xt,t] \quad (3.5)
$$

where Appell's function $F_1$ is given by (2.3).

(ii) Consider the generating function [28,p.185(Q.45(ii));18,p.290(8); see also 3].

$$
\sum_{n=0}^{\infty} g_{n+1}(x)\frac{t^n}{n!} = 2\,x\,e^t\,{}_1F_1\left[\begin{array}{lc} 1-x & ; \\ & -2t \\ 2 & ; \end{array}\right] \quad (3.6)
$$

where $g_{n+1}(x)$ is a Mittag-Leffler Polynomial given by (2.18).

Then for $F(x,t) = 2xe^t \, {}_1F_1\left[\begin{array}{ll} 1-x & ; \\ & \quad -2t \\ 2 & ; \end{array}\right]$, the application of corollary of Theorem 1 would give us

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(1+p-q)_n \, n!} \, {}_{1+A+C}F_{1+D+F}\left[\begin{array}{ll} p+n,\,(a),\,(c) & ; \\ & \quad y \\ q,\,(d),\,(f) & ; \end{array}\right] g_{n+1}(x)\, t^n$$

$$= 2\,x\,F^{(3)}\left[\begin{array}{l} p \;::\text{—};\text{————};\text{—}:\;(a),\,(c)\;;1-x\,;\text{—}; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad y,\,-2t,\,t \\ \text{—}::\text{—};1+p-q\,;\text{—}:q,\,(d),\,(f)\,;\;\;2\;\;;\text{—}; \end{array}\right] \tag{3.7}$$

which for $y = 0$ gives:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(1+p-q)_n \, n!} \, g_{n+1}(x)\, t^n = 2x\,F^{(2)}\left[\begin{array}{l} \qquad p \quad :1-x\,;\text{—}; \\ \qquad\qquad\qquad\qquad\qquad -2t,\,t \\ 1+p-q:\;\;2\;\;;\text{—}; \end{array}\right] \tag{3.8}$$

Further equation (3.7) for $A = C = D = 1,\ F = 0$, would give us:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(1+p-q)_n \, n!} \, {}_3F_2\left[\begin{array}{ll} p+n,\,\alpha,\,\beta & ; \\ & \quad y \\ q,\,\gamma & ; \end{array}\right] g_{n+1}(x)\, t^n$$

$$= 2\,x\,F^{(3)}\left[\begin{array}{l} p \;::\text{—};\text{————};\text{—}:\alpha,\,\beta\,;1-x\,;\text{—}; \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad y,\,-2t,\,t \\ \text{—}::\text{—};1+p-q\,;\text{—}:q,\,\gamma\,;\;\;2\;\;;\text{—}; \end{array}\right] \tag{3.9}$$

(iii) Consider the generating function [2,p.140; see also 28,p.214(Q.No.10)]

$$\sum_{n=0}^{\infty} \frac{(\lambda)_n}{(\mu)_n} P_n^{(\alpha-n,\,\beta-n)}(x)\, t^n = F_1\left[\lambda, -\alpha, -\beta;\ \mu;\ -\frac{1}{2}(x+1)t, -\frac{1}{2}(x-1)t\right] \tag{3.10}$$

where $P_n^{(\alpha-n,\,\beta-n)}(x)$ is the restricted Jacobi Polynomial given by (2.8).

In (1.1), we take $F(x,t) = F_1[\lambda, -\alpha, -\beta;\ \mu;\ -\frac{1}{2}(x+1)t, -\frac{1}{2}(x-1)t]$, combining (3.10) with (1.1), then using corollary of Theorem 1, we get

