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HYPERGEOMETRIC FORMS OF SOME FUNCTIONS INVOLVING ARCSINE(x) USING DIFFERENTIAL EQUATION APPROACH

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Abstract: In this paper, by changing the independent and dependent variables in the suitable ordinary differential equations of second and third order and comparing the resulting ordinary differential equations with standard ordinary hypergeometric differential equations of Gauss and Clausen, we obtain the hypergeometric forms of following functions:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}, \quad [\sin^{-1}(x)]^2 \quad \textit{and} \quad \sin^{-1}(x).$$

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1. Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \ \{1, 2, 3, \cdots\}\,; \mathbb{N}_0 := \mathbb{N}\bigcup\{0\}\,; \mathbb{Z}_0^- := \mathbb{Z}^-\bigcup\{0\} = \{0, -1, -2, -3, \cdots\}\,.$
The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol:

The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ [9, p. 22 eq(1), p. 32 Q. N.(8) and Q. N.(9)], see also [11, p. 23, eq(22) and eq(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \prod\limits_{j=0}^{n-1}(\lambda+j) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda = -n;\ \nu = k;\ n,k \in \mathbb{N}_0;\ 0 \leqq k \leqq n) \\ 0 & (\lambda = -n;\ \nu = k;\ n,k \in \mathbb{N}_0;\ k > n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{lr} (\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{lr} \alpha_1, \alpha_2, \ldots, \alpha_p; & \\ & z \\ \beta_1, \beta_2, \ldots, \beta_q; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots\ ; \quad j = 1, 2, \ldots, q.$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$,

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$,

(iv) converges absolutely for $|z| = 1$, if $p = q + 1$, and $\Re(\omega) > 0$,

(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.2}$$

$\Re(\omega)$ being the real part of complex number ω.

(I) When $x = \sqrt{t}$, then

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \tag{1.3}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2\sqrt{t}\ \frac{dy}{dt}, \tag{1.4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(2\sqrt{t}\ \frac{dy}{dt}\right)\frac{dt}{dx}$$

after simplification, we get

$$\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}, \tag{1.5}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dt}\left(4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)\frac{dt}{dx}$$

after simplification, we get

$$\frac{d^3y}{dx^3} = 8t^{\frac{3}{2}}\frac{d^3y}{dt^3} + 12\sqrt{t}\frac{d^2y}{dt^2}. \tag{1.6}$$

(II) When $y = z(\sqrt{t})$, where z is the function of t then

$$\frac{dy}{dt} = \sqrt{t}\ \frac{dz}{dt} + \frac{z}{2\sqrt{t}}, \tag{1.7}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\sqrt{t}\ \frac{dz}{dt} + \frac{z}{2\sqrt{t}}\right)$$

after simplification, we get

$$\frac{d^2y}{dt^2} = \sqrt{t}\,\frac{d^2z}{dt^2} + \frac{1}{\sqrt{t}}\frac{dz}{dt} - \frac{z}{4t^{\frac{3}{2}}}\ . \tag{1.8}$$

(III) When $y = zt$, where z is the function of t then

$$\frac{dy}{dt} = z + t\,\frac{dz}{dt}, \tag{1.9}$$

$$\frac{d^2y}{dt^2} = 2\frac{dz}{dt} + t\,\frac{d^2z}{dt^2}, \tag{1.10}$$

$$\frac{d^3y}{dt^3} = 3\frac{d^2z}{dt^2} + t\,\frac{d^3z}{dt^3}\ . \tag{1.11}$$

(IV) We know that

$$z = {}_2F_1\left[\begin{matrix} a,\ b; & \\ & t \\ c; & \end{matrix}\right], \tag{1.12}$$

is one of the series solution of the following Gauss' ordinary hypergeometric homogeneous linear differential equation of second order with variable coefficients

$$t(1-t)\frac{d^2z}{dt^2} + [c-(a+b+1)t]\frac{dz}{dt} - abz = 0\ . \tag{1.13}$$

(V) We know that

$$z = {}_3F_2\left[\begin{matrix} \alpha,\ \beta,\ \gamma; & \\ & t \\ \lambda,\ \mu; & \end{matrix}\right], \tag{1.14}$$

is one of the series solution of the following Clausen's ordinary hypergeometric homogeneous linear differential equation of third order with variable coefficients

$$t^2(1-t)\frac{d^3z}{dt^3} + [(1+\lambda+\mu)-(3+\alpha+\beta+\gamma)t]t\frac{d^2z}{dt^2} +$$

$$+[\lambda\mu-(1+\alpha+\beta+\gamma+\alpha\beta+\alpha\gamma+\beta\gamma)t]\frac{dz}{dt} - \alpha\beta\gamma z = 0\ . \tag{1.15}$$

The present article is organized as follows. In section 3, we have derived the hypergeometric forms of some functions involving arcsine function, using differential equation approach. For hypergeometric forms of other mathematical functions and functions of mathematical physics, one can refer the literature [1], [2], [3], [4], [5], [6], [7], [8], [10] and [12], where the proof of hypergeometric forms of related functions are not given. So we are interested to give the proof of hypergeometric forms of some arcsine function using differential equation approach.

