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# LIAR'S DOMINATION IN SIERPIŃSKI-LIKE GRAPHS

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**Abstract:** The vertex set $L \subseteq V(G)$ is a liar's dominating set if and only if it satisfies the following two conditions: (i) $L$ double dominates every $v \in V(G)$ and (ii) for every pair $u, v$ of distinct vertices, $|(N[u] \cup N[v]) \cap L| \geq 3$. The liar's domination number for a graph $G$ is denoted by $\gamma_L(G)$ which is the minimum cardinality of the liar's dominating set $L$. Liar's domination was introduced by P. J. Slater. In a liar's dominating set it is assumed that any one protective device in its neighborhood of the intruder vertex might misreport the location of an intruder vertex in its closed neighborhood. In this paper, we determine the liar's domination set for Sierpiński-like graphs.

**Keywords and Phrases:** Domination, Liar's domination, Sierpiński graphs, Sierpiński cycle graphs, Sierpiński complete graphs.

**2010 Mathematics Subject Classification:** 97K30, 05C85.

## 1. Introduction

Domination in graphs is a widely researched topic because of its applications in many fields. There are different variations of domination existing in literature that motivates one to explore its applications in any graph or network. In the year 2009 Slater introduced liar's domination. This concept was introduced in such a way that a network is modeled as a graph and all its vertices are the possible locations for the intruder to enter and a dominating set as a set of protection devices placed at a vertex $v$ so that the intruder and its exact location can be detected in its closed neighbourhood even if a protection device is allowed to lie or becomes faulty.

Consider the graph $G = (V, E)$, for $u \in V(G)$ we denote the open and closed neighbourhoods of $v$ as $N(v) = \{u|uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ respectively. A vertex $u$ is said to be dominated by $v$ if $u \in N[v]$. A set $D \subseteq V$ is a dominating set if each vertex $v \in V$ is dominated by a vertex in $D$ or $|N[v] \cap D| \geq 1$ for all $v \in V$. A set $D \subseteq V$ is a double dominating set if each vertex $v \in V$ is dominated by atleast two vertices in $D$ or $|N[v] \cap D| \geq 2$ for all $v \in V$. A set $D \subseteq V$ is a $k$-tuple dominating set if each vertex $v \in V$ is dominated by atleast $k$ vertices in $D$ or $|N[v] \cap D| \geq k$ for all $v \in V$. P. J. Slater introduced liar's domination with the following characterization: A set $L \in V(G)$ is a liar's dominating set if and only if it satisfies these two conditions (i) $|N[v] \cap L| \geq 2$ and (ii) for any two distinct vertices $u, v$, $|(N[u] \cup N[v]) \cap L| \geq 3$ [9].

Slater [9] has showed that the liar's dominating set problem is NP-*hard* for general graphs and has specified a lower bound for trees. Roden and Slater [8] proved that the problem is NP-*hard* even for bipartite graphs. Panda and Paul [5,6] have proved that the problem is NP-*hard* for split graphs and chordal graphs and later they suggested a linear time algorithm for proper interval graphs. Liar dominating set for Circulant networks was given by Paul Manuel [4]. B. S. Panda et al. [7] studied the problem for bounded degree graphs and $p$-claw free graphs. Alimadadi et al. [1] have given the characterization of graphs and trees such that the liar's domination number is $| V |$ and $| V | -1$ respectively. In this paper we determine the liar's domination number for Sierpiński cycle graphs and Sierpiński complete graphs.

## 2. Sierpiński Graphs

Consider the Sierpiński graph $S(n, G)$ to be a finite undirected graph with the set of vertices $\{1, 2, \ldots, k\}$ where $k$ is an integer, with vertex set $\{1, 2, \ldots, k\}^n$ and edge set $\{u, v\}$ is defined if and only if there exists an $h \in \{1, 2, \ldots, n\}$ such that:

- $u_t = v_t$ for $t = 1, 2, \ldots, h - 1$;
- $u_h \neq v_h$;
- $u_t = v_h$ and $v_t = u_h$ for $t = h + 1, \ldots, n$.