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\mu)_n} \, {}_{1+A+C}F_{1+D+F}\left[\begin{array}{ll} p+n,\,(a),\,(c) & ; \\ & \quad y \\ q,\,(d),\,(f) & ; \end{array}\right] P_n^{(\alpha-n,\,\beta-n)}(x)\, t^n$$

$$= F^{(3)} \left[ \begin{array}{l} p \;::\text{-};\text{-};\text{-}:\; (a),\,(c) \;\; ;\, -\alpha\, ;\, -\beta\, ; \\ \\ \text{-}::\text{-};\, \mu\, ;\text{-}:q,\, (d),\, (f)\, ;\text{—};\text{—}; \end{array} \quad y,\, -\tfrac{1}{2}(x+1)t, -\tfrac{1}{2}(x-1)t \right] \tag{3.11}$$

Now for $C = D = F = 0$, $A = 1$, above equation reduces to:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\mu)_n} \, {}_2F_1 \left[ \begin{array}{ll} p+n,\, \gamma & ; \\ & \quad y \\ q & ; \end{array} \right] P_n^{(\alpha-n,\, \beta-n)}(x)\, t^n$$

$$= F_G \left[ p, p, p, \gamma, -\alpha, -\beta;\; q, \mu, \mu;\; y,\; -\frac{1}{2}(x+1)t, -\frac{1}{2}(x-1)t \right] \tag{3.12}$$

Again equation (3.12) in the light of the result (2.9) gives an elegant generating relation in the form:

$$\sum_{n=0}^{\infty} \frac{(p)_n}{(\mu)_n} \left(\frac{1-x}{2}\right)^n {}_2F_1 \left[ \begin{array}{ll} p+n,\, \gamma & ; \\ & \quad y \\ q & ; \end{array} \right] P_n^{(-\alpha-\beta-1,\, \beta-n)} \left(\frac{x+3}{x-1}\right) t^n$$

$$= F_G \left[ p, p, p, \gamma, -\alpha, -\beta;\; q, \mu, \mu;\; y,\; -\frac{1}{2}(x+1)t, -\frac{1}{2}(x-1)t \right] \tag{3.13}$$

Further by means of the relations (2.10), (2.11) and (2.12), equation (3.12) reduces to generating functions involving Ultraspherical, Legendre and Laguerre polynomials.

(iv) Consider the generating function [15,p.421; see also 28,p.197(Q.64(ii))]

$$\sum_{n=0}^{\infty} J_n^{(\alpha,\, \beta-n)}(x;k)\, \frac{t^n}{(\alpha+1)_{kn}} = e^t \, {}_kF_k \left[ \begin{array}{ll} \Delta(k;\alpha+\beta+1) & ; \\ & \quad -(\frac{1-x}{2})^k t \\ \Delta(k;\alpha+1) & ; \end{array} \right] \tag{3.14}$$

where $k$ is a positive integer, $\Delta(k;b)$ represents an array of $k$ parameters given by $\frac{b}{k}, \frac{b+1}{k}, \ldots, \frac{b+k-1}{k}$ and $J_n^{(\alpha,\, \beta-n)}(x;k)$ is a Madhekar-Thakare Polynomial given by (2.14).

In (1.1), we take

$$G(x,t) = e^t \, {}_kF_k \left[ \begin{array}{ll} \Delta(k;\alpha+\beta+1) & ; \\ & \quad -(\frac{1-x}{2})^k t \\ \Delta(k;\alpha+1) & ; \end{array} \right]$$

Combining (3.14) with (1.3) in Theorem 2, we get

$$\sum_{n=0}^{\infty} \frac{(b)_n}{(\alpha+1)_{kn}} \, {}_2F_1\left[\begin{matrix} b+n,\, a \;; \\ \\ d \;; \end{matrix} \; z\right] J_n^{(\alpha,\beta-n)}(x;k)\, t^n$$

$$= F^{(3)}\left[\begin{matrix} b \;::\; \text{—}\,;\text{—}\,;\text{—}:a\,;\text{—}\,;\Delta(k;\alpha+\beta+1)\,; \\ \\ \text{—}::\text{—}\,;\text{—}\,;\text{—}:d\,;\text{—}\,;\;\Delta(k;\alpha+1)\;\;; \end{matrix} \; z,\, t,\, -(\tfrac{1-x}{2})^k t\right] \tag{3.15}$$

Now expanding $F^{(3)}$ of (3.15) for $k=1$, multiplying by $e^u\, u^{-\delta}$, replacing $t$ by $\frac{t}{u}$ and then evaluating the result obtained with the help of contour integral for Gamma function [10,p.32(1.5.1.5)]:

$$\frac{1}{2\pi i}\int_C e^v\, v^{-a-m}\mathrm{d}v = \frac{1}{\Gamma(a+m)} \tag{3.16}$$

(where $m$ is a non-negative integer and $a$ does not take non-positive integer values), we get a bilateral generating function involving Jacobi Polynomials:

$$\sum_{n=0}^{\infty} \frac{(b)_n}{(\delta)_n(\alpha+1)_n} \, {}_2F_1\left[\begin{matrix} b+n,\, a \;; \\ \\ d \;; \end{matrix} \; z\right] P_n^{(\alpha,\,\beta-n)}(x)\, t^n$$

$$= F^{(3)}\left[\begin{matrix} b \;::\; \text{—}\,;\text{—}\,;\text{—}:a\,;\text{—}\,;\alpha+\beta+1\,; \\ \\ \text{—}::\text{—}\,;\;\delta\;;\text{—}:d\,;\text{—}\,;\;\alpha+1\;\;; \end{matrix} \; z,\, t,\, (\tfrac{x-1}{2})t\right] \tag{3.17}$$

Further equation (3.15) for $k=2$, reduces to

$$\sum_{n=0}^{\infty} \frac{(b)_n}{(\alpha+1)_{2n}} \, {}_2F_1\left[\begin{matrix} b+n,\, a \;; \\ \\ d \;; \end{matrix} \; z\right] J_n^{(\alpha,\,\beta-n)}(x;2)\, t^n$$

$$= F^{(3)}\left[\begin{matrix} b \;::\text{—}\,;\text{—}\,;\text{—}:a\,;\text{—}\,;\frac{\alpha+\beta+1}{2},\,\frac{\alpha+\beta+2}{2}\,; \\ \\ \text{—}::\text{—}\,;\text{—}\,;\text{—}:d\,;\text{—}\,;\;\frac{\alpha+1}{2},\,\frac{\alpha+2}{2}\;\;; \end{matrix} \; z,\, t,\, -\frac{(1-x)^2}{4}t\right] \tag{3.18}$$

(v) Consider the generating function [11, p. 454(Eq.2.3); 18, p.303 (Q.No.10); 28, p.186 (Q.No.47(ii))]

$$\sum_{n=0}^{\infty} \ell_n(x;\lambda)\frac{t^n}{n!} = e^t \, {}_1F_1\left[\begin{matrix} x+1 \;; \\ \\ 1 \;; \end{matrix} \; -(1-e^{-\lambda})t\right] \tag{3.19}$$

where $\ell_n(x;\lambda)$ is the Gottlieb Polynomials given by (2.16).

In (1.3), we take $G(x,t) = e^t\, {}_1F_1\left[\begin{matrix} x+1 & ; & \\ & & -(1-e^{-\lambda})t \\ 1 & ; & \end{matrix}\right]$, combining (3.19) with (1.3) in Theorem 2, we get

$$\sum_{n=0}^{\infty} \frac{(b)_n}{n!}\, {}_2F_1\left[\begin{matrix} b+n,\, c & ; & \\ & & y \\ b & ; & \end{matrix}\right] \ell_n(x;\lambda)\, t^n$$

$$= F^{(3)}\left[\begin{matrix} b :: —;—;— : -x;—;c; & \\ & (1-e^{-\lambda})t,\, e^{-\lambda}t,\, y \\ — :: —;—;— : 1 \;;—;b; & \end{matrix}\right] \tag{3.20}$$

which on using (2.17), replacing $t$ by $\frac{t}{u}$, multiplying by $e^u u^{-\mu}$ and then evaluating with the help of the integral (3.16), gives a result involving Jacobi Polynomial:

$$\sum_{n=0}^{\infty} \frac{(b)_n}{(\mu)_n\, n!}\, {}_2F_1\left[\begin{matrix} b+n,\, c & ; & \\ & & y \\ b & ; & \end{matrix}\right] P_n^{(0,\, x-n)}(2e^{-\lambda}-1)\, t^n$$

$$= F^{(3)}\left[\begin{matrix} b :: —;—;— : c;—;x+1; & \\ & y,\, t,\, (e^{-\lambda}-1)\,t \\ — :: —;\ \mu\ ;— : b;—;\ 1\ ; & \end{matrix}\right] \tag{3.21}$$

(vi) Consider the generating function [13, p.304 (Eq.5); 26, p.490; 28, p.198 (Q.No.65(ii))]

$$\sum_{n=0}^{\infty} Z_n^{\alpha}(x;k)\, \frac{t^n}{(\alpha+1)_{kn}} = e^t\, {}_0F_k\left[\begin{matrix} \text{———} & ; & \\ & & -(\frac{x}{k})^k t \\ \Delta(k;\alpha+1) & ; & \end{matrix}\right] \tag{3.22}$$

where $Z_n^{\alpha}(x;k)$ is the Konhauser Polynomial given by (2.15). Then, the application of Theorem 3 would give us:

$$\sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(\alpha+1)_{kn}}\, {}_2F_1\left[\begin{matrix} a+n,\, b+n & ; & \\ & & y \\ c & ; & \end{matrix}\right] Z_n^{\alpha}(x;k)\, t^n$$

$$= F^{(3)} \left[ \begin{array}{l} a,\, b :: \text{---} ; \text{---} ; \text{---} : \text{---} ; \text{---} ; \text{------} ; \\ \text{---} :: \text{---} ; \text{---} ; \text{---} : \ c \ ; \text{---} ; \Delta(k; \alpha+1) ; \end{array} \ y,\, t,\, -(\tfrac{x}{k})^k t \right] \quad (3.23)$$

which for $k = 2$ gives:

$$\sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(\alpha+1)_{2n}}\, {}_2F_1 \left[ \begin{array}{l} a+n,\, b+n \ ; \\ c \ ; \end{array} \ y \right] Z_n^{\alpha}(x; 2)\, t^n$$

$$= F^{(3)} \left[ \begin{array}{l} a,\, b :: \text{---} ; \text{---} ; \text{---} : \text{---} ; \text{---} ; \text{------} ; \\ \text{---} :: \text{---} ; \text{---} ; \text{---} : \ c \ ; \text{---} ; \frac{\alpha+1}{2},\, \frac{\alpha+2}{2} ; \end{array} \ y,\, t,\, -\tfrac{x^2}{4} t \right] \quad (3.24)$$

Further multiplying by $e^u u^{-\beta}$, replacing $t$ by $\frac{t}{u}$ and evaluating the result obtained with the help of the integral (3.16), equation (3.23) for $k = 1$ gives:

$$\sum_{n=0}^{\infty} \frac{(a)_n\, (b)_n}{(\beta)_n\, (\alpha+1)_n}\, {}_2F_1 \left[ \begin{array}{l} a+n,\, b+n \ ; \\ c \ ; \end{array} \ y \right] L_n^{(\alpha)}(x)\, t^n$$

$$= F^{(3)} \left[ \begin{array}{l} a,\, b :: \text{---} ; \text{---} ; \text{---} : \text{---} ; \text{---} ; \text{------} ; \\ \text{---} :: \text{---} ; \ \beta \ ; \text{---} : \ c \ ; \text{---} ; \alpha+1 ; \end{array} \ y,\, t,\, -xt \right] \quad (3.25)$$

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