2. Some Hypergeometric Forms

When $|x| < 1$, then following hypergeometric forms hold true:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}} = x\ {}_2F_1\left[\begin{array}{c} 1,\ 1; \\ \frac{3}{2}; \end{array} x^2\right]. \tag{2.1}$$

$$[\sin^{-1}(x)]^2 = x^2\ {}_3F_2\left[\begin{array}{c} 1,\ 1,\ 1; \\ 2,\ \frac{3}{2}; \end{array} x^2\right]. \tag{2.2}$$

$$\sin^{-1}(x) = x\ {}_2F_1\left[\begin{array}{c} \frac{1}{2},\ \frac{1}{2}; \\ \frac{3}{2}; \end{array} x^2\right]. \tag{2.3}$$

3. Proof of Hypergeometric Forms

Proof of hypergeometric form (2.1):

Consider the following function

$$y = \frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}$$

$$or \quad \sqrt{(1-x^2)}\ y = \sin^{-1}(x). \tag{3.1}$$

Differentiate the equation (3.1) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)\frac{dy}{dx} - xy = 1. \tag{3.2}$$

Again differentiate the equation (3.2) w.r.t. x and apply product rule, after simplification we have

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0. \tag{3.3}$$

Put $x = \sqrt{t}$, then use values of equations (1.4) and (1.5) in above differential equation (3.3), after simplification we get

$$t(1-t)\frac{d^2y}{dt^2} + \left\{\frac{1}{2} - 2t\right\}\frac{dy}{dt} - \frac{1}{4}y = 0\ . \tag{3.4}$$

Now substitute $y = z(\sqrt{t})$ and put the values of equations (1.7) and (1.8) in above differential equation (3.4), after simplification we obtain

$$t(1-t)\frac{d^2z}{dt^2} + \left\{\frac{3}{2} - 3t\right\}\frac{dz}{dt} - z = 0 \ . \tag{3.5}$$

Now compare the coefficients of above differential equation (3.5) with Gauss' standard differential equation (1.13), we get

$$c = \frac{3}{2},\ a + b + 1 = 3 \ and\ ab = 1 \ .$$

Now solve the above algebraic equations simultaneously, we get

$$a = 1,\ b = 1 \ .$$

Therefore the solution of above differential equation (3.5) is given by

$$z = {}_2F_1\left[\begin{matrix} 1,\ 1; \\ \frac{3}{2}; \end{matrix}\ t\right],$$

$$y = \sqrt{t}\ {}_2F_1\left[\begin{matrix} 1,\ 1; \\ \frac{3}{2}; \end{matrix}\ t\right],$$

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}} = x\ {}_2F_1\left[\begin{matrix} 1,\ 1; \\ \frac{3}{2}; \end{matrix}\ x^2\right].$$

This completes the proof of hypergeometric form (2.1).

Proof of hypergeometric form (2.2):
Consider the following function

$$y = [\sin^{-1}(x)]^2. \tag{3.6}$$

Differentiate the equation (3.6) w.r.t. x, we get

$$\sqrt{(1-x^2)}\frac{dy}{dx} = 2[\sin^{-1}(x)]. \tag{3.7}$$

Again differentiate the equation (3.7) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2 \ . \tag{3.8}$$

Now again differentiate the equation (3.8) w.r.t. x and apply product rule, after simplification we obtain

$$(1-x^2)\frac{d^3y}{dx^3} - 3x\frac{d^2y}{dx^2} - \frac{dy}{dx} = 0 \ . \tag{3.9}$$

Put $x = \sqrt{t}$, then use the values of equations (1.4), (1.5) and (1.6) in above differential equation (3.9), after simplification we get

$$t(1-t)\frac{d^3y}{dt^3} + \left\{\frac{3}{2} - 3t\right\}\frac{d^2y}{dt^2} - \frac{dy}{dt} = 0 \ . \tag{3.10}$$

Now substitute $y = tz$ and put the values of equations (1.9), (1.10) and (1.11) in above differential equation (3.10), after simplification we have

$$t^2(1-t)\frac{d^3z}{dt^3} + \left\{\frac{9}{2} - 6t\right\}t\frac{d^2z}{dt^2} + \{3-7t\}\frac{dz}{dt} - z = 0 \ . \tag{3.11}$$

Now compare the coefficients of above differential equation (3.11) with Clausen's standard differential equation (1.15), we get

$$1+\lambda+\mu = \frac{9}{2}, 3+\alpha+\beta+\gamma = 6, \lambda\mu = 3, 1+\alpha+\beta+\gamma+\alpha\beta+\alpha\gamma+\beta\gamma = 7 \ and \ \alpha\beta\gamma = 1.$$

Now solve the above algebraic equations simultaneously, we get

$$\lambda = 2,\ \mu = \frac{3}{2},\ \alpha = 1,\ \beta = 1,\ \gamma = 1 \quad .$$

Therefore the solution of above differential equation (3.11) is given by

$$z = {}_3F_2\left[\begin{matrix} 1,1,1; \\ 2,\ \frac{3}{2}; \end{matrix} \ t \right],$$

$$y = t\ {}_3F_2\left[\begin{matrix} 1,1,1; \\ 2,\ \frac{3}{2}; \end{matrix} \ t \right],$$

$$[\sin^{-1}(x)]^2 = x^2\ {}_3F_2\left[\begin{matrix} 1,1,1; \\ 2,\ \frac{3}{2}; \end{matrix} \ x^2 \right].$$

This completes the proof of hypergeometric form (2.2).

Proof of hypergeometric form (2.3):
Consider the following function

$$y = \sin^{-1}(x) \ . \tag{3.12}$$

Differentiate the equation (3.12) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

$$or \quad \sqrt{(1-x^2)}\ \frac{dy}{dx} = 1 \quