

**CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY
CONVOLUTION**

N. Magesh, S. M. El-Deeb* and R. Themangani**

Post-Graduate and Research Department of Mathematics,
Government Arts College for Men,
Krishnagiri - 635001, Tamil Nadu, INDIA

E-mail : nmagi_2000@yahoo.co.in

*Department of Mathematics, Faculty of Science,
Damietta University, New Damietta - 34517, EGYPT

*Department of Mathematics,
College of Science and Arts in Badaya,
Qassim University, Qassim, SAUDI ARABIA

E-mail : shezaeldeen@yahoo.com

**Post-Graduate and Research Department of Mathematics
Voorhees College, Vellore - 632001, Tamil Nadu, INDIA

E-mail : rthema2011@yahoo.com

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Abstract: In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficient $|a_2|$ and we obtain the Fekete-Szegő inequalities for certain classes $Q_{\Sigma}(\alpha; \Psi(z))$ and $\mathcal{P}_{\Sigma}(\beta; \Psi(z))$ of bi-univalent functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \quad (1.1)$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$, which are univalent functions.

Let $\Psi \in \mathcal{S}$ be given by

$$\Psi(z) = z + \sum_{k=2}^{\infty} b_k z^k, \quad b_k \neq 0, \quad (1.2)$$

the Hadamard product (or convolution) of f and Ψ is given by

$$(f * \Psi)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (\Psi * f)(z). \quad (1.3)$$

For the parameter λ ($\lambda > 0$) and an analytic function $f \in \mathcal{A}$, we obtain the following integral operator (see Irmak [3, with $p = 1$])

$$\mathcal{Q}^\lambda[f](z) = \frac{2^\lambda}{z\Gamma(\lambda)} \int_0^z \left[\log\left(\frac{z}{t}\right) \right]^{\lambda-1} f(t) dt, \quad (1.4)$$

where the function Γ is well-known gamma function. $f(z) \in \mathcal{A}$, we can easily determine that

$$\mathcal{Q}^\lambda[f](z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^\lambda a_k z^k \quad (\lambda > 0). \quad (1.5)$$

It is readily verified from (1.5) that

$$z \left(\mathcal{Q}^\lambda[f](z) \right)' = 2\mathcal{Q}^{\lambda-1}[f](z) - \mathcal{Q}^\lambda[f](z) \quad (\lambda > 0). \quad (1.6)$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1} , defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); r_0(f) \geq \frac{1}{4} \right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples in the class Σ (see [4]).

Brannan and Taha [5] (see also [31]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α ($0 \leq \alpha < 1$), respectively (see [4]). Thus, following Brannan and Taha [5], a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}_\Sigma^*(\alpha)$ of strongly bi-starlike functions of order α ($0 < \alpha \leq 1$) if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{and} \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; z \in \mathbb{U}) \quad (1.7)$$

and

$$\left| \arg \left(\frac{zg'(w)}{g(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; w \in \mathbb{U}), \quad (1.8)$$

where the function g is given by

$$g(w) = w - a_2 w^2 + (2a_2^2 - a_3) w^3 - (5a_2^3 - 5a_2 a_3 + a_4) w^4 + \cdots \quad (1.9)$$

and g is the extension of f^{-1} to $\mathbb{U}$. The classes $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α ($0 < \alpha \leq 1$), corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were also introduced analogously. For each of the function classes $\mathcal{S}_\Sigma^*(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [5] and [31]).

Recently, the subject of bi-univalent functions actually, revived by Srivastava et al. [29] by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1-3, 6-20, 24-38, 30, 32]) and including the references therein. The afore-cited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [29]. However, the problem to find the coefficient bounds on $|a_n|$ ($n = 3, 4, \dots$) for functions $f \in \Sigma$ is still an open problem.

Motivated in this line we consider the following comprehensive classes of bi-univalent function:

Definition 1.1. Let f given by (1.1) and Ψ given by (1.2), respectively, then f is said to be in the class $Q_{\Sigma}(\alpha; \Psi(z))$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left| \arg \left(\frac{z(f * \Psi)'(z)}{(f * \Psi)(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; z \in \mathbb{U}) \quad (1.10)$$

and

$$\left| \arg \left(\frac{w(g * \Psi)'(w)}{(g * \Psi)(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; w \in \mathbb{U}), \quad (1.11)$$

where the function g is given by (1.9).