Here vertex $(u_1, u_2, ..., u_n)$ is represented as $(u_1 u_2 \ldots u_n)$ and in Figures as $u_1 u_2 \ldots u_n$. The vertices $(1 \ldots 1)$, $(2 \ldots 2)$,$\ldots$,$(k \ldots k)$ are called extreme vertices of $S(n, G)$. Let $n \geq 2$, for $i \in \{1, \ldots, k\}$ $S_i(n - 1, G)$ be the subgraph of $S(n, G)$ induced by the vertices of the form $(iv_2 \ldots v_n)$. Note that $S_i(n-1, G)$ is isomorphic to $S(n-1, G)$ [2].

**Remark 2.1.** *$S(1, G)$ is isomorphic to the graph $G$ and we can construct $S(n +$*

*$1, G)$ by copying $|V(G)|$ times $S(n, G)$ and adding an edge between the vertices $ijj...j$ and $jii...i$ which is called as the linking edge in $S(n+1, G)$ [2].*

Sierpiński graphs have played an important role in the growing literature of research. The variants of these graphs are numerous and have applications in different fields of mathematics. Klavžar and Milutinović [3] proved that the Sierpiński graphs $S(n, K_3)$ are isomorphic to the Tower of Hanoi graphs on 3 pegs. Many authors have discussed, investigated and given many results regarding the chromatic number, vertex cover number, clique number, domination number and many more.

## 3. Sierpiński Cycle Graphs

In this section we consider $G$ to be isomorphic to $C_4$.

**Theorem 3.1.** [9] *For a cycle $C_n$ we have $\gamma_L(C_n) = \lceil \frac{3n}{4} \rceil$.*

By the above result for $n = 1$, the result is obvious.

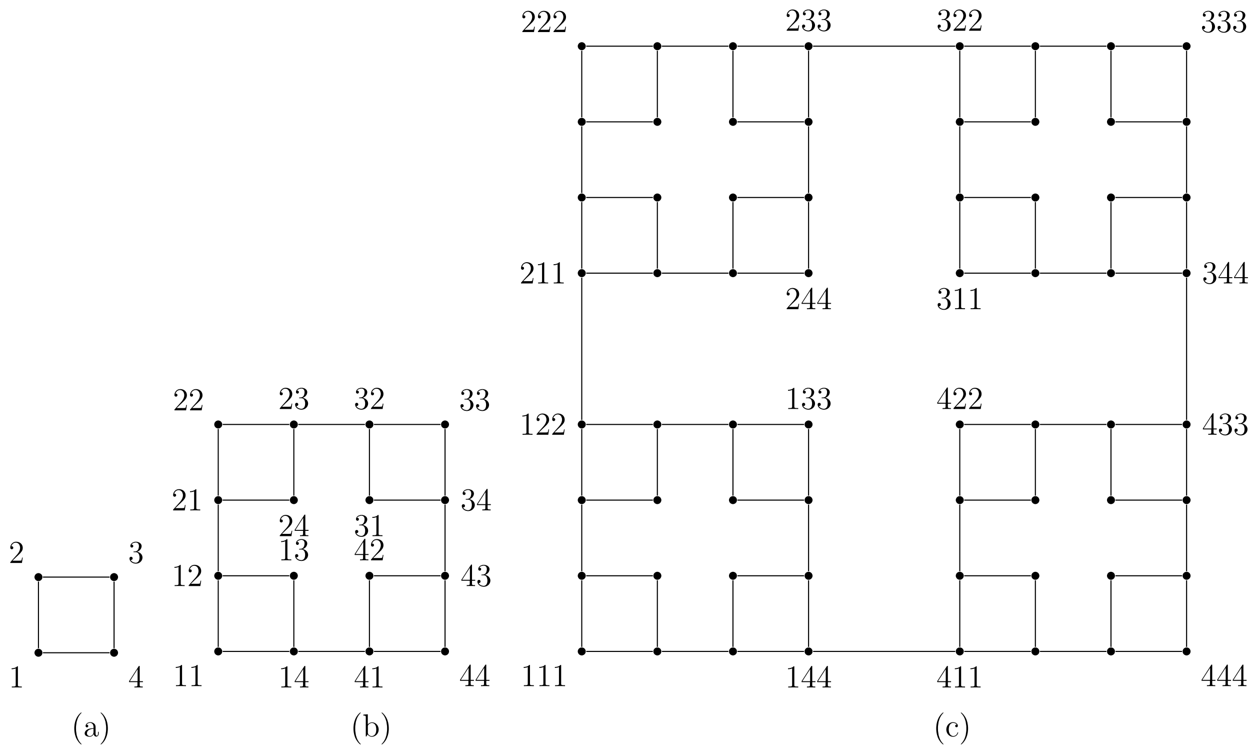


Figure 1: Sierpiński Cycle Graphs $S(1, C_4)$, $S(2, C_4)$, $S(3, C_4)$

**Theorem 3.2.** *Let $G$ be a Sierpiński cycle graph $S(n, C_4)$, $n \geq 2$ then $\gamma_L S(n, C_4) = 4[\gamma_L S(n-1, C_4)] - 4$.*

