South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 2 (2020), pp. 219-240

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

EXISTENCE AND UNIQUENESS SOLUTIONS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE POINT CONDITIONS

Deepak Dhiman, Ashok Kumar and Lakshmi Narayan Mishra*

Department of Mathematics,
H. N. B. Garhwal University, Srinagar - 246174, INDIA

E-mail : deepakdhiman09@gmail.com, ashrsdma@gmail.com

*Department of Mathematics,
School of Advanced Sciences,
Vellore Institute of Technology (VIT) University
Vellore 632 014, Tamil Nadu, INDIA

E-mail : lakshminarayanmishra04@gmail.com

(**Received:** *Mar. 04, 2020* **Accepted:** *Jun. 07, 2020* **Published:** *Aug. 30, 2020)*

Abstract: In this article, we prove the existence of solutions of fractional integro-differential equations with infinite point conditions by using fractional calculus and fixed point theorems. Further continuous dependence on initial point, on nonlocal data, on the functional is also studied. Finally, the obtained results are verified with the help of some examples.

Keywords and Phrases: Functional-differential equations with fractional derivatives, Nonlinear differential equations in abstract spaces, Initial value problems, Fixed point theorems.

2010 Mathematics Subject Classification: 34K37, 34G60, 34A12, 47H10.

1. Introduction

The subject of fractional calculus and fractional differential equations is a rapidly growing area of mathematics. There are many applications of this subject in many field such as engineering, viscoelasticity, economics and biological sciences. There are many remarkable research articles in which theory regarding the existence and uniqueness of solutions established. One can see the research articles [3, 5, 11, 13, 15] for more details. The basic theory of fractional calculus and fractional differential can be found in many books like [4, 6, 9, 16, 19, 30, 32]. In the literature, it has been seen that functional integral and fractional differential equations are closely related. For detailed work one can see the references [8, 16, 26, 27]. Fixed point theory is a great tool to study the existence and uniqueness of solutions of fractional differential equations. Theory and applications of fixed point theory can be found in [1, 7, 14, 29] and the references therein. For some interesting recent work one can see the research articles [18, 20, 21, 22, 23, 26, 27, 28].

Very recently Al-Syed and Ahmad [2] discussed the existence of solutions for the following initial value problems of the functional integro-differential equation

$$\frac{du}{dt} = h(t, u(t), \int_0^t g(s, u(s))ds),$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_i u(\sigma_i) = u_0, \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T].$$

Motivated by this work we study fractional case of the above work and consider the following functional fractional integro-differential equations

$$^{C}D^{p}u(t) = h(t, u(t), \int_0^t g(s, u(s))ds) \tag{1.1}$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_i u(\sigma_i) = u_0, \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T]. \tag{1.2}$$

Where $^{C}D^{p}$ denotes the Caputo fractional derivative of order $p \in (0, 1]$, $t \in J = [0, T]$, $u : J \to X$, $C[J, X]$ denote the Banach space of all continuous functions from J to X with the norm $\|u\| = \sup_{t \in J} |u(t)|, h : J \times X \times X \to X; g : J \times X \to X$ are given functions. We will prove the existence and uniqueness of solution $u \in C[J, X]$, under certain conditions. Where X is the Banach space with the norm $\| \cdot \|$. Also we will study the continuous dependence of the solution on u_0, on the nonlocal-data q_j and on the functional g.

For application point of view, we also study the initial value problem (1.1)-(1.2) if $\sum_{i=1}^{n} q_i$ is convergent.

2. Preliminaries

Definition 2.1. *The fractional integral operator (in Riemann-Liouville sense) of order $p > 0$ of the function u is defined as*

$$I^p u(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s) ds,$$

where $\Gamma(.)$ denotes the Euler gamma function.

Definition 2.2. *We define the fractional derivative of u of order $p > 0$ in Caputo sense as*

$$^{C}D^p u(t) = \frac{1}{\Gamma(1-p)} \int_0^t (t-s)^{-p} u'(s) ds, \tag{2.1}$$

where $0 < p \leq 1$ and $u'(s) = \frac{du(s)}{ds}$.

Consider the initial value problem (1.1)-(1.2) with the following assumptions

H_1. Let $h : J \times X \times X \to X$ satisfies the Carathedory condition. There exist a function $\phi \in L^1[0,T]$ and a positive constant $k_1 > 0$, such that

$$|h(t,x,y)| \leq \phi(t) + k_1|x| + k_1|y|.$$

H_2. Let $g : J \times X \to X$ satisfies the Carathedory condition. There exist a function $\psi \in L^1[0,T]$ and a positive constant $k_2 > 0$, such that

$$|g(t,y)| \leq \psi(t) + k_2|y|.$$

H_3. $\sup\limits_{\sigma_i \in [0,1]} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \phi(s) ds \leq M_1$, $\sup\limits_{\sigma_i \in [0,1]} \int_0^{\sigma_i} \int_0^{s} (\sigma_i - s)^{p-1} \psi(\theta) d\theta ds \leq M_2$.

H_4. $\frac{1}{\Gamma(p+1)} \left(1 + E \sum_{i=1}^{n} q_i\right) \left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right) < 1$, where $E = (1 + \sum_{i=1}^{n} q_i)^{-1}$.

Definition 2.3. *A function $u \in C[J,X]$ is said to be the solution of the initial value problem (1.1)-(1.2) if it satisfies the equations (1.1)-(1.2).*

Lemma 2.4. *The solution of initial value problem (1.1)-(1.2) can be represented* *by the following integral equation*

$$u(t) = E\big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\big] + \frac{1}{\Gamma p} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds, \quad (2.2)$$

where $E = (1 + \sum_{i=1}^{n} q_i)^{-1}$.

Proof. Let u be a solution of the fractional initial value problem (1.1)-(1.2). Applying Riemann-Liouville operator on both sides of (1.1). We get

$$u(t) = u(0) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds. \quad (2.3)$$

Using the nonlocal condition (1.2), we get

$$\sum_{i=1}^{n} q_i u(\sigma_i) = u_0 \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds,$$

since, $\sum_{i=1}^{n} q_i u(\sigma_i) = u_0 - u(0)$, we get

$$u_0 - u(0) = u_0 \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds,$$

which gives

$$u(0) = \frac{1}{1 + \sum_{i=1}^{n} q_i} \big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\big]. \quad (2.4)$$

Using (2.3) and (2.4), we obtain

$$u(t) = \frac{1}{1 + \sum_{i=1}^{n} q_i} \big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\big] + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds.$$

3. Existence of Solution

Theorem 3.1. *Let the assumptions* $H_1 - H_4$ *are satisfied. Then initial value* *problem (1.1)-(1.2) has at least one solution* $u \in C[J, X]$.

Proof. Define the operator associated with the integral equation (2.2)

$$\begin{aligned} Fu(t) &= E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta)) d\theta) ds\Big] \\ &\quad + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta)) d\theta) ds. \end{aligned}$$

Let $Q_r = \{u \in \mathbb{R} : \|u\| \le r\}$, where $r = \frac{E|u_0| + \frac{1}{\Gamma(p+1)}\left(1 + E\sum_{i=1}^n q_i\right)\left(M_1 + k_1 M_2\right)}{1 - \left[\frac{1}{\Gamma(p+1)}\left(1 + E\sum_{i=1}^n q_i\right)\left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right)\right]}$, it is clear that Q_r is nonempty, closed, bounded and convex subset of $C[0, T]$. Then we have, for $u \in Q_r$

$$\begin{aligned} |Fu(t)| &\le E\Big[|u_0| + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} |h(s, u(s), \int_0^s g(\theta, u(\theta)) d\theta)| ds\Big] \\ &\quad + \frac{1}{\Gamma p} \int_0^t (t-s)^{p-1} |h(s, u(s), \int_0^s g(\theta, u(\theta)) d\theta)| ds \\ &\le E\Big[|u_0| + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Big(\phi(s) + k_1 |u(s)| + k_1 \int_0^s |g(\theta, u(\theta))| d\theta\Big) ds\Big] \\ &\quad + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big(\phi(s) + k_1 |u(s)| + \int_0^s |g(\theta, u(\theta))| d\theta\Big) ds \\ &\le E\Big[|u_0| \\ &\quad + \sum_{i=1}^{n} q_i \Big(\frac{M_1}{\Gamma(p+1)} + \frac{k_1 T^p r}{\Gamma(p+1)} + \frac{k_1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Big(\int_0^s (\psi(\theta) + k_2 |u(\theta)|) d\theta\Big) ds\Big)\Big] \\ &\quad + \frac{M_1}{\Gamma(p+1)} + \frac{k_1 T^p r}{\Gamma(p+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big(\int_0^s (\psi(\theta) + k_2 |u(\theta)|) d\theta\Big) ds \\ &\le E|u_0| + \frac{E}{\Gamma(p+1)} \sum_{i=1}^{n} q_i \Big(M_1 + k_1 T^p r + k_1 M_2 + \frac{k_1 k_2 T^{p+1} r}{p+1}\Big) \\ &\quad + \frac{1}{\Gamma(p+1)} \Big(M_1 + k_1 T^p r + k_1 M_2 + \frac{k_1 k_2 T^{p+1} r}{p+1}\Big) \\ &\le E|u_0| + \frac{1}{\Gamma(p+1)} \Big(1 + E\sum_{i=1}^{n} q_i\Big) \Big(M_1 + k_1 T^p r + k_1 M_2 + \frac{k_1 k_2 T^{p+1} r}{p+1}\Big) = r. \end{aligned}$$

Then $F : Q_r \to Q_r$ and the class of functions $\{Fu\}$ is uniformly bounded in Q_r.

Now, let $t_1, t_2 \in (0,1]$ such that $|t_2 - t_1| < \delta$, then

$$|Fu(t_2) - Fu(t_1)| = \frac{1}{\Gamma(p)}\bigg|\int_0^{t_2}(t_2-s)^{p-1}h\Big(s,u(s),\int_0^s g(\theta,u(\theta))d\theta\Big)ds$$

$$-\int_0^{t_1}(t_1-s)^{p-1}h\Big(s,u(s),\int_0^s g(\theta,u(\theta))d\theta\Big)ds\bigg|$$

$$\leq \frac{1}{\Gamma(p)}\int_0^{t_1}\bigg|\big[(t_2-s)^{p-1}-(t_1-s)^{p-1}\big]h\Big(s,u(s),\int_0^s g(\theta,u(\theta))d\theta\Big)\bigg|ds$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t_2}\bigg|(t_2-s)^{p-1}h\Big(s,u(s),\int_0^s g(\theta,u(\theta))d\theta\Big)\bigg|ds$$

$$\leq \frac{1}{\Gamma(p)}\int_0^{t_1}\big|(t_2-s)^{p-1}-(t_1-s)^{p-1}\big|$$

$$\times\Big(\phi(s)+k_1|u(s)|+k_1\int_0^s|g(\theta,u(\theta))|d\theta\Big)ds$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t_2}\big|(t_2-s)^{p-1}\big|\Big(\phi(s)+k_1|u(s)|+k_1\int_0^s|g(\theta,u(\theta))|d\theta\Big)ds$$

$$\leq \frac{1}{\Gamma(p)}\int_0^{t_1}\big[(t_2-s)^{p-1}-(t_1-s)^{p-1}\big]\phi(s)ds$$

$$+\frac{k_1 r}{\Gamma(p)}\int_0^{t_1}\big[(t_2-s)^{p-1}-(t_1-s)^{p-1}\big]ds$$

$$+\frac{k_1}{\Gamma(p)}\int_0^{t_1}\big[(t_2-s)^{p-1}-(t_1-s)^{p-1}\big]\int_0^s\big(\psi(\theta)+k_2|u(\theta)|\big)d\theta ds$$

$$+\frac{1}{\Gamma(p)}\int_{t_1}^{t_2}\big[(t_2-s)^{p-1}\big]\phi(s)ds$$

$$+\frac{k_1 r}{\Gamma(p)}\int_{t_1}^{t_2}\big[(t_2-s)^{p-1}\big]ds$$

$$+\frac{k_1}{\Gamma(p)}\int_{t_1}^{t_2}\big[(t_2-s)^{p-1}\int_0^s\big(\psi(\theta)+k_2|u(\theta)|\big)d\theta ds.$$

We see that $(t-s)^{p-1} \in L^{\frac{1}{1-p_1}}[0,t]$ for $t \in [0,T]$ and $p_1 \in [0,p)$. Let $d = \frac{p-1}{1-p_1}$, $N_1 = \|\phi(s)\|_{L^{\frac{1}{1-p_1}}[0,T]}$ and $N_2 = \|\int_0^s\big(\psi(\theta)+k_2|u(\theta)|\big)d\theta\|_{L^{\frac{1}{1-p_1}}[0,T]}$.
Now we apply the Hölder inequality [32]

$$|Fu(t_2) - Fu(t_1)| \leq \frac{1}{\Gamma(p)}\Big(\int_0^{t_1}\big((t_2-s)^{p-1}-(t_1-s)^{p-1}\big)^{\frac{1}{1-p_1}}ds\Big)^{1-p_1}\|\phi(s)\|_{L^{\frac{1}{1-p_1}}[0,t]}$$

$$
\begin{aligned}
& +\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p - t_2^p + t_1^p\big) \\
& +\frac{k_1}{\Gamma(p)}\bigg(\int_0^{t_1} \big((t_2-s)^{p-1}-(t_1-s)^{p-1}\big)^{\frac{1}{1-p_1}} ds\bigg)^{1-p_1} \\
& \times\Big\| \int_0^s \big(\psi(\theta)+k_2|u(\theta)|\big)d\theta\Big\|_{L^{\frac{1}{1-p_1}}[0,t]} \\
& +\frac{1}{\Gamma(p)}\bigg(\int_{t_1}^{t_2} (t_2-s)^{\frac{p-1}{1-p_1}} ds\bigg)^{1-p_1} \|\phi(s)\|_{L^{\frac{1}{1-p_1}}[0,T]} - \frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p \\
& +\frac{k_1}{\Gamma(p)}\bigg(\int_{t_1}^{t_2} (t_2-s)^{\frac{p-1}{1-p_1}} ds\bigg)^{1-p_1} \Big\| \int_0^s \big(\psi(\theta)+k_2|u(\theta)|\big)d\theta\Big\|_{L^{\frac{1}{1-p_1}}[0,t]} \\
\leq\ & \frac{N_1}{\Gamma(p)}\bigg(\int_0^{t_1} \big((t_2-s)^d-(t_1-s)^d\big)ds\bigg)^{1-p_1} \\
& +\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p - t_2^p + t_1^p\big) \\
& +\frac{k_1 N_2}{\Gamma(p)}\bigg(\int_0^{t_1} \big((t_2-s)^d-(t_1-s)^d\big)ds\bigg)^{1-p_1} \\
& -\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)} - \frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p \\
& -\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)} \\
\leq\ & \frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}\Big((t_1)^{1+d}-(t_2)^{1+d}+(t_2-t_1)^{1+d}\Big)^{1-p_1} \\
& +\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p - t_2^p + t_1^p\big) \\
& +\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}\Big((t_1)^{1+b}-(t_2)^{1+b}+(t_2-t_1)^{1+b}\Big)^{1-p_1} \\
& -\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)} - \frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p \\
& -\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)} \\
\leq\ & \frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}
\end{aligned}
$$

$$+\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p - t_2^p + t_1^p\big)$$
$$+\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}$$
$$-\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)} - \frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p$$
$$-\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}.$$

Which shows that the class of functions $\{Fu\}$ is equi-continuous in Q_r.
Let $u_n \in Q_r$ such that $u_n \to u$ as $n \to \infty$. Then by the assumption $H_1 - H_2$, it is clear that $h(t, u_n(t), v_n(t)) \to h(t, u(t), v(t))$ and $g(t, u_n(t)) \to g(t, u(t))$. Also

$$\lim_{n\to\infty} Fu_n(t) = \lim_{n\to\infty}\Bigg[E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Bigg] + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} g(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Bigg]. \tag{3.1}$$

By using assumption $H_1 - H_2$ and Lebesgue Dominated convergence Theorem [12], from (3.1) we obtain

$$\lim_{n\to\infty} Fu_n(t) = \lim_{n\to\infty}\Bigg[E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Bigg] + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Bigg] = Fu(t).$$

Which shows that $Fu_n \to Fu$ as $n \to \infty$. Therefore F is continuous.

$$\lim_{t\to 0} u(t) = E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Bigg] \in C[0,T].$$

Then by Schauder fixed point Theorem [1] there exist at least one solution $u \in C[J, X]$ of the integral equation (2.2).

4. Infinite-Point Boundary Condition

Theorem 4.1. *Let assumption $H_1 - H_4$ are satisfied and*

$$M = M_1 + \frac{k_1\|u\|}{p} + k_2 M_2 + \frac{k_1 k_2 \|u\|}{p(p+1)}.$$

Then the initial value problem (1.1)-(1.2) has at least one solution $u \in C[J, X]$.
Proof. Let the assumptions of Theorem 3.1 be satisfied. Let $t_n, t_n = \sum_{i=1}^{n} q_i$ be convergent sequence, then

$$u_n(t) = \frac{1}{1+\sum_{i=1}^{n} q_i}\left[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\right] + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds. \tag{4.1}$$

Taking the limit to (4.1), as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} u_n(t) &= \lim_{n\to\infty}\Bigg[\frac{1}{1+\sum_{i=1}^{n} q_i}\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u(s), \\ &\int_0^s g(\theta, u(\theta))d\theta)ds\Bigg] + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Bigg] \\ &= \lim_{n\to\infty}\frac{1}{1+\sum_{i=1}^{n} q_i}\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u(s), \\ &\int_0^s g(\theta, u(\theta))d\theta)ds\Bigg] + \lim_{n\to\infty}\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds.\end{aligned} \tag{4.2}$$

Now $|q_i u(\sigma_i)| \leq |q_i|\|u\|$, therefore by the comparison test $\sum_{i=1}^{\infty} q_i u(\sigma_i)$ is convergent. Also

$$\begin{aligned}\left|\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\right| &\leq \int_0^{\sigma_i}(\sigma_i - s)^{p-1}\big(\phi(s) + k_1|u(s)| \\ &\quad + k_1\int_0^s g(\theta, u(\theta)d\theta)\big)ds \\ &\leq \int_0^{\sigma_i}(\sigma_i - s)^{p-1}\big(\phi(s) + k_1|u(s)| \\ &\quad + k_2\int_0^s (\psi(s) + k_2|u(s)|)d\theta\big)ds \\ &\leq M_1 + \frac{k_1\|u\|}{p} + k_2 M_2 + \frac{k_1 k_2\|u\|}{p(p+1)} \\ &\leq M,\end{aligned}$$

then $\left|q_i \int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\right| \leq |q_i|.M$ and by the comparison test $\sum_{i=1}^{\infty} q_i \int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds$ is convergent.
Now, using assumption $H_1 - H_2$ and Lebesgue Dominated convergence Theorem [12], from (4.2) we obtain

$$u(t) = \frac{1}{1+\sum_{i=1}^{\infty} q_i}\left[u_0 - \sum_{i=1}^{\infty} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\right] + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds. \tag{4.3}$$

Hence, the theorem is proved.

5. Uniqueness of Solution

Consider the following assumptions
H_5. Let $h : J \times X \times X \to X$ is measurable in t for any $x, y \in X$ and satisfies the Lipschitz condition

$$|h(t,x,y) - h(t,u,v)| \leq k_1|x-u| + k_1|y-v|, \tag{5.1}$$

H_6. Let $g : J \times X \to X$ is measurable in t for any $x \in X$ and satisfies the Lipschitz condition

$$|g(t,x) - g(t,u)| \leq k_2|x-u|, \tag{5.2}$$

H_7. Let there exists constants L_1 and L_2 such that

$$\sup_{\sigma_i\in[0,T]}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}|g(s,0,0)|ds \leq L_1, \ \sup_{\sigma_i\in[0,T]}\int_0^{\sigma_i}\int_0^s(\sigma_i - s)^{p-1}|h(s,0)|d\theta ds \leq L_2,$$

Theorem 5.1. *Let the assumptions H_5-H_7 are satisfied. Then the initial value problem (1.1)-(1.2) has a unique solution.*
Proof. From assumption H_5 we have h is measurable in t for any $u, v \in \mathbb{R}$ and satisfies the lipschitz condition, then it is continuous for $x, y \in \mathbb{R}, \forall t \in [0,T]$, and

$$|h(t,x,y)| \leq k_1|x| + k_1|y| + |g(t,0,0)|.$$

Which shows that assumption H_1 is satisfied. In a similar way, we can show that assumption H_2 is also satisfied with the help of assumption H_6. Therefore Theorem

3.1 ensures the existence of solution of initial value problem (1.1)-(1.2). Let u, v be two solutions of (1.1)-(1.2), then

$$\begin{aligned}
|u(t)-v(t)| &= \Bigg|E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\Bigg] \\
&\quad + \frac{1}{\Gamma p}\int_0^t (t-s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds \\
&\quad - E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,v(s),\int_0^s g(\theta,v(\theta))d\theta)ds\Bigg] \\
&\quad + \frac{1}{\Gamma p}\int_0^t (t-s)^{p-1}h(s,v(s),\int_0^s g(\theta,v(\theta))d\theta)ds\Bigg| \\
&\le E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}\Bigg|h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta) \\
&\quad - h(s,v(s),\int_0^s g(\theta,v(\theta))d\theta)\Bigg|ds \\
&\quad + \frac{1}{\Gamma p}\int_0^t (t-s)^{p-1}\Bigg|h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta) - h(s,v(s),\int_0^s g(\theta,v(\theta))d\theta)\Bigg|ds \\
&\le E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}\left(k_1\|u-v\| + k_1\int_0^s |g(\theta,u(\theta)) - g(\theta,v(\theta))|d\theta\right)ds \\
&\quad + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}\left(k_1\|u-v\| + k_1\int_0^s |g(\theta,u(\theta)) - g(\theta,v(\theta))|d\theta\right)ds \\
&\le \frac{k_1 E T^p \|u-v\| \sum_{i=1}^{n} q_i}{\Gamma(p+1)} + \frac{k_1 k_2 E T^{p+1}\|u-v\|\sum_{i=1}^{n} q_i}{\Gamma(p+2)} \\
&\quad + \frac{k_1 T^p\|u-v\|}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1}\|u-v\|}{\Gamma(p+2)} \\
&= \frac{1}{\Gamma(p+1)}\left(1 + E\sum_{i=1}^{n} q_i\right)\left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right)\|u-v\|.
\end{aligned}$$

Which gives

$$\left(1 - \frac{1}{\Gamma(p+1)}\left(1 + E\sum_{i=1}^{n} q_i\right)\left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right)\right)\|u-v\| \le 0.$$

Since $\frac{1}{\Gamma(p+1)}\left(1 + E\sum_{i=1}^{n} q_i\right)\left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right) < 1$, therefore $u(t) = v(t)$ and the solution of the initial value problem (1.1)-(1.2) is unique.

6. Continuous Dependence

6.1. Continuous Dependence on u_0

Definition 6.1. *Let u^* is the solution of the initial value problem*

$$^{C}D^{p}u^*(t) = h(t, u^*(t), \int_0^t f(s, u^*(s))ds) \tag{6.1}$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_i u^*(\sigma_i) = u_0^*, \quad \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T]. \tag{6.2}$$

Then, the solution $u \in C[J, X]$ of initial value problem (1.1)-(1.2) is said to be continuously depends on u_0, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) > 0 \quad s.t. \quad |u_0 - u_0^*| < \delta \implies \|u - u^*\| < \epsilon.$$

Theorem 6.2. *Let the assumptions H_5-H_7 are satisfied. Then the solution of initial value problem (1.1)-(1.2) continuously depends on u_0.*

Proof. Let u, u^* be two solutions of the initial value problem (1.1)-(1.2) and (6.1)-(6.2) respectively. Then

$$\begin{aligned}
|u(t) - u^*(t)| = &\Bigg| E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Bigg] \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&- E\Bigg[u_0^* - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds\Bigg] \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds\Bigg| \\
\leq \;& E|u_0 - u_0^*| + E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&- h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds\Bigg| + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta \\
&- h(s, u^*(s), \int_0^s f(\theta, u^*(\theta))d\theta\Bigg| ds
\end{aligned}$$

$$\begin{aligned}
\leq\ & E|u_0 - u_0^*| + E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1}\Big(k_1\|u-u^*\| \\
& + k_2 \int_0^s |g(\theta, u(\theta)) - g(\theta, u^*(\theta))| d\theta\Big) ds + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1}\Big(k_1\|u-u^*\| \\
& + k_2 \int_0^s |g(\theta, u(\theta)) - g(\theta, u^*(\theta))| d\theta\Big) ds \\
\leq\ & E|u_0 - u_0^*| + \frac{k_1 T^p \|u-u^*\| E\sum_{i=1}^n q_i}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1} \|u-u^*\| E\sum_{i=1}^n q_i}{\Gamma(p+2)} \\
& + \frac{k_1 T^p \|u-u^*\|}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1}\|u-u^*\|}{\Gamma(p+2)} \\
\leq\ & E\delta + \frac{1}{\Gamma(p+1)}\Big(1 + E\sum_{i=1}^n q_i\Big)\Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\Big)\|u-u^*\|.
\end{aligned}$$

Which gives

$$\|u-u^*\| \leq \frac{E\delta}{\Big[1 - \frac{1}{\Gamma(p+1)}\Big(1 + E\sum_{i=1}^n q_i\Big)\Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{\alpha+1}\Big)\Big]} = \epsilon$$

Thus, the solution of initial value problem (1.1)-(1.2) continuously depends on u_0^*.

6.2. Continuous Dependence on the Nonlocal Data q_i

Definition 6.3. *Let u^* is the solution of the initial value problem*

$$^{C}D^p u^*(t) = h(t, u^*(t), \int_0^t g(s, u^*(s)) ds) \tag{6.3}$$

with nonlocal condition

$$u(0) + \sum_{i=1}^n q_i^* u^*(\sigma_i) = u_0, \quad \sum_{i=1}^n q_i^* > 0, \quad \sigma_i \in (0, T]. \tag{6.4}$$

Then, the solution $u \in C[J, X]$ of initial value problem (1.1)-(1.2) is said to be continuously depends on nonlocal data q_i, if

$$\forall \epsilon > 0, \quad \exists \ \ \delta(\epsilon) > 0 \ \ s.t. \ \ |u_0 - u_0^*| < \delta \implies \|u - u^*\| < \epsilon.$$

Theorem 6.4. *Let the assumptions H_5-H_7 are satisfied. Then the solution of initial value problem (1.1)-(1.2) continuously depends on the nonlocal data q_i.*

Proof. Let u, u^* be two solutions of the initial value problem (1.1)-(1.2) and (6.3)-(6.4) respectively. Then

$$\begin{aligned}
&|u(t)-u^*(t)| = \Big|E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\Big] \\
&+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds \\
&-E^*\Big[u_0^* - \sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds\Big] \\
&+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds\Big| \\
\leq\ & EE^* n\delta|u_0| + \Big|E^*\sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds \\
&-E\sum_{j=1}^{n} q_i^* \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds \\
&+E\Big[\sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds \\
&-\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\Big]\Big| \\
&+\frac{k_1T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}\|u-u^*\|}{\Gamma(p+2)} \\
\leq\ & EE^* n\delta|u_0| + n\delta\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{\alpha+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big]\sum_{i=1}^{n} q_i \\
&+E\Big|\Big[\sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds \\
&-\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds \\
&+\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u^*(s),\int_0^s g(\theta,u^*(\theta))d\theta)ds \\
&-\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\Big]\Big| \\
&+\frac{k_1T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}\|u-u^*\|}{\Gamma(p+2)}
\end{aligned}$$

$$
\begin{aligned}
&\leq \quad EE^*n\delta|u_0| + n\delta\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{\alpha+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big]\sum_{i=1}^{n} q_i \\
&\quad +E\Big[n\delta\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big] \\
&\quad +\sum_{i=1}^{n} q_i\Big(\frac{k_1T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}\|u-u^*\|}{\Gamma(p+2)}\Big)\Big] \\
&\quad +\frac{k_1T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}\|u-u^*\|}{\Gamma(p+2)} \\
&\leq \quad EE^*n\delta|u_0| + n\delta\Big[\Big(\frac{a_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big] \\
&\quad \Big(E + \sum_{i=1}^{n} q_i\Big) + \Big(1 + E\sum_{i=1}^{n} q_i\Big)\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u-u^*\|.
\end{aligned}
$$

Hence

$$
\|u-u^*\| \leq \frac{EE^*n\|u_0\| + n\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big]\Big(E + \sum_{i=1}^{n} q_i\Big) + \Big(1 + E\sum_{i=1}^{n} q_i\Big)}{1 - \Big(1 + E\sum_{i=1}^{n} q_i\Big)\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)}\delta = \epsilon,
$$

where $E^* = (1 + \sum_{i=1}^{n} q_i^*)^{-1}$. Then the solution of the initial value problem (1.1)-(1.2) continuously depends on the nonlocal data q_i.

6.3. Continuous Dependence on the Functional g

Definition 6.5. *Let u^* is the solution of the initial value problem*

$$
{}^CD^pu^*(t) = h(t, u^*(t), \int_0^t g^*(s, u^*(s))ds) \tag{6.5}
$$

with nonlocal condition

$$
u(0) + \sum_{i=1}^{n} q_iu^*(\sigma_i) = u_0, \quad \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T]. \tag{6.6}
$$

Then, the solution $u \in C[J, X]$ of initial value problem (1.1)-(1.2) is said to be continuously depends on the functional g, if

$$
\forall\epsilon > 0, \quad \exists \quad \delta(\epsilon) > 0 \quad s.t. \quad |g - g^*| < \delta \implies \|u - u^*\| < \epsilon.
$$

Theorem 6.6. *Let the assumptions H_5-H_7 are satisfied. Then the solution of initial value problem (1.1)-(1.2) continuously depends on the functional g.*

Proof. Let u, u^* be two solutions of the initial value problem (1.1)-(1.2) and (6.5)-(6.6) respectively. Then

$$\begin{aligned}
|u(t)-u^*(t)| &= \Big| E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big] \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&- E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta)ds\Big] \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta)ds \Big| \\
\leq\ & E \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Big| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta) \\
&- h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta) \Big| ds + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta \\
&- h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta \Big| ds \\
\leq\ & E \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Big(k_1 \|u-u^*\| + k_1 \int_0^s |g(\theta, u(\theta)) - g^*(\theta, u^*(\theta))| d\theta \Big) ds \\
&+ \int_0^t (t-s)^{p-1} \Big(k_1 \|u-u^*\| + k_1 \int_0^s |g(\theta, u(\theta)) - g^*(\theta, u^*(\theta))| d\theta \Big) ds \\
\leq\ & \frac{1}{\Gamma(p+1)} \Big(1 + E \sum_{i=1}^{n} q_i \Big) \frac{k_1 T^{p+1} \delta}{p+1} \\
&+ \frac{1}{\Gamma(p+1)} \Big(1 + E \sum_{i=1}^{n} q_i \Big) \Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1} \Big) \|u-u^*\|.
\end{aligned}$$

Hence

$$\|u-u^*\| \leq \frac{\frac{1}{\Gamma(p+1)} \Big(1 + E \sum_{i=1}^{n} q_i \Big) \frac{k_1 T^{p+1} \delta}{p+1}}{1 - \frac{1}{\Gamma(p+1)} \Big(1 + E \sum_{i=1}^{n} q_i \Big) \Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1} \Big)} = \epsilon.$$

Then the solution of the initial value problem (1.1)-(1.2) continuously depends upon the functional g.

7. Examples

Example 7.1. Consider the following nonlinear fractional order integro-differential equation for $p \in (0,1]$

$$^{C}D^{p}u(t) = (1+t)^2 + \frac{u(t)}{3+t^2} + \int_0^t \frac{1}{4}\left(\sin(2s+2) + \frac{s^2u(s)}{3(1+u(s))}\right)ds, \quad a.e\ t \in (0,1] \tag{7.1}$$

with infinite point boundary condition

$$u(0) + \sum_{i=1}^{\infty} \frac{1}{3^i} u\left(\frac{i}{i+1}\right) = u_0. \tag{7.2}$$

Set

$$h(t, u(t), \int_0^t g(s,u(s))ds) = (1+t)^2 + \frac{u(t)}{3+t^2} + \int_0^t \frac{1}{4}\left(\sin(2s+2) + \frac{s^2u(s)}{3(1+u(s))}\right)ds.$$

Then

$$\begin{aligned} |h(t,u(t), \int_0^t g(s,u(s))ds)| &= (1+t)^2 \\ &+\frac{1}{3}\left(|u(t)| + \int_0^t \frac{3}{4}\left|\left(\sin(2s+2) + \frac{s^2u(s)}{3(1+u(s))}\right)\right|ds\right), \end{aligned}$$

and also

$$|g(s,u(s))| = \frac{3}{4}|\sin(2s+2)| + \frac{3}{12}|u(s)|.$$

With $\phi(t) = (1+t)^2 \in L^1[0,1]$, $\varphi(t) = \frac{3}{4}|\sin(2s+2)| \in L^1[0,1]$, $k_1 = \frac{1}{3}, k_2 = \frac{3}{12}$, $\frac{1}{\Gamma(\alpha+1)}\left(1 + E\sum_{i=1}^{n} q_i\right)\left(k_1T^p + \frac{k_1k_2T^{p+1}}{p+1}\right) = \frac{1}{\Gamma(p+1)}\left(1 + \frac{\frac{1}{2}}{1+\frac{1}{2}}\right)\left(\frac{1}{3} + \frac{\frac{1}{3}\cdot\frac{3}{12}}{p+1}\right) < 1$, $\forall p \in (0,1]$, all the assumption $H_1 - H_4$ of Theorem 3.1 are satisfied. Therefore by applying the Theorem 3.1 with convergent series $\sum_{i=1}^{\infty} \frac{1}{3^i}$, IVP (7.1)-(7.2) has a solution u.

Example 7.2. Consider the following nonlinear fractional order integro-differential equation for $\alpha \in (0,1]$

$$^{C}D^{p}u(t) = t^7 + t^3e^{-2t} + 1 + \frac{u(t)}{t+2}$$

$$+\int_0^t \frac{1}{5}\left(\cos^2(2s+2)+\frac{s^3u(s)}{4e^{|u(s)|}}\right)ds, \quad a.e\ t\in(0,1], \tag{7.3}$$

with infinite point boundary condition

$$u(0)+\sum_{i=1}^{\infty}\frac{1}{4^i}u\left(\frac{i^3+i^2-1}{i^3+i^2}\right)=u_0. \tag{7.4}$$

Set

$$\begin{aligned} h(t,u(t),\int_0^t g(s,u(s))ds) &= t^7+t^3e^{-2t}+1+\frac{u(t)}{t+2} \\ &\quad +\int_0^t \frac{1}{5}\left(\cos^2(2s+2)+\frac{s^3u(s)}{4e^{|u(s)|}}\right)ds. \end{aligned}$$

Then

$$\begin{aligned} |h(t,u(t),\int_0^t g(s,u(s))ds)| &\le t^7+t^3e^{-2t}+1 \\ &\quad +\frac{1}{2}\left(|u|+\frac{2}{5}\int_0^t\left|\left(\cos^2(2s+2)+\frac{s^3u(s)}{4e^{|u(s)|}}\right)\right|ds\right) \end{aligned}$$

and also

$$|g(s,u(s))|=\frac{2}{5}|\cos^2(2s+2)|+\frac{1}{10}|u|.$$

All the assumption H_1-H_4 of Theorem 3.1 are satisfied with $\phi(t)=t^7+t^3e^{-2t}+1\in L^1[0,1]$, $\varphi(t)=\frac{2}{5}|\cos^2(2s+2)|\in L^1[0,1]$, $k_1=\frac{1}{2}, k_2=\frac{1}{10}$, $\frac{1}{\Gamma(\alpha+1)}\left(1+E\sum_{i=1}^n q_i\right)\left(k_1T^p+\frac{k_1k_2T^{p+1}}{p+1}\right)=\frac{1}{\Gamma(p+1)}\left(1+\frac{\frac{1}{3}}{1+\frac{1}{3}}\right)\left(\frac{1}{3}+\frac{\frac{1}{3}\cdot\frac{1}{10}}{p+1}\right)<1$, $\forall p\in(0,1]$, all the assumption H_1-H_4 of Theorem 3.1 are satisfied. Therefore by applying the Theorem 3.1 with convergent series $\sum_{i=1}^{\infty}\frac{1}{4^i}$, IVP (7.1)-(7.2) has a solution u.

8. Conclusion

In this paper Caputo fractional differential equations are studied with infinite point boundary conditions. The statement of the initial value problem is set up and an interpretation of the solutions is given. Further continuous dependence on initial point, on nonlocal data, on the functional is also studied. The fixed point theorems are used to prove main results. The obtained results are verified by some examples.

References

[1] R. P. Agarwal, M. Meehan and D. O'regan, Fixed point theory and applications, Cambridge university press, 141, (2001).

[2] R. G. Ahmed, Existence of Solutions for a Functional Integro-Differential Equations, International Journal of Applied and Computational Mathematics, Springer India, 123, (2019).

[3] K. Balachandran and J. Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, Nonlinear Analysis, Theory, Methods and Applications, (2009).

[4] M. Benchohra, A. Cabada and J. Henderson, Fractional differential equations and their applications, (2013).

[5] A. Chadha and D. N. Pandey, Existence results for an impulsive neutral fractional integro-differential equation with infinite delay, International Journal of Differential Equations, (2014).

[6] S. Das, Functional Fractional Calculus, (2011).

[7] Deepmala, A Study on Fixed Point Theorems for Nonlinear Contractions and its Applications, Ph. D. Thesis, (2014).

[8] Deepmala and H. K. Pathak, A study on some problems on existence of solutions for nonlinear functional-integral equations, Acta Mathematica Scientia Elsevier, 33 (5) (2013), 1305-1313.

[9] K. Diethelm, The Analysis of Fractional Differential Equations, Lecture Notes in Mathematics, (2011).

[10] R. Dubey, Deepmala and V. N. Mishra, Higher-order symmetric duality in nondifferentiable multiobjective fractional programming problem over cone constraints, Stat. Optim. Inf. Comput., 8 (2020), 187-205.

[11] A. M. A. El-Sayed, On the fractional differential equations, (1992).

[12] D. J. H. Garling, A. N. Kolmogorov, S. V. Fomin and R. A. Silverman, Introductory Real Analysis, The Mathematical Gazette, (2009).

[13] G. R. Gautam and J. Dabas, A study on existence of solutions for fractional functional differential equations, (2018).

[14] A. Granas and J. Dugundji, Fixed point theory, Springer Science & Business Media, (2013).

[15] P. Kumar, D. N. Pandey and D. Bahuguna, On a new class of abstract impulsive functional differential equations of fractional order, (2014).

[16] V. Lakshmikantham, Theory of fractional functional differential equations, Nonlinear Analysis, Theory, Methods and Applications, (2008).

[17] C. Li, D. Qian and Y. Chen, On Riemann-Liouville and Caputo derivatives, (2011).

[18] X. Liu, M. Zhou, L. N. Mishra, V. N. Mishra and B. Damjanović, Common fixed point theorem of six self-mappings in Menger spaces using (CLR_{ST}) property, Open Mathematics, 16(2018), 1423-1434.

[19] K. S. Miller and B. Ross, An introduction to the fractional calculus and fractional differential equations, John Wiley and Sons New York, (1993).

[20] L. N. Mishra, S. K. Tiwari, V. N. Mishra and I. A. Khan, Unique Fixed Point Theorems for Generalized Contractive Mappings in Partial Metric Spaces, Journal of Function Spaces, Volume 2015, Article ID 960827, 8 pages, (2015).

[21] L. N. Mishra, S. K. Tiwari and V. N. Mishra, Fixed point theorems for generalized weakly S-contractive mappings in partial metric spaces, Journal Of Applied Analysis and Computation, 5 (4) (2015), 600-612.

[22] L. N. Mishra and M. Sen, On the concept of existence and local attractivity of solutions for some quadratic Volterra integral equation of fractional order, Applied Mathematics and Computation, Elsevier, 285(2016), 174-183.

[23] L. N. Mishra, R. P. Agarwal and M. Sen, Solvability and asymptotic behavior for some nonlinear quadratic integral equation involving Erdélyi-Kober fractional integrals on the unbounded interval, Progress in Fractional Differentiation and Applications, 2 (3) (2016), 153-168.

[24] L. N. Mishra, H. M. Srivastava and M. Sen, Existence results for some nonlinear functional-integral equations in Banach algebra with applications, International Journal of Analysis and Applications, 11 (1) (2016), 1-10.

[25] L. N. Mishra, M. Sen and R. N. Mohpatra, On existence theorems for some generalized nonlinear functional-integral equations with applications, Filomat, 31 (7) (2017), 2081-2091.

[26] L. N. Mishra and R. P. Agarwal, On existence theorems for some nonlinear functional-integral equations, Dynamic systems and Applications, 25 (3) (2016), 303-320.

[27] L. N. Mishra, On existence and behavior of solutions to some nonlinear integral equations with Applications, National Institute of Technology, Silchar, 788 (010), (2017).

[28] L. N. Mishra, V. N. Mishra, P. Gautam and K. Negi, Fixed Point Theorems for Cyclic-Ćirić-Reich-Rus contraction mappings in Quasi-Partial b-metric spaces, Scientific Publication of the State University of Novi Pazar Ser. A, Appl. Math. Inform. and Mech., 12 (1) (2020), 47-56.

[29] H. K. Pathak and Deepmala, Common fixed point theorems for PD-operator pairs under relaxed conditions with applications, Journal of Computational and Applied Mathematics, Elsevier, 239(2013), 103-113.

[30] I. Podlubny, Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, Methods of their Solution and some of their Applications, (1999).

[31] D. Seba, An existence result for nonlinear fractional differential equations on Banach spaces, Boundary Value Problems, (2009).

[32] Z. Yong, J. Wang and L. Zhang, Basic theory of fractional differential equations, World Scientific, (2016).