Definition 2.2. Let f given by (1.1) and Ψ given by (1.2), respectively, then f is said to be in the class $\mathcal{P}_{\Sigma}(\beta; \Psi(z))$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \Re \left(\frac{z(f * \Psi)'(z)}{(f * \Psi)(z)} \right) > \beta \quad (0 \leq \beta < 1; z \in \mathbb{U}) \quad (1.12)$$

and

$$\Re \left(\frac{w(g * \Psi)'(w)}{(g * \Psi)(w)} \right) > \beta \quad (0 \leq \beta < 1; w \in \mathbb{U}), \quad (1.13)$$

where the function g is given by (1.9).

We note that for suitable choices of Ψ we obtain the following subclasses:

(i) Putting $\Psi(z) = \frac{z}{1-z}$ in Definition 1.1, we obtain the following subclass $Q_{\Sigma}(\alpha)$, defined as follows:

$$\left\{ f \in \Sigma \quad \text{and} \quad \left| \arg \left(\frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; z \in \mathbb{U}) \right\}$$

and

$$\left| \arg \left(\frac{wf'(w)}{f(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; w \in \mathbb{U}).$$

(ii) Putting $\Psi(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{\lambda} z^k$ in Definition (1.1), we obtain the following subclass $Q_{\Sigma}(\lambda; \alpha)$, defined as follows:

$$\left\{ f \in \Sigma \quad \text{and} \quad \left| \arg \left(\frac{z(\mathcal{Q}^{\lambda}[f](z))'}{\mathcal{Q}^{\lambda}[f](z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; \lambda > 0; z \in \mathbb{U}) \right\}$$

and

$$\left| \arg \left(\frac{w (\mathcal{Q}^\lambda [f](w))'}{\mathcal{Q}^\lambda [f](w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; \lambda > 0; w \in \mathbb{U}),$$

where $\mathcal{Q}^\lambda$ is defined by (1.5).

(iii) Putting $\Psi(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k$ ($m > 0$) in Definition 1.1, we obtain the following subclass $\mathcal{D}_\Sigma^m(\alpha)$, defined as follows:

$$\left\{ f \in \Sigma \text{ and } \left| \arg \left(\frac{z (\mathcal{I}^m f(z))'}{\mathcal{I}^m f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; m > 0; z \in \mathbb{U}) \right\}$$

and

$$\left| \arg \left(\frac{w (\mathcal{I}^m f(w))'}{\mathcal{I}^m f(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; m > 0; w \in \mathbb{U}),$$

where $\mathcal{I}^m$ is a Poisson distribution series [22].

(iv) Putting

$$\Psi(z) = z + \sum_{k=2}^{\infty} \left(\frac{\ell + \gamma(k-1) + 1}{1 + \ell} \right)^n z^k \quad (\gamma \geq 0; \ell > -1; n \in \mathbb{Z} = \{0, \pm 1, \dots\})$$

in Definition 1.1, we obtain the following subclass $\mathcal{P}_\Sigma^n(\gamma, \ell; \alpha)$, defined as follows:

$$\left\{ f \in \Sigma \text{ and } \left| \arg \left(\frac{z (\mathcal{J}^n(\gamma, \ell) f(z))'}{\mathcal{J}^n(\gamma, \ell) f(z)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; \gamma \geq 0; \ell > -1; n \in \mathbb{Z}; z \in \mathbb{U}) \right\},$$

and

$$\left| \arg \left(\frac{w (\mathcal{J}^n(\gamma, \ell) f(w))'}{\mathcal{J}^n(\gamma, \ell) f(w)} \right) \right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leq 1; \gamma \geq 0; \ell > -1; n \in \mathbb{Z}; w \in \mathbb{U}),$$

where $\mathcal{J}^n(\gamma, \ell)$ is the Prajapat operator [23].