**Proof.** Let us prove the result by the method of induction. For $n = 2$, the vertices are labeled as shown in Figure 1(b). Since the corner vertices $ii$, $i = 1, 2, 3, 4$ and its diagonally opposite vertices are of degree two, among $(i1, i2, i3, i4)$; $i = 1, 2, 3, 4$ any three of the vertices should be in liar's dominating set. Therefore $L = \{11, 12, 14, 21, 22, 23, 32, 33, 34, 41, 43, 44\}$ is a minimum liar's dominating set. For $n = 3$, we know that $S(3, C_4)$ is constructed from 4 copies of $S(2, C_4)$ namely $S_i(2, C_4)$, $i = 1, 2, 3, 4$. We can obtain the liar's dominating set for $S_i(2, C_4)$ from $S(2, C_4)$. By construction, the extreme vertex $(i \quad i+1 \quad i+1)$ of $S_i(2, C_4)$ is joined by an edge with the extreme vertex $(i+1 \quad i \quad i)$, $i = 1, 2, 3, 4 (i \ mod \ 4)$ and in $S(2, C_4)$ all the extreme vertices are in $L$. In $S(3, C_4)$ we can either have $(i \quad i+1 \quad i+1)$ or $(i+1 \quad i \quad i)$. Without loss of generality suppose $(i \quad i+1 \quad i+1) \in L$ and $(i+1 \quad i \quad i) \notin L$, then also the closed neighbourhood of the vertices $(i+1 \quad i \quad i)$ and its diagonally opposite vertex have atleast 3 vertices in $L$. Thus $\gamma_L S(3, C_4) = 4[\gamma_L S(2, C_4)] - 4$. Let us assume that the result is true for $S(n, C_4)$, $n < k$. Let $n = k$. Since $S(k, C_4)$ is obtained from 4 copies of $S(k-1, C_4)$ by joining the edge $(i \quad i+1 \quad i+1 \ldots i+1)$ with $(i+1 \quad i \quad i \ldots i)$, $i = 1, 2, 3, 4 (i \ mod \ 4)$. Also since minimum liar's dominating set of $S(k-1, C_4)$ includes all the corner vertices, by induction hypothesis in a similar manner $\gamma_L S(k, C_4) = 4[\gamma_L S(k-1, C_4)] - 4$.

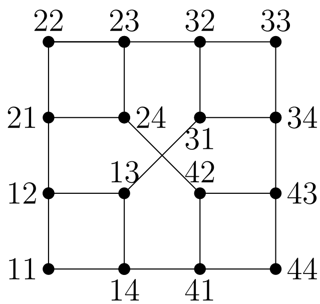


Figure 2: Sierpiński Cycle Graph $S'(2, C_4)$

The graph $S'(2, C_4)$ shown in Figure 2 is obtained by joining the inner vertices 13, 31, 24, 42 of $S(2, C_4)$ with edges called cross edges namely (13, 31), (24, 42). Similarly $S'(3, C_4)$ is obtained by taking four copies of $S'(2, C_4)$ along with linking edges in $S(3, C_4)$ and also the vertices (133, 311) and (244, 422) are joined with edges. In general $S'(n, C_4)$ is obtained in a similar manner from $S'(n-1, C_4)$ which include linking edges and the cross edges (244...4, 422...2) and (133...3, 311...1).

**Theorem 3.3.** *Let $G$ be a Sierpiński cycle graph $S'(2, C_4)$ then $\gamma_L S'(2, C_4) = 10$.*

**Proof.** In view of Theorem 3.2, $\gamma_L S(2, C_4) = 12$ where $L$ should have atleast three vertices from each $S_i(1, C_4)$ of $S(2, C_4)$ and since $d(i \quad i+2) = 3$ for $i = 1, 2, 3, 4 (i \ mod \ 4)$ let us take $L = \{12, 14, 21, 23, 32, 34, 41, 43, 13, 31, 24, 42\}$ as a

liar's dominating set. In order to get minimum, either (24 and 31) or (13 and 42) can be removed from $L$. Without loss of generality let (13 and 42) $\in L$. Then also $|(N[31] \cup N[33]) \cap L| = 3$ and $|(N[24] \cup N[22]) \cap L| = 3$. Thus $\gamma_L S'(2, C_4) = 10$.
**Note**: There are number of minimum liar's dominating sets for $S'(2, C_4)$. In which by symmetry the minimum liar's dominating set that includes all its extreme vertices is $\{11, 12, 13, 22, 23, 32, 33, 42, 43, 44\}$.

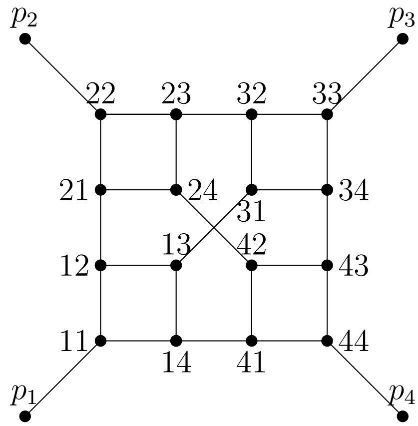


Figure 3: $S'_H(2, C_4)$

$\alpha$-graph shown in Figure 3 is obtained from $S'(2, C_4)$ by attaching pendant edge at each of the extreme vertices namely $ii$, $i = 1, 2, 3, 4$ *mod* 4. The end vertex (pendant vertex) in each of the pendant edge are labeled as $p_1$, $p_2$, $p_3$ and $p_4$.

**Remark 3.4.** *We prove that the minimum liar's dominating set of $S'(2, C_4)$ in $\alpha$-graph is same as that of $S'(2, C_4)$. The only case for which we have to verify is for the minimum liar's dominating set which include the corner vertices (i.e) for $L = \{11, 12, 13, 22, 23, 32, 33, 42, 43, 44\}$. Now we prove that instead of $ii \in L$ we cannot include $p_i \in L$. If $p_1 \in L$ and 11 $\notin L$ then $|(N[12] \cup N[13]) \cap L| = 2$. And if $p_2 \in L$ and $22 \notin L$ we have $|N[21] \cap L| = 1$. Thus if any of the $p_i$'s are included then minimum liar's dominating set whose cardinality is 10 cannot be obtained.*
**Theorem 3.5.** *Let $G$ be a Sierpiński cycle graph $S'(n, C_4)$ then $\gamma_L S'(n, C_4) = 4[\gamma_L S'(n-1, C_4)]$.*
**Proof.** In view of Theorem 3.3, the result is true for $n = 2$. For $n = 3$, consider $S'_1(2, C_4)$ since $S'(3, C_4)$ consists of 4 copies of $S'(2, C_4)$ namely $S'_i(2, C_4)$, $i = 1, 2, 3, 4$. The graph induced by $\langle S'_1(2, C_4), 211, 311, 411\rangle$ is an $\alpha \setminus p_1$-graph. In view of Remark 3.4, the minimum liar's dominating set for this $\alpha \setminus p_1$-graph lies in $S'_1(2, C_4)$. Similar cases are dealt for $S'_i(2, C_4)$, $i = 2, 3, 4$. Therefore $\gamma_L S'(3, C_4) =$

$4[\gamma_L S'(2, C_4)]$. Let us assume that the result is true for $S'(n, C_4)$, $n < k$. Let $n = k$, since $S'(k, C_4)$ is constructed from 4 copies of $S'(k-1, C_4)$ by induction hypothesis we have $\gamma_L S'(k, C_4) = 4[\gamma_L S'(k-1, C_4)]$.

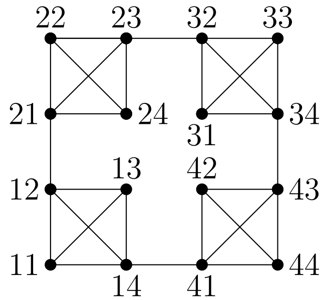


Figure 4: Sierpiński Complete Graph $S(2, K_4)$

## 4. Sierpiński Complete Graph

The Sierpiński complete graph $S(2, K_4)$ is formed by 4 copies of $K_4$ by adding linking edges $(12, 21), (23, 32), (34, 43), (14, 41)$ between them. We observe that the minimum liar's dominating set of $S(n, K_4)$ is the same as that of the minimum liar's dominating set of $S(n, C_4)$. $S(1, K_4) \cong K_4$ and by [8] $\gamma_L(K_n) = 3$. Consider $n = 2$, each $S_i(1, C_4)$ should have 3 vertices in $L$, otherwise $|(N[11] \cup N[13]) \cap L| < 3$. Similarly for the case $S_i(1, K_4)$. Thus in view of Theorem 3.2 we have the following result.

**Theorem 4.1.** *Let $G$ be a Sierpinski complete graph $S(n, K_4)$ then $\gamma_L S(n, K_4) = 4[\gamma_L S(n-1, K_4)] - 4$.*

Let $S'(n, K_4)$ be obtained from 4 copies of $S'(n-1, K_4)$ together with the linking edges and cross edges.

**Theorem 4.2.** *Let $G$ be a Sierpiński complete graph $S'(2, K_4)$ then $\gamma_L S'(2, K_4) = 9$.*

**Proof.** Since $d(ii) = 3$, $L$ should have atleast two and atmost three vertices from each $S'_i(1, K_4)$, $i = 1, 2, 3, 4$. Thus $8 \leq \gamma_L S'(2, K_4) \leq 12$. Now we prove that $\gamma_L S'(2, K_4) > 8$. Suppose $\gamma_L S'(2, K_4) = 8$ then two vertices from $S'_1(1, K_4)$ will be in $L$. It can be either (11, 12) or (11, 13) or (11, 14) or (12, 13) or (12, 14) or (13, 14). In order to satisfy (ii) condition of minimum liar's dominating set the adjacent vertices of $S'_1(1, K_4)$ should be in $L$ namely 21, 31 and 41. Thus now $|L| = 5$.

**Case 1**: $11 \notin L$

Without loss of generality let us take $12, 13 \in L$ along with 21, 31, 41. Now we can take exactly one vertex from each $S'_i(1, K_4)$, $i = 2, 3, 4$. In $S'_2(1, K_4)$ it can be either

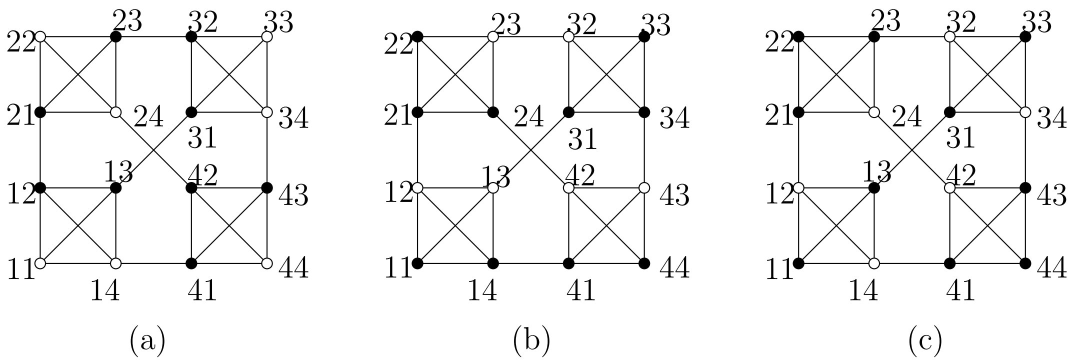


Figure 5: Bold vertices in each case represent the liar's dominating set

22 or 23 or 24. Since 23 and 24 are adjacent vertices of $S'_3(1, K_4)$ and $S'_4(1, K_4)$ respectively, let $23 \in L$. Then 42 should be in $L$. Also because the adjacent vertex of $S'_3(1, K_4)$ should be in $L$, $43 \in L$. Thus $|L| > 8$.

