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# SOME COMMON FIXED POINT RESULTS IN 2-BANACH SPACES

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**Abstract:** In this paper, we have proved some common fixed point theorems of a family of self maps without continuity in 2-Banach space. We have used functions on ${\mathbb{R}_+}^5$ to $\mathbb{R}_+$ and also generalize many existing results.

**Keywords and Phrases:** 2-norm, 2-Banach.

**2010 Mathematics Subject Classification:** 54H25, 47H10.

## 1. Introduction

In 1965, Gahler ([5], [6]) introduced 2-Banach space and Iseki [7] obtained some results on fixed point theorems in 2-Banach spaces. After the introduction of 2-Banach space many research workers have extended fixed point theorems of metric, Banach spaces etc. in the new setup of 2-Banach spaces. Mishra et al. [10], Khan and Khan [8], Saha et al. [12], Mishra et al. [11], Saluja [13], Saluja and Dhakde [14], Das et al. [1], Shrivas [15], Das et al, [2] - [3], Liu et al. [9] and etc. have worked on fixed point and common fixed point theorems in this space. In this paper we also have proved some unique common fixed point theorems in 2-Banach spaces.

## 2. Definitions and Preliminaries

Gahler [5] has introduced the notion of 2-norm as follows:

**2-norm:** Let $X$ be a linear space and $\|.,.\|$ is a real valued function defined on $X$ where

i) $\|a,b\| = 0$ if and only if $a$ and $b$ are linearly dependent;

ii) $\|a,b\| = \|b,a\|$;

iii) $\|a,xb\| = |x|\ \|a,b\|$;

iv) $\|a,b+c\| \leq \|a,b\| + \|a,c\|$

for all $a,b,c \in X$ and $x \in \mathbb{R}$. Then $\|.,.\|$ is called a 2-norm and the pair $(X,\|.,.\|)$ is called a 2-norm space.

In this paper, we denote $X$ as a 2-normed space unless otherwise stated.

**Convergent:** A sequence $\{x_n\}$ in a 2-norm space $X$ is said to be convergent if there is a point $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$ for all $a \in X$.

**Cauchy Sequence:** A sequence $\{x_n\}$ in a 2-norm space $X$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$ for all $a \in X$.

**2-Banach Space:** A linear 2-norm space is said to be complete if every Cauchy sequence in $X$ is convergent in $X$. Then we say $X$ is a 2-Banach Space.

Let us consider a function $f : {\mathbb{R}_+}^5 \to \mathbb{R}_+$ given by

$$f(t_1,t_2,t_3,t_4,t_5) = \max\{t_1, \frac{t_2+t_3}{2}, \frac{t_4+t_5}{2}\}; \tag{2.1}$$

$$f(t_1,t_2,t_3,t_4,t_5) = \max\{\frac{t_1+t_2+t_3}{3}, \frac{t_4+t_5}{3}\}. \tag{2.2}$$

## 3. Main Part

In this part we have proved some unique common fixed point theorems in 2-Banach spaces.

**Theorem 3.1.** *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X,\|.,.\|)$ satisfying*
$\|F_ix - F_jy, p\| \leq \alpha f(\|x-y,p\|, \|x-F_ix,p\|, \|y-F_jy,p\|, \|x-F_jy,p\|, \|y-F_ix,p\|)$, *where $\alpha < 1$ and $f$ satisfies the relation (2.1). Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.*

**Proof.** Let $\{x_n\}$ be sequence of points of $X$ given by $x_{n+1} = F_ix_n$ with the initial approximation $x_0 \in X$ for a fixed $i$. If $F_ix_n = x_n$ i.e., $x_{n+1} = x_n$, then $x_n$ is a common fixed point of $\{F_n\}$. So without loss of generality assume $x_{n+1} \neq x_n$.

We now show that $\lim_{n\to\infty} \|x_n - x, p\| = 0$.

Since,

$\|x_{n+1} - x_n, p\| = \|F_ix_n - F_jx_{n-1}, p\|$
$\leq \alpha f(\|x_n - x_{n-1}, p\|, \|x_n - F_ix_n, p\|, \|x_{n-1} - F_jx_{n-1}, p\|, \|x_n - F_jx_{n-1}, p\|, \|x_{n-1} -$

$F_i x_n, p\|)$
$= \alpha f(\|x_n - x_{n-1}, p\|, \|x_n - x_{n+1}, p\|, \|x_{n-1} - x_n, p\|, \|x_n - x_n, p\|, \|x_{n-1} - x_{n+1}, p\|)$
$= \alpha \max\{\|x_n - x_{n-1}, p\|, \frac{\|x_n - x_{n+1}, p\| + \|x_{n-1} - x_n, p\|}{2}, \frac{0 + \|x_{n-1} - x_{n+1}, p\|}{2}\}$
$\leq \alpha \max\{\|x_n - x_{n-1}, p\|, \frac{\|x_n - x_{n+1}, p\| + \|x_{n-1} - x_n, p\|}{2}, \frac{\|x_{n-1} - x_n, p\| + \|x_n - x_{n+1}, p\|}{2}\}$

$$\leq \alpha \max\{\|x_n - x_{n-1}, p\|, \|x_n - x_{n+1}, p\|\}. \tag{3.1}$$

If $\|x_n - x_{n-1}, p\| \leq \|x_n - x_{n+1}, p\|$, then from (3.1), we have
$\|x_{n+1} - x_n, p\| \leq \alpha \|x_{n+1} - x_n, p\|$
implies $1 \leq \alpha$, which is a contradiction.
Therefore $\{\|x_n - x_{n-1}, p\|\}$ is a sequence of real numbers monotone decreasing and bounded below.

Suppose $\lim_{n\to\infty} \|x_n - x_{n-1}, p\| = s$.
Since,
$s = \lim_{n\to\infty} \|x_n - x_{n-1}, p\|$
$= \lim_{n\to\infty} \|F_i x_{n-1} - F_j x_{n-2}, p\|$
$\leq \lim_{n\to\infty} \alpha f(\|x_{n-1} - x_{n-2}, p\|, \|x_{n-1} - F_i x_{n-1}, p\|, \|x_{n-2} - F_j x_{n-2}, p\|,$
$\|x_{n-1} - F_j x_{n-2}, p\|, \|x_{n-2} - F_i x_{n-1}, p\|)$
$\leq \alpha \lim_{n\to\infty} f(\|x_{n-1} - x_{n-2}, p\|, \|x_{n-1} - x_n, p\|, \|x_{n-2} - x_{n-1}, p\|, \|x_{n-1} - x_{n-1}, p\|,$
$\|x_{n-2} - x_n, p\|)$
$= \alpha \lim_{n\to\infty} \max\{\|x_{n-1} - x_{n-2}, p\|, \frac{\|x_{n-1} - x_n, p\| + \|x_{n-2} - x_{n-1}, p\|}{2}, \frac{0 + \|x_{n-2} - x_n, p\|}{2}\}$
$\leq \alpha \lim\limits_{n\to\infty} \max\{\|x_{n-1} - x_{n-2}, p\|, \frac{\|x_{n-1} - x_n, p\| + \|x_{n-2} - x_{n-1}, p\|}{2}, \frac{\|x_{n-2} - x_{n-1}, p\| + \|x_{n-1} - x_n, p\|}{2}\}$
$\leq \alpha\ s$
implies, $s = 0$
i.e., $\lim_{n\to\infty} \|x_n - x, p\| = 0$.

Now, let $n \geq m \in \mathbb{N} \cup \{0\}$. Then
$\|x_{n+1} - x_{m+1}, p\| = \|F_i x_n - F_j x_m, p\|$
$\leq \alpha f(\|x_n - x_m, p\|, \|x_n - F_i x_n, p\|, \|x_m - F_j x_m, p\|, \|x_n - F_j x_m, p\|, \|x_m - F_i x_n, p\|)$
$= \alpha f(\|x_n - x_m, p\|, \|x_n - x_{n+1}, p\|, \|x_m - x_{m+1}, p\|, \|x_n - x_{m+1}, p\|, \|x_m - x_{n+1}, p\|)$
$= \alpha \max\{\|x_n - x_m, p\|, \frac{\|x_n - x_{n+1}, p\| + \|x_m - x_{m+1}, p\|}{2}, \frac{\|x_n - x_{m+1}, p\| + \|x_m - x_{n+1}, p\|}{2}\}$.
Taking limit as $n, m \to \infty$ on the both sides of the above inequality, we get
$\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, p\|$
$\leq \alpha \max\{\lim\limits_{n,m\to\infty} \|x_n - x_m, p\|, 0, \lim\limits_{n,m\to\infty} \frac{\|x_n - x_m, p\| + \|x_m - x_{m+1}, p\| + \|x_m - x_n, p\| + \|x_n - x_{n+1}, p\|}{2}\}$
$= \alpha \max\{\lim_{n,m\to\infty} \|x_n - x_m, p\|, \lim_{n,m\to\infty} \|x_n - x_m, p\|\}$
$= \alpha \lim_{n,m\to\infty} \|x_n - x_m, p\|$,
which implies, $\lim_{n,m\to\infty} \|x_n - x_m, p\| = 0$ [ since $\alpha \neq 0$].
Thus $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there exists an $x \in X$ such that $\lim_{n,m\to\infty} \|x_n - x, p\| = 0$.

Now we show that $x$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.
Again,

$\|F_ix - x, p\| \leq \|F_ix - x_n, p\| + \|x_n - x, p\|$
$= \|F_ix - F_jx_{n-1}, p\| + \|x_n - x, p\|$
$\leq \alpha f(\|x - x_{n-1}, p\|, \|x - F_ix, p\|, \|x_{n-1} - F_jx_{n-1}, p\|, \|x - F_jx_{n-1}, p\|, \|x_{n-1} - F_ix, p\|) + \|x_n - x, p\|$
$= \alpha f(\|x_n - x_{n-1}, p\|, \|x - F_ix, p\|, \|x_{n-1} - x_n, p\|, \|x - x_n, p\|, \|x_{n-1} - F_ix, p\|) + \|x_n - x, p\|$
$= \alpha \max\{\|x_n - x_{n-1}, p\|, \frac{\|x - F_ix, p\| + \|x_{n-1} - x_n, p\|}{2}, \frac{\|x - x_n, p\| + \|x_{n-1} - F_ix, p\|}{2}\} + \|x_n - x, p\|$.
Taking limit as $n \to \infty$ we get from above

$\lim_{n\to\infty} \|F_ix - x, p\| \leq \alpha \max\{0, \frac{\|F_ix - x, p\|}{2}, \frac{\|F_ix - x, p\|}{2}\} + 0$
i.e., $\|F_ix - x, p\| \leq \alpha \frac{\|F_ix - x, p\|}{2} \leq \alpha \|F_ix - x, p\|$
implies, $\|F_ix - x, p\| = 0$
i.e., $F_ix = x$.
Thus $x$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

To show the uniqueness, let $x'$ be another fixed point of $\{F_n\}_{n=1}^{\infty}$.
Since,

$\|x - x', p\| = \|F_ix - F_jx', p\|$
$\leq \alpha f(\|x - x', p\|, \|x - F_ix, p\|, \|x' - F_jx', p\|, \|x - F_jx', p\|, \|x' - F_ix, p\|)$
$= \alpha f(\|x - x', p\|, \|x - x, p\|, \|x' - x', p\|, \|x - x', p\|, \|x' - x, p\|)$
$= \alpha \max\{\|x - x', p\|, 0, \frac{\|x - x', p\| + \|x - x', p\|}{2}\}$
$= \alpha \|x - x', p\|$,
which implies, $\|x - x', p\| = 0$ [ since $\alpha \neq 0$ ]
i.e., $x = x'$.

Hence $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.

**Corollary 3.1.** *Let $F_1$ and $F_2$ be two self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|F_1x - F_2y, p\| \leq \alpha f(\|x - y, p\|, \|x - F_1x, p\|, \|y - F_2y, p\|, \|x - F_2y, p\|, \|y - F_1x, p\|)$,
*where $\alpha < 1$ and $f$ satisfies the relation (2.1). Then $F_1$ and $F_2$ have a unique common fixed point in $X$.*
**Proof.** Putting $F_i = F_1$ and $F_j = F_2$ in the **Theorem 3.1** we get the result.

**Corollary 3.2.** *Let $F$ be a self map on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|Fx - Fy, p\| \leq \alpha f(\|x - y, p\|, \|x - Fx, p\|, \|y - Fy, p\|, \|x - Fy, p\|, \|y - Fx, p\|)$,
*where $\alpha < 1$ and $f$ satisfies the relation (2.1). Then $F$ have a unique fixed point in $X$.*
**Proof.** Putting $F_i = F_j = F$ in the **Theorem 3.1** we get the result.

**Theorem 3.2.** *Let* $\{F_n\}_{n=1}^{\infty}$ *be sequence of self maps on 2-Banach space* $(X, \|., .\|)$ *satisfying*
$\|F_ix - F_jy, p\| \leq \beta f(\|x - F_ix, p\|, \|y - F_jy, p\|, \|x - F_jy, p\|, \|y - F_ix, p\|, \|x - y, p\|)$, *where* $\beta < 1$ *and* $f$ *satisfy the relation (2.2). Then* $\{F_n\}_{n=1}^{\infty}$ *have a unique common fixed point in* $X$.

**Proof.** Let $x_0 \in X$ be an initial point. Construct a sequence $\{x_n\}$ in $X$, for a fixed $i$, such that $x_{n+1} = F_ix_n$. If $x_{n+1} = x_n$ i.e., $F_ix_n = x_n$, then $x_n$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$. So without loss of generality, suppose $x_{n+1} \neq x_n \forall n \in \mathbb{N} \cup \{0\}$.
Since,

$\|x_{n+1} - x_n, p\| = \|F_ix_n - F_jx_{n-1}, p\|$
$\leq \beta f(\|x_n - F_ix_n, p\|, \|x_{n-1} - F_jx_{n-1}, p\|, \|x_n - F_jx_{n-1}, p\|, \|x_{n-1} - F_ix_n, p\|, \|x_n - x_{n-1}, p\|)$
$= \beta f(\|x_n - x_{n+1}, p\|, \|x_{n-1} - x_n, p\|, \|x_n - x_n, p\|, \|x_{n-1} - x_{n+1}, p\|, \|x_n - x_{n-1}, p\|)$
$= \beta \max\{\frac{\|x_n - x_{n+1}, p\| + \|x_{n-1} - x_n, p\| + \|x_n - x_n, p\|}{3}, \frac{\|x_{n-1} - x_{n+1}, p\| + \|x_n - x_{n-1}, p\|}{3}\}$
$\leq \beta \max\{\frac{\|x_n - x_{n+1}, p\| + \|x_{n-1} - x_n, p\|}{3}, \frac{\|x_{n-1} - x_n, p\| + \|x_n - x_{n+1}, p\| + \|x_n - x_{n-1}, p\|}{3}\}$

$$\leq \beta \max\{\|x_n - x_{n+1}, p\|, \|x_n - x_{n-1}, p\|\}. \tag{3.2}$$

If $\|x_n - x_{n-1}, p\| \leq \|x_n - x_{n+1}, p\|$, then from (3.2) we get
$\|x_{n+1} - x_n, p\| \leq \beta \|x_{n+1} - x_n, p\|$
which implies, $1 \leq \beta$, a contradiction.
Therefore,
$\|x_{n+1} - x_n, p\| \leq \|x_n - x_{n-1}, p\|$.
Thus $\{\|x_n - x_{n-1}, p\|\}$ is a monotone decreasing sequence of non-negative real numbers. Suppose $\lim_{n\to\infty} \|x_n - x_{n-1}, p\| = r$.
Thus

$r = \lim_{n\to\infty} \|x_n - x_{n-1}, p\| = \lim_{n\to\infty} \|F_ix_{n-1} - F_jx_{n-2}, p\|$
$\leq \beta \lim_{n\to\infty} f(\|x_{n-1} - F_ix_{n-1}, p\|, \|x_{n-2} - F_jx_{n-2}, p\|, \|x_{n-1} - F_jx_{n-2}, p\|,$
$\|x_{n-2} - F_ix_{n-1}, p\|,$
$\|x_{n-1} - x_{n-2}, p\|)$
$= \lim_{n\to\infty} \beta f(\|x_{n-1} - x_n, p\|, \|x_{n-2} - x_{n-1}, p\|, \|x_{n-1} - x_{n-1}, p\|, \|x_{n-2} - x_n, p\|,$
$\|x_{n-1} - x_{n-2}, p\|)$
$= \lim_{n\to\infty} \beta \max\{\frac{(\|x_{n-1} - x_n, p\| + \|x_{n-2} - x_{n-1}, p\| + \|x_{n-1} - x_{n-1}, p\|}{3}, \frac{\|x_{n-2} - x_n, p\| + \|x_{n-1} - x_{n-2}, p\|}{3}\}$
$\leq \beta \lim_{n\to\infty} \max\{\frac{(\|x_{n-1} - x_n, p\| + \|x_{n-2} - x_{n-1}, p\| + \|x_{n-1} - x_{n-1}, p\|}{3},$
$\frac{\|x_{n-2} - x_{n-1}, p\| + \|x_{n-1} - x_n, p\| + \|x_{n-1} - x_{n-2}, p\|}{3}\}$
$\leq \beta \lim_{n\to\infty} \max\{\|x_n - x_{n-1}, p\|, \|x_{n-1} - x_{n-2}, p\|\}$
$= \beta \max\{r, r\}$
$= \beta r$

implies, $r = 0$ [as $\beta < 1$]
i.e., $\lim_{n\to\infty} \|x_n - x_{n-1}, p\| = 0$.
Now, for $n \geq m \in \mathbb{N}$,

$\|x_{n+1} - x_{m+1}, p\| = \|F_i x_n - F_j x_m, p\|$
$\leq \beta f(\|x_n - F_i x_n, p\|, \|x_m - F_j x_m, p\|, \|x_n - F_j x_m, p\|, \|x_m - F_i x_n, p\|, \|x_n - x_m, p\|)$
$= \beta f(\|x_n - x_{n+1}, p\|, \|x_m - x_{m+1}, p\|, \|x_n - x_{m+1}, p\|, \|x_m - x_{n+1}, p\|, \|x_n - x_m, p\|)$
$= \beta \max\{\frac{\|x_n - x_{n+1},p\| + \|x_m - x_{m+1},p\| + \|x_n - x_{m+1},p\|}{3}, \frac{\|x_m - x_{n+1},p\| + \|x_n - x_m,p\|}{3}\}$
$\leq \beta \max\{\frac{\|x_n - x_{n+1},p\| + \|x_m - x_{m+1},p\| + \|x_n - x_m,p\| + \|x_m - x_{m+1},p\|}{3}, \frac{\|x_m - x_n,p\| + \|x_n - x_{n+1},p\| + \|x_n - x_m,p\|}{3}\}$
$\leq \beta \max\{\|x_n - x_{n+1}, p\|, \|x_m - x_{m+1}, p\|, \|x_n - x_m, p\|\}$.

Taking limit as $n, m \to \infty$ on the both sides of the above inequality, we get

$\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, p\| \leq \beta \lim_{n,m\to\infty} \|x_n - x_m, p\|$

implies, $\lim_{n,m\to\infty} \|x_n - x_m, p\| = 0$.
Thus $\{x_n\}$ is a Cauchy sequence in $X$. Since $X$ is complete, there is an $z \in X$ such that $\lim_{n\to\infty} \|x_n - z, p\| = 0$.
Since $\|F_i z - z, p\| \leq \|F_i z - x_n, p\| + \|x_n - z, p\|$
$= \|F_i z - F_j x_{n-1}, p\| + \|x_n - z, p\|$
$\leq \beta f(\|z - F_i z, p\|, \|x_{n-1} - F_j x_{n-1}, p\|, \|z - F_j x_{n-1}, p\|, \|x_{n-1} - F_i z, p\|, \|z - x_{n-1}, p\|) + \|x_n - z, p\|$
$= \beta f(\|z - F_i z, p\|, \|x_{n-1} - x_n, p\|, \|z - x_n, p\|, \|x_{n-1} - F_i z, p\|, \|z - x_{n-1}, p\|) + \|x_n - z, p\|$
$= \beta \max\{\frac{\|z - F_i z,p\| + \|x_{n-1} - x_n,p\| + \|z - x_n,p\|}{3}, \frac{\|x_{n-1} - F_i z,p\| + \|z - x_{n-1},p\|}{3}\} + \|x_n - z, p\|$.

Taking $\lim_{n\to\infty}$ on the both sides of the above inequality, we have
$\lim\limits_{n\to\infty} \|F_i z - z, p\|$
$\leq \beta \lim\limits_{n\to\infty} \max\{\frac{\|z - F_i z,p\| + \|x_{n-1} - x_n,p\| + \|z - x_n,p\|}{3}, \frac{\|x_{n-1} - F_i z,p\| + \|z - x_{n-1},p\|}{3}\} + \lim\limits_{n\to\infty} \|x_n - z, p\|$
$= \beta \max\{\frac{\|F_i z - z,p\|}{3}, \frac{\|F_i z - z,p\|}{3}\}$
$\leq \beta \|F_i z - z, p\|$
which implies, $(1 - \beta)\|F_i z - z, p\| \leq 0$
i.e., $\|F_i z - z, p\| = 0$
i.e., $F_i z = z$.
So $z$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Let $z'$ be another common fixed point of $\{F_n\}_{n=1}^{\infty}$.
Then ,

$\|z - z', p\| \leq \|F_i z - F_j z', p\|$
$\leq \beta f(\|z - F_i z, p\|, \|z' - F_j z', p\|, \|z - F_j z', p\|, \|z' - F_i z, p\|, \|z - z', p\|)$
$= \beta f(\|z - z, p\|, \|z' - z', p\|, \|z - z', p\|, \|z' - z, p\|, \|z - z', p\|)$
$= \beta \max\{\frac{0 + 0 + \|z - z',p\|}{3}, \frac{\|z - z',p\| + \|z - z',p\|}{3}\}$
$\leq \beta \|z - z', p\|$

implies, $\|z - z', p\| = 0$ i.e., $z = z'$.

Hence $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.

**Corollary 3.3** *Let $F_1$ and $F_2$ be two self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|F_1x - F_2y, p\| \leq \beta f(\|x - F_1x, p\|, \|y - F_2y, p\|, \|x - F_2y, p\|, \|y - F_1x, p\|, \|x - y, p\|)$, *where $\beta < 1$ and $f$ satisfy the relation (2.2). Then $F_1$ and $F_2$ have a unique common fixed point in $X$.*

**Proof.** Put $F_i = F_1$ and $F_j = F_2$ in the above **Theorem 3.2** we get the result.

**Corollary 3.4.** *Let $F$ be a self map on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|Fx - Fy, p\| \leq \beta f(\|x - Fx, p\|, \|y - Fy, p\|, \|x - Fy, p\|, \|y - Fx, p\|, \|x - y, p\|)$, *where $\beta < 1$ and $f$ satisfy the relation (2.2). Then $F$ have a unique fixed point in $X$.*

**Proof.** Put $F_i = F_j = F$ in the above **Theorem 3.2** we get the result.

**Theorem 3.3.** *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|F_ix - F_jy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_jy,p\|+\|y-F_ix,p\|}{1+\|x-F_jy,p\|+\|y-F_ix,p\|} + \beta \max\{\|x - F_jy, p\|, \|y - F_ix, p\|\} + \gamma\|y - F_jy, p\|$, *where $\alpha, \beta, \gamma$ are non-negative real numbers and $3\alpha + 2\beta + \gamma < 1$. Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.*

**Proof.** For an initial approximation $y_0 \in X$ construct a sequence $\{y_n\}$ in $X$ such that $y_{n+1} = F_iy_n$ for a fixed $i = 1, 2, 3, \ldots$. If $y_n = F_iy_n$ i.e., $y_n = y_{n+1}$, $n = 0, 1, 2, \ldots$ then $y_n$ is common fixed point of $\{F_n\}_{n=1}^{\infty}$ for all $n = 0, 1, 2, \ldots$ and the proof is completed.

So we assume that $y_{n+1} \neq y_n \ \ \forall \ \ n \in \mathbb{N} \cup \{0\}$.

Now we show that $\{y_n\}$ is a Cauchy sequence.

Since,

$\|y_{n+1} - y_n, p\| = \|F_iy_n - F_jy_{n-1}, p\|$
$\leq \alpha(\frac{\|y_n-y_{n-1},p\|+\|y_n-F_jy_{n-1},p\|+\|y_{n-1}-F_iy_n,p\|}{1+\|y_n-F_jy_{n-1},p\|+\|y_{n-1}-F_iy_n,p\|}) + \beta \max\{\|y_n - F_jy_{n-1}, p\|, \|y_{n-1} - F_iy_n, p\|\}$
$+ \gamma\|y_{n-1} - F_jy_{n-1}, p\|$
$\leq \alpha(\|y_n - y_{n-1}, p\| + \|y_n - y_n, p\| + \|y_{n-1} - y_{n+1}, p\|) + \beta \max\{\|y_n - y_n, p\|, \|y_{n-1} - y_{n+1}, p\|\} + \gamma\|y_{n-1} - y_n, p\|$
$\leq \alpha(\|y_n - y_{n-1}, p\| + \|y_{n-1} - y_n, p\| + \|y_n - y_{n+1}, p\|) + \beta[\|y_{n-1} - y_n, p\| + \|y_n - y_{n+1}, p\|] + \gamma\|y_{n-1} - y_n, p\|$
implies, $(1 - \alpha - \beta)\|y_{n+1} - y_n, p\| \leq (2\alpha + \beta + \gamma)\|y_n - y_{n-1}, p\|$
i.e., $\|y_{n+1} - y_n, p\| \leq (\frac{2\alpha+\beta+\gamma}{1-\alpha-\beta})\|y_n - y_{n-1}, p\|$
$= k\|y_n - y_{n-1}, p\|$ [ where $\frac{2\alpha+\gamma+\beta}{1-\alpha-\beta} = k < 1$ ]

$\leq k^2\|y_{n-1} - y_{n-2}, p\|$
$\vdots$
$\leq k^n\|y_1 - y_0, p\|$.
Taking limit as $n \to \infty$, we get
$\lim_{n\to\infty} \|y_{n+1} - y_n, p\| = 0$ [ as $k < 1$].
Now, let $n \geq m \in \mathbb{N}$. Then
$\|y_n - y_m, p\| = \|F_i y_{n-1} - F_j y_{m-1}, p\|$
$\leq \alpha(\frac{\|y_{n-1}-y_{m-1},p\|+\|y_{n-1}-F_j y_{m-1},p\|+\|y_{m-1}-F_i y_{n-1},p\|}{1+\|y_{n-1}-F_j y_{m-1},p\|+\|y_{m-1}-F_i y_{n-1},p\|})$
$+ \beta \max\{\|y_{n-1} - F_j y_{m-1}, p\|, \|y_{m-1} - F_i y_{n-1}, p\|\} + \gamma\|y_{m-1} - F_j y_{m-1}, p\|$
$\leq \alpha(\|y_{n-1}-y_{m-1}, p\|+\|y_{n-1}-y_m, p\|+\|y_{m-1}-y_n, p\|)+\beta \max\{\|y_{n-1}-y_m, p\|, \|y_{m-1}-y_n, p\|\} + \gamma\|y_{m-1} - y_m, p\|$
$\leq \alpha(\|y_{n-1}-y_{m-1}, p\|+\|y_{n-1}-y_n, p\|+\|y_n-y_m, p\|+\|y_{m-1}-y_m, p\|+\|y_m-y_n, p\|)+\beta \max\{\|y_{n-1}-y_n, p\|+\|y_n-y_m, p\|, \|y_{m-1}-y_m, p\|+\|y_m-y_n, p\|\}+\gamma\|y_{m-1}-y_m, p\|$.
Let $\lim_{n,m\to\infty} \|y_m - y_n, p\| = r$.
Then from above we get
$r \leq \alpha(r + 0 + r + 0 + r) + \beta \max\{0 + r, 0 + r\} + \gamma.0$
implies, $r \leq 3\alpha r + \beta r$
i.e.,$(1 - 3\alpha - \beta)r \leq 0$
i.e., $r = 0$ [ since $1 - 3\alpha - \beta \neq 0$ ]
i.e., $\lim_{n,m\to\infty} \|y_n - y_m, p\| = 0$.
Thus $\{y_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists a $y \in X$ such that $\lim_{n\to\infty} \|y_n - y, p\| = 0$.
Since,
$\|F_i y - y, p\| \leq \|F_i y - y_n, p\| + \|y_n - y, p\|$
$= \|F_i y - F_j y_{n-1}, p\| + \|y_n - y, p\|$
$\leq \alpha(\frac{\|y-y_{n-1},p\|+\|y-F_j y_{n-1},p\|+\|y_{n-1}-F_i y,p\|}{1+\|y-F_j y_{n-1},p\|+\|y_{n-1}-F_i y,p\|}) + \beta \max\{\|y - F_j y_{n-1}, p\|, \|y_{n-1} - F_i y, p\|\} + \gamma\|y_{n-1} - F_j y_{n-1}, p\| + \|y_n - y, p\|$
$\leq \alpha(\|y - y_{n-1}, p\| + \|y - y_n, p\| + \|y_{n-1} - F_i y, p\|) + \beta \max\{\|y - y_n, p\|, \|y_{n-1} - F_i y, p\|\} + \gamma\|y_{n-1} - y_n, p\| + \|y_n - y, p\|$.
Taking limit as $n \to \infty$ we get from the above inequality,
$\lim_{n\to\infty} \|F_i y - y, p\| \leq \alpha\|y - F_i y, p\| + \beta\|y - F_i y, p\| + \gamma.0 + 0$
implies, $(1 - \alpha - \beta)\|y - F_i y, p\| \leq 0$
i.e., $\|y - F_i y, p\| = 0$
i.e., $F_i y = y$.
Thus $y$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.
Let $z$ be another common fixed point of $\{F_n\}_{n=1}^{\infty}$. Then
$\|y - z, p\| = \|F_i y - F_j z, p\|$

$\leq \alpha \frac{\|y-z,p\|+\|y-F_jz,p\|+\|z-F_iy,p\|}{1+\|y-F_jz,p\|+\|z-F_iy,p\|} + \beta \max\{\|y-F_jz,p\|, \|z-F_iy,p\|\} + \gamma\|z-F_jz,p\|$
$\leq \alpha \frac{\|y-z,p\|+\|y-z,p\|+\|z-y,p\|}{1+\|y-z,p\|+\|z-y,p\|} + \beta \max\{\|y-z,p\|, \|z-y,p\|\} + \gamma\|z-z,p\|$
$\leq (3\alpha+\beta+\gamma)\|y-z,p\|$
implies, $(1-3\alpha-\beta-\gamma)\|y-z,p\| \leq 0$
i.e., $\|y-z,p\| = 0$
i.e., $y = z$.

Hence $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.

**Corollary 3.5.** *Let $F_1$ and $F_2$ be two self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_1x - F_2y, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_2y,p\|+\|y-F_1x,p\|}{1+\|x-F_2y,p\|+\|y-F_1x,p\|} + \beta \max\{\|x-F_2y,p\|, \|y-F_1x,p\|\} + \gamma\|y-F_2y,p\|,$
*where $\alpha, \beta, \gamma$ are non-negative real numbers and $3\alpha+2\beta+\gamma < 1$. Then $F_1$ and $F_2$ have a unique common fixed point in $X$.*

**Proof.** Putting $F_i = F_1$ and $F_j = F_2$ in the **Theorem 3.3** we get the desired result.

**Corollary 3.6.** *Let $F$ be a self map on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|Fx - Fy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-Fy,p\|+\|y-Fx,p\|}{1+\|x-Fy,p\|+\|y-Fx,p\|} + \beta \max\{\|x-Fy,p\|, \|y-Fx,p\|\} + \gamma\|y-Fy,p\|,$
*where $\alpha, \beta, \gamma$ are non-negative real numbers and $3\alpha+2\beta+\gamma < 1$. Then $F$ have a unique fixed point in $X$.*

**Proof.** Putting $F_i = F_j = F$ in the **Theorem 3.3** we get the desired result.

**Theorem 3.4.** *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_ix - F_jy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_jy,p\|+\|y-F_ix,p\|}{1+\|x-F_jy,p\|+\|y-F_ix,p\|} + \beta \min\{\|x-F_jy,p\|, \|y-F_ix,p\|\} + \gamma\|y-F_jy,p\|,$
*where $\alpha, \beta, \gamma$ are non-negative real numbers and $3\alpha+2\beta+\gamma < 1$. Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.*

**Proof.** Since $\min\{\|x-F_jy,p\|, \|y-F_ix,p\|\} \leq \max\{\|x-F_jy,p\|, \|y-F_ix,p\|\}$, the result follows from the **Theorem 3.3**.

**Theorem 3.5.** *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_ix - F_jy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_ix,p\|}{1+\|y-F_ix,p\|} + \beta \max\{\|x-F_jy,p\|, \|y-F_jy,p\|\} + \gamma[\|x-F_ix,p\|+\|y-F_jy,p\|],$
*where $\alpha, \beta, \gamma$ are non-negative real numbers and $2\alpha+\beta+2\gamma < 1$. Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in $X$.*

**Proof.** With an initial approximation $y_0 \in X$, construct a sequence $\{y_n\}$ such that $y_{n+1} = F_i y_n;\ n = 0, 1, 2, ...$ for a fixed $i$. Similarly as previous theorems, assume $y_{n+1} \neq y_n, \forall\ \ n\ \ \in \mathbb{N} \cup \{0\}$.

First of all we show that $\{y_n\}$ is a Cauchy sequence.
Since,
$\|y_{n+1} - y_n, p\| = \|F_i y_n - F_j y_{n-1}, p\|$
$\leq \alpha(\frac{\|y_n - y_{n-1}, p\| + \|y_n - F_i y_n, p\|}{1 + \|y_{n-1} - F_i y_n, p\|}) + \beta \max\{\|y_n - F_j y_{n-1}, p\|, \|y_{n-1} - F_j y_{n-1}, p\|\}$
$+\gamma[\|y_n - F_j y_n, p\| + \|y_{n-1} - F_j y_{n-1}, p\|]$
$\leq \alpha(\|y_n - y_{n-1}, p\| + \|y_n - y_{n+1}, p\|) + \beta \max\{\|y_n - y_n, p\|, \|y_{n-1} - y_n, p\|\} + \gamma[\|y_n - y_{n+1}, p\| + \|y_{n-1} - y_n, p\|]$
$= \alpha\|y_n - y_{n-1}, p\| + \alpha\|y_n - y_{n+1}, p\| + \beta\|y_{n-1} - y_n, p\| + \gamma\|y_n - y_{n+1}, p\| + \gamma\|y_{n-1} - y_n, p\|$
implies, $(1 - \alpha - \gamma)\|y_{n+1} - y_n, p\| \leq (\alpha + \beta + \gamma)\|y_n - y_{n-1}, p\|$

$$\text{i.e., } \|y_{n+1} - y_n, p\| \leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma})\|y_n - y_{n-1}, p\|$$

$$\leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma})^2\|y_{n-1} - y_{n-2}, p\|$$

$$\vdots$$

$$\leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma})^n\|y_1 - y_0, p\|.$$

Taking $\lim_{n\to\infty}$ on the both sides of the above inequality, we get
$\lim_{n\to\infty} \|y_{n+1} - y_n, p\| = 0.$
Now, let $n \geq m \in \mathbb{N}$. Then
$\|y_{n+1} - y_{m+1}, p\|$
$= \|F_i y_n - F_j y_m, p\|$
$\leq \alpha\frac{\|y_n - y_m, p\| + \|y_n - F_i y_n, p\|}{1 + \|y_m - F_i y_n, p\|} + \beta \max\{\|y_n - F_j y_m, p\|, \|y_m - F_j y_m, p\|\} + \gamma[\|y_n - F_i y_n, p\| + \|y_m - F_j y_m, p\|]$
$\leq \alpha(\|y_n - y_m, p\| + \|y_n - y_{n+1}, p\|) + \beta \max\{\|y_n - y_{m+1}, p\|, \|y_m - y_{m+1}, p\|\} + \gamma[\|y_n - y_{n+1}, p\| + \|y_m - y_{m+1}, p\|]$.
Taking limit as $n, m \to \infty$ we get from above
$\lim_{n,m\to\infty} \|y_{n+1} - y_{m+1}, p\|$
$\leq \alpha \lim_{n,m\to\infty} \|y_n - y_m, p\| + \beta \lim_{n,m\to\infty} \|y_n - y_{m+1}, p\| + \gamma.0$
$\leq \alpha \lim_{n,m\to\infty} \|y_n - y_m, p\| + \beta \lim_{n,m\to\infty}[\|y_n - y_m, p\| + \|y_m - y_{m+1}, p\|]$
$= (\alpha + \beta) \lim_{n,m\to\infty} \|y_n - y_m, p\|$
implies, $\lim_{n,m\to\infty} \|y_n - y_m, p\| = 0$

i.e., $\{y_n\}$ is a Cauchy sequence. Since $X$ is complete, there exists an $y \in X$ such that $\lim_{n\to\infty} \|y_n - y, p\| = 0$.

Now we show that $y$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Since

$\|F_i y - y, p\| \leq \|F_i y - y_n, p\| + \|y_n - y, p\|$
$= \|F_i y - F_j y_{n-1}, p\| + \|y_n - y, p\|$
$\leq \alpha \frac{\|y - y_{n-1}, p\| + \|y - F_i y, p\|}{1 + \|y_{n-1} - F_i y, p\|} + \beta \max\{\|y - F_j y_{n-1}, p\|, \|y_{n-1} - F_j y_{n-1}, p\|\} + \gamma[\|y - F_i y, p\| + \|y_{n-1} - F_j y_{n-1}, p\|] + \|y_n - y, p\|$
$\leq \alpha(\|y - y_{n-1}, p\| + \|y - F_i y, p\|) + \beta \max\{\|y - y_n, p\|, \|y_{n-1} - y_n, p\|\} + \gamma[\|y - F_i y, p\| + \|y_{n-1} - y_n, p\|] + \|y_n - y, p\|$.

Taking $\lim_{n\to\infty}$ on the both sides of above inequality, we get

$\lim_{n\to\infty} \|F_i y - y, p\| \leq \alpha \|y - F_i y, p\| + \beta.0 + \gamma \|y - F_i y, p\| + 0$
which implies, $(1 - \alpha - \gamma)\|F_i y - y, p\| \leq 0$
i.e., $\|F_i y - y, p\| = 0$
i.e., $F_i y = y$.
Thus $y$ is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Let $y'$ be another fixed point of $\{F_n\}_{n=1}^{\infty}$. Then

$\|y - y', p\| = \|F_i y - F_j y', p\|$
$\leq \alpha(\frac{\|y - y', p\| + \|y - F_i y, p\|}{1 + \|y' - F_i y, p\|}) + \beta \max\{\|y - F_j y', p\|, \|y' - F_j y', p\|\} + \gamma[\|y - F_i y, p\| + \|y' - F_j y', p\|]$
$\leq \alpha(\|y - y', p\| + \|y - y, p\|) + \beta \max\{\|y - y', p\|, \|y' - y', p\|\} + \gamma[\|y - y, p\| + \|y' - y', p\|]$
$= \alpha\|y - y', p\| + \beta\|y - y', p\|$
that implies, $(1 - \alpha - \beta)\|y - y', p\| \leq 0$ i.e., $\|y - y', p\| = 0$
i.e., $y = y'$.

Hence the result.

**Corollary 3.7.** *Let $F_1$ and $F_2$ be two self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_1 x - F_2 y, p\|$
$\leq \alpha \frac{\|x - y, p\| + \|x - F_1 x, p\|}{1 + \|y - F_1 x, p\|} + \beta \max\{\|x - F_2 y, p\|, \|y - F_2 y, p\|\} + \gamma[\|x - F_1 x, p\| + \|y - F_2 y, p\|]$,

*where $\alpha, \beta, \gamma$ are non-negative real numbers and $2\alpha + \beta + 2\gamma < 1$. Then $F_1$ and $F_2$ have a unique common fixed point in $X$.*

**Proof.** Putting $F_i = F_1$ and $F_j = F_2$ in the **Theorem 3.5** the result holds.

**Corollary 3.8.** *Let $F$ be a self map on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|Fx - Fy, p\|$
$\leq \alpha \frac{\|x - y, p\| + \|x - Fx, p\|}{1 + \|y - Fx, p\|} + \beta \max\{\|x - Fy, p\|, \|y - Fy, p\|\} + \gamma[\|x - Fx, p\| + \|y - Fy, p\|]$,

*where $\alpha, \beta, \gamma$ are non-negative real numbers and $2\alpha + \beta + 2\gamma < 1$. Then $F$ have a unique fixed point in $X$.*

**Proof.** Putting $F_i = F_j = F$ in the **Theorem 3.5** the result holds.

The next theorem is the generalization of Saluja [13] theorem 3.1. In that theorem $T$ was a continuous self map on $X$. We have proved it to a family of self maps without continuity as follows:

**Theorem 3.6.** *Let $X$ be a 2-Banach space(with $dimX \geq 2$) and $\{T_i\}_{i=1}^{\infty}$ be a family of self maps on $X$ satisfying*

$$\|T_ix - T_jy, a\| \leq h\max\{\|x - y, a\|, \tfrac{\|x-T_ix,a\|+\|y-T_jy,a\|}{2}, \tfrac{\|x-T_jy,a\|+\|y-T_ix,a\|}{2}\},$$

*where $0 < h < 1$. Then $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point in $X$.*

**Proof.** Let $x_0 \in X$ be arbitrary. Then we construct a sequence $\{x_n\}$ such that $x_{n+1} = T_ix_n$ for a fixed $i$.

We now show that $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = 0$.

Now,

$$\|x_{n+1} - x_n, a\| = \|T_ix_n - T_{n-1}, a\|$$
$$\leq h\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-T_ix_n,a\|+\|x_{n-1}-T_jx_{n-1},a\|}{2}, \tfrac{\|x_n-T_jx_{n-1},a\|+\|x_{n-1}-T_ix_n,a\|}{2}\}$$
$$= h\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-x_{n+1},a\|+\|x_{n-1}-x_n,a\|}{2}, \tfrac{\|x_n-x_n,a\|+\|x_{n-1}-x_{n+1},a\|}{2}\}$$
$$\leq h\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-x_{n+1},a\|+\|x_{n-1}-x_n,a\|}{2}, \tfrac{\|x_{n-1}-x_n,a\|+\|x_n-x_{n+1},a\|}{2}\}$$
$$= h\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-x_{n+1},a\|+\|x_{n-1}-x_n,a\|}{2}\}$$
$$\leq h\max\{\|x_n - x_{n-1}, a\|, \|x_n - x_{n+1}, a\|\}. \tag{3.3}$$

Suppose $\|x_{n-1} - x_n, a\| \leq \|x_n - x_{n+1}, a\|$.