To prove our results, we need the following lemmas.

Lemma 1.1. [21] *If $h \in P$ then $|d_k| \leq 2$ for each k , where P is the family of all functions h analytic in $\mathbb{U}$ for which $\Re(h(z)) > 0$, $h(z) = 1 + d_1 z + d_2 z^2 + \dots$ for $z \in \mathbb{U}$.*

Lemma 1.2. [32] *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < \mathcal{R}$ and $|z_2| < \mathcal{R}$, then*

$$|(k+l)z_1 + (k-l)z_2| \leq \begin{cases} 2|k|\mathcal{R} & ; |k| \geq |l|, \\ 2|l|\mathcal{R} & ; |k| \leq |l|. \end{cases} \quad (1.14)$$

2. Fekete-Szegő inequalities for the function class $Q_{\Sigma}(\alpha; \Psi(z))$

Unless otherwise mentioned, we assume throughout this paper that $0 < \alpha \leq 1$, $0 \leq \beta < 1$, $z \in \mathbb{U}$, λ , $m > 0$ and Ψ is given by (1.2).

Theorem 2.1. *Let f given by (1.1) belongs to the class $Q_{\Sigma}(\alpha; \Psi(z))$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{b_2^2 + \alpha[4b_3 - 3b_2^2]}}, \quad (2.1)$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha^2}{(\alpha+1)b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) & ; \frac{\alpha^2}{(\alpha+1)b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) > \frac{\alpha}{4b_3} \\ \frac{\alpha}{b_3} & ; \frac{\alpha^2}{(\alpha+1)b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) < \frac{\alpha}{4b_3} \end{cases}. \quad (2.2)$$

Proof. It follows from (1.10) and (1.11) that

$$\frac{z(f * \Psi)'(z)}{(f * \Psi)(z)} = [p(z)]^{\alpha} \quad (2.3)$$

and

$$\frac{w(g * \Psi)'(w)}{(g * \Psi)(w)} = [q(w)]^{\alpha}, \quad (2.4)$$

where p and q in P and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \dots \quad (2.5)$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \dots. \quad (2.6)$$

Now, equating the coefficients in (2.3) and (2.4), we get

$$a_2 b_2 = p_1 \alpha, \quad (2.7)$$

$$2a_3 b_3 - a_2^2 b_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \quad (2.8)$$

$$-a_2 b_2 = q_1 \alpha, \quad (2.9)$$

and

$$3a_2^2b_2^2 - 2a_3b_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \quad (2.10)$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \quad (2.11)$$

and

$$2a_2^2b_2^2 = \alpha^2(p_1^2 + q_1^2). \quad (2.12)$$

From (2.8), (2.10) and (2.12), we obtain

$$a_2^2 = \frac{\alpha^2(p_2 + q_2)}{b_2^2 + \alpha[4b_3 - 3b_2^2]}. \quad (2.13)$$

Applying Lemma 1.1 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \leq \frac{2\alpha}{\sqrt{b_2^2 + \alpha[4b_3 - 3b_2^2]}}.$$

This gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3 - \mu a_2^2|$, by summing and subtracting (2.8) and (2.10), we get

$$4a_3b_3 - 4a_2^2b_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2 - q_1^2). \quad (2.14)$$

It follows from (2.11), (2.12) and (2.14) that

$$a_3 = \frac{a_2^2b_2^2}{b_3} + \frac{\alpha(p_2 - q_2)}{4b_3}. \quad (2.15)$$

Applying (2.7) and (2.9), we dispose of p_1 and q_1 , we have

$$a_3 - \mu a_2^2 = p_2 \left[\frac{\alpha^2}{(\alpha+1)b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) + \frac{\alpha}{4b_3} \right] + q_2 \left[\frac{\alpha^2}{(\alpha+1)b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) - \frac{\alpha}{4b_3} \right].$$

From Lemma 1 and Lemma 1.2, we get the desired result (2.2). This completes the proof of Theorem 2.1.

Putting $\Psi(z) = \frac{z}{1-z}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let f given by (1.1) belongs to the class $Q_\Sigma(\alpha, \gamma)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}, \quad (2.16)$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha^2}{\alpha+1} (1-\mu) & ; \frac{\alpha^2}{(\alpha+1)} (1-\mu) > \frac{\alpha}{4} \\ \alpha & ; \frac{\alpha^2}{(\alpha+1)} (1-\mu) < \frac{\alpha}{4} \end{cases}. \quad (2.17)$$

Putting $\Psi(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^{\lambda} z^k$ ($\lambda > 0$) in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let f given by (1.1) belongs to the class $Q_{\Sigma}(\alpha; \lambda)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left(\frac{2}{3}\right)^{2\lambda} + \alpha \left[4 \left(\frac{2}{4}\right)^{\lambda} - 3 \left(\frac{2}{3}\right)^{2\lambda}\right]}}, \quad (2.18)$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha^2}{(\alpha+1) \left(\frac{2}{3}\right)^{2\lambda}} \left(\frac{\left(\frac{2}{3}\right)^{2\lambda}}{\left(\frac{2}{4}\right)^{\lambda}} - \mu \right) & ; \frac{\alpha^2}{(\alpha+1) \left(\frac{2}{3}\right)^{2\lambda}} \left(\frac{\left(\frac{2}{3}\right)^{2\lambda}}{\left(\frac{2}{4}\right)^{\lambda}} - \mu \right) > \frac{\alpha}{4 \left(\frac{2}{4}\right)^{\lambda}} \\ \frac{\alpha}{\left(\frac{2}{4}\right)^{\lambda}} & ; \frac{\alpha^2}{(\alpha+1) \left(\frac{2}{3}\right)^{2\lambda}} \left(\frac{\left(\frac{2}{3}\right)^{2\lambda}}{\left(\frac{2}{4}\right)^{\lambda}} - \mu \right) < \frac{\alpha}{4 \left(\frac{2}{4}\right)^{\lambda}} \end{cases}. \quad (2.19)$$

Putting $\Psi(z) = z + \sum_{k=2}^{\infty} \frac{m^{k-1}}{(k-1)!} e^{-m} z^k$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.3. *Let f given by (1.1) belongs to the class $\mathcal{D}_{\Sigma}^m(\alpha)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{m^2 e^{-2m} + \alpha [2m^2 e^{-m} - 3m^2 e^{-2m}]}}, \quad (2.20)$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha^2}{(\alpha+1)m^2e^{-2m}}(2e^{-m} - \mu) & ; \frac{\alpha^2}{(\alpha+1)m^2e^{-2m}}(2e^{-m} - \mu) > \frac{\alpha}{2m^2e^{-m}} \\ \frac{2\alpha}{m^2e^{-m}} & ; \frac{\alpha^2}{(\alpha+1)m^2e^{-2m}}(2e^{-m} - \mu) < \frac{\alpha}{2m^2e^{-m}} \end{cases}. \quad (2.21)$$

Putting $\Psi(z) = z + \sum_{k=2}^{\infty} \left(\frac{\ell+1+\gamma(k-1)}{1+\ell} \right)^n z^k$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.4. *Let f given by (1.1) belongs to the class $\mathcal{P}_{\Sigma}^n(\gamma, \ell; \alpha)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left(\left(\frac{\gamma+\ell+1}{1+\ell} \right)^n + \alpha \left[4 \left(\frac{2\gamma+\ell+1}{1+\ell} \right)^n - 3 \left(\frac{\gamma+\ell+1}{1+\ell} \right)^{2n} \right]}}}, \quad (2.22)$$

and

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{4\alpha^2}{(\alpha+1)} \left(\left(\frac{1+\ell}{2\gamma+\ell+1} \right)^n - \mu \left(\frac{1+\ell}{\gamma+\ell+1} \right)^{2n} \right); \\ \frac{\alpha^2}{(\alpha+1)} \left(\left(\frac{1+\ell}{2\gamma+\ell+1} \right)^n - \mu \left(\frac{1+\ell}{\gamma+\ell+1} \right)^{2n} \right) > \frac{\alpha}{4} \left(\frac{1+\ell}{2\gamma+\ell+1} \right)^n \\ \alpha \left(\frac{1+\ell}{2\gamma+\ell+1} \right)^n; \\ \frac{4\alpha^2}{(\alpha+1)} \left(\left(\frac{1+\ell}{2\gamma+\ell+1} \right)^n - \mu \left(\frac{1+\ell}{\gamma+\ell+1} \right)^{2n} \right) < \frac{\alpha}{4} \left(\frac{1+\ell}{2\gamma+\ell+1} \right)^n \end{cases}. \quad (2.23)$$

3. Fekete-Szegő inequalities for the function class $P_{\Sigma}(\beta; \Psi(z))$

This section begins by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $P_{\Sigma}(\beta; \Psi(z))$.

Theorem 3.1. *Let f given by (1.1) belongs to the class $P_{\Sigma}(\beta; \Psi(z))$, then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2b_3 - b_2^2}}, \quad (3.1)$$

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\gamma)^2 b_2^2} + \frac{(1-\beta)}{(1+2\gamma)b_3}. \quad (3.2)$$

and for $\mu \in \mathbb{R}$

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(1-\beta)}{2b_3 - b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) & ; \frac{1}{4b_3 - 2b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) \geq \frac{1}{4b_3} \\ \frac{1-\beta}{b_3} & ; \frac{1}{4b_3 - 2b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) \leq \frac{1}{4b_3} \end{cases}. \quad (3.3)$$

Proof. It follows from (1.12) and (1.13) that

$$\frac{z(f * \Psi)'(z)}{(f * \Psi)(z)} = \beta + (1 - \beta)p(z) \quad (3.4)$$

and

$$\frac{w(g * \Psi)'(w)}{(g * \Psi)(w)} = \beta + (1 - \beta)q(w), \quad (3.5)$$

where p and q have the forms (2.5) and (2.6), respectively. Equating the coefficients in (3.4) and (3.5), we get

$$a_2 b_2 = (1 - \beta)p_1, \quad (3.6)$$

$$2a_3 b_3 - a_2^2 b_2^2 = (1 - \beta)p_2, \quad (3.7)$$

$$-a_2 b_2 = (1 - \beta)q_1, \quad (3.8)$$

and

$$3a_2^2 b_2^2 - 2a_3 b_3 = (1 - \beta)q_2. \quad (3.9)$$

From (3.6) and (3.8), we get

$$p_1 = -q_1 \quad (3.10)$$

and

$$2a_2^2 b_2^2 = (1 - \beta)^2 (p_1^2 + q_1^2). \quad (3.11)$$

From (3.7) and (3.9), we obtain

$$a_2^2 = \frac{(1 - \beta)(p_2 + q_2)}{4b_3 - 2b_2^2}. \quad (3.12)$$

Applying Lemma 1.1 for the coefficients p_2 and q_2 , we immediately have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2b_3 - b_2^2}}.$$

This gives the bound on $|a_2|$ as asserted in (3.1).

Next, in order to find the bound on $|a_3 - \mu a_2^2|$, by summing and subtracting (3.7) and (3.9), we get

$$4a_3b_3 - 4a_2^2b_2^2 = (1 - \beta)(p_2 - q_2). \quad (3.13)$$

It follows from (3.10), (3.11) and (3.13) that

$$a_3 = \frac{a_2^2b_2^2}{b_3} + \frac{(1 - \beta)(p_2 - q_2)}{4b_3}. \quad (3.14)$$

We have

$$a_3 - \mu a_2^2 = p_2 \left[\frac{(1 - \beta)}{4b_3 - 2b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) + \frac{1 - \beta}{4b_3} \right] + q_2 \left[\frac{(1 - \beta)}{4b_3 - 2b_2^2} \left(\frac{b_2^2}{b_3} - \mu \right) - \frac{1 - \beta}{4b_3} \right].$$

From Lemma 1 and Lemma 1.2, we get the desired result (3.3). This completes the proof of Theorem 3.1.

Remark 3.1. By specializing the parameter b_k , we obtain various results for different operators defined in the introduction.

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