**Case 2**: $11 \in L$

Without loss of generality let us take 11, 12 along with 21, 31, 41 in $L$. Again either 22 or 23 or $24 \in L$. Suppose $23 \in L$ then 32, 42 should be in $L$. Now each $S'_i(1, K_4)$ has exactly two vertices. If it is so then the adjacent vertices of $S'_i(1, K_4)$ should be in $L$. Thus 13 and 43 the adjacent vertices of $S'_3(1, K_4) \in L$. In this case $|L| = 10 > 8$.

Hence $\gamma_L S'(2, K_4) > 8$ in both the cases. See Figure 5(a) in which $\gamma_L S'(2, K_4) = 9$.

**Remark 4.3.**

1. *In view of case* (2) *in Theorem* 4.2, *we observe that if* $ii \in L$ *then* $|L| \geq 10$. *If* $ii \in L$ *and* $|N(ii) \cap L| = 2$ *then in* $S'_i(1, K_4)$ *all the adjacent vertices of* $S'_j(1, K_4)$ *does not belong to* $L$. *Also if* $ii \in L$ *and* $|N(ii) \cap L| = 1$ *in* $S'_i(1, K_4)$ *then all the adjacent vertices of* $S'_j(1, K_4)$ *belong to* $L$. *Moreover in* $S'(2, K_4)$ *exactly two* $S'_i(1, K_4)$ *contain three vertices and other two has two vertices in* $L$. *Now let us consider the cases in which* $S'(2, K_4)$ *has all corner vertices in* $L$, *it can be either Figure* 5(*b*) *or* 5(*c*).

2. *Consider* $S'(3, K_4)$ *which can be constructed from Figure* 5(*b*) *or* 5(*c*) *or both.* $S'(3, K_4)$ *obtained so consists of a path* $P_4$ *in* $G[L]$ *namely* $(111, 114, 141, 144)$ *or* $(111, 113, 131, 133)$ *then we can remove a vertex which will be corner vertex in* $S'_i(2, K_4)$ *of* $S'(3, K_4)$ *other than* $iii$, $i = 1, 2, 3, 4$ *since the adjacent vertex of* $S'_i(2, K_4)$ *namely the corner vertex of* $S'_j(2, K_4)$, $j \neq i$ *are in* $L$. *By doing so, we can remove maximum of 4 vertices in* $S'(3, K_4)$ *which is obtained by*

*considering Figure 5(b) and 5(c) which is equal to obtaining minimum liar's dominating set through 5(a). Thus there are* 36 *vertices in* $S'(3, K_4)$ *and paths* $P_4$ *in* $G[L]$ *are reduced to* $P_3$.

In view of Remark 4.3 we have the following result.

**Theorem 4.4.** *Let* $G$ *be a Sierpiński complete graph* $S'(n, K_4)$, $n \geq 2$ *then* $\gamma_L S'(n, K_4) = 4[\gamma_L S'(n-1, K_4)]$.

**Proof.** Let us prove the result by the method of induction. Consider $n = 3$. In view of Theorem 4.2, $\gamma_L S'(2, K_4) = 9$. Since $S'(3, K_4)$ is made up of 4 copies of $S'(2, K_4)$, the minimum liar's dominating set in each of $S'_i(2, K_4)$ is same as that of $S'(2, K_4)$. Thus $\gamma_L S'(3, K_4) = 4\gamma_L S'(2, K_4)$. Suppose in $S'_i(2, K_4)$ the vertices selected in liar's dominating set are by Remark 4.3, (i.e) all the corner vertices are in $L$ then $\gamma_L S'(3, K_4) = 4(10) - 4 = 36$. Since $S'(3, K_4)$ consists of only $P_3$ in $G[L]$, $\gamma_L S'(4, K_4) = 4[\gamma_L S'(3, K_4)]$. Let us assume that the result is true for $n < k$. Let $n = k$. By induction hypothesis, $\gamma_L S'(n, K_4) = 4[\gamma_L S'(n-1, K_4)]$.

## 5. Conclusion

Liar's domination is applied to protect a network even when one protection device becomes faulty or is allowed to lie. This paper provides the liar's domination number for Sierpiński-cycle graphs $S(n, C_4)$, and $S'(n, C_4)$ as well as Sierpiński-complete graphs $S(n, K_4)$ and $S'(n, K_4)$. The further work can be extended to families of Sierpiński-graphs $S(n, G)$ namely $G$ isomorphic to other cycle, complete graph, path, star graphs etc.

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