Then from (3.3), $\|x_{n+1} - x_n, a\| \leq h\|x_{n+1} - x_n, a\|$

implies, $1 \leq h$, a contradiction.

Thus $\|x_{n+1} - x_n, a\| \leq \|x_n - x_{n-1}, a\|$.

Therefore, $\{\|x_{n+1} - x_n, a\|\}$ is a sequence of real numbers monotone decreasing and bounded below. Suppose $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = r$.

Suppose $r \neq 0$. Then,

$$r = \lim_{n\to\infty} \|x_{n+1} - x_n, a\| = \lim_{n\to\infty} \|T_ix_n - T_jx_{n-1}, a\|$$
$$\leq \lim_{n\to\infty} h\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-T_ix_n,a\|+\|x_{n-1}-T_jx_{n-1},a\|}{2}, \tfrac{\|x_n-T_jx_{n-1},a\|+\|x_{n-1}-T_ix_n,a\|}{2}\}$$
$$= h\lim_{n\to\infty}\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-x_{n+1},a\|+\|x_{n-1}-x_n,a\|}{2}, \tfrac{\|x_n-x_n,a\|+\|x_{n-1}-x_{n+1},a\|}{2}\}$$
$$\leq h\lim_{n\to\infty}\max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n-x_{n+1},a\|+\|x_{n-1}-x_n,a\|}{2}, \tfrac{\|x_{n-1}-x_n,a\|+\|x_n-x_{n+1},a\|}{2}\}\}$$
$$= h\lim_{n\to\infty}\max\{r, \tfrac{r+r}{2}, \tfrac{r+r}{2}\} = hr$$

implies, $1 \leq h$, a contradiction.

Therefore, $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence.

Since for $n > m \in \mathbb{N}$,

$\lim_{n,m\to\infty}\|x_n - x_m, a\|$
$\le \lim_{n,m\to\infty}[\|x_n - x_{n-1}, a\| + \|x_{n-1} - x_m, a\|]$
$= \lim_{n,m\to\infty}\|x_{n-1} - x_m, a\|$
$\vdots$
$\le \lim_{n,m\to\infty}\|x_m - x_m, a\|$
$= 0.$
Therefore, $\{x_n\}$ is a Cauchy sequence. Since $X$ is a complete space, there exist a $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Next, we show that $x$ is a fixed point of $\{T_i\}_{i=1}^{\infty}$.
Since
$\lim_{n\to\infty}\|T_ix - x, a\| \le \lim_{n\to\infty}[\|T_ix - x_n, a\| + \|x_n - x, a\|]$
$= \lim_{n\to\infty}\|T_ix - T_jx_{n-1}, a\| + \lim_{n\to\infty}\|x_n - x, a\|$
$\le \lim_{n\to\infty} h\max\{\|x - x_{n-1}, a\|, \frac{\|x-T_ix,a\|+\|x_{n-1}-T_jx_{n-1},a\|}{2}, \frac{\|x-T_jx_{n-1},a\|+\|x_{n-1}-T_ix,a\|}{2}\}$
$= h\lim_{n\to\infty}\max\{\|x - x_{n-1}, a\|, \frac{\|x-T_ix,a\|+\|x_{n-1}-x_n,a\|}{2}, \frac{\|x-x_n,a\|+\|x_{n-1}-T_ix,a\|}{2}\}$
$\le h\|T_ix - x, a\|$
implies, $\|T_ix - x, a\| \ne 0$,
i.e., $T_ix = x$.
Thus $x$ is fixed point of $X$.

Now we show that $x$ is a unique common fixed point of $\{T_i\}_{i=1}^{\infty}$. Let $y$ be another common fixed point. Then by the given condition, we get
$\|x - y, a\| = \|T_ix - T_jy, a\|$
$\le h\max\{\|x - y, a\|, \frac{\|x-T_ix,a\|+\|y-T_jy,a\|}{2}, \frac{\|x-T_jy,a\|+\|y-T_ix,a\|}{2}\}$
$= h\max\{\|x - y, a\|, \frac{\|x-x,a\|+\|y-y,a\|}{2}, \frac{\|x-y,a\|+\|y-x,a\|}{2}\}$
$= h\|x - y, a\|$
implies, $\|x - y, a\| = 0$
i.e., $x = y$.
Thus $x$ is a unique common fixed point of $\{T_i\}_{i=1}^{\infty}$.
Hence the theorem.

**Corollary 3.9.** *Let $X$ be a 2-Banach space(with dim$X \ge 2$) and $T_1$ and $T_2$ be two self maps on $X$ satisfying*
$\|T_1x - T_2y, a\| \le h\max\{\|x - y, a\|, \frac{\|x-T_1x,a\|+\|y-T_2y,a\|}{2}, \frac{\|x-T_2y,a\|+\|y-T_1x,a\|}{2}\}$,
*where $0 < h < 1$. Then $T_1$ and $T_2$ have a unique common fixed point in $X$.*
**Proof.** Putting $T_i = T_1$ and $T_j = T_2$ in the above **Theorem 3.6** we have the required result.

This result is same as Saluja ([13]) theorem 3.1 without continuity.

**Corollary 3.10.** *Let $X$ be a 2-Banach space(with dim$X \ge 2$) and $T$ be a self maps* *on X satisfying*

$$\|Tx - Ty, a\| \leq h \max\{\|x - y, a\|, \frac{\|x-Tx,a\|+\|y-Ty,a\|}{2}, \frac{\|x-Ty,a\|+\|y-Tx,a\|}{2}\},$$

*where* $0 < h < 1$. *Then* $T$ *have a unique fixed point in* $X$.

**Proof.** Putting $T_i = T_j = T$ in the above **Theorem 3.6** we have the desired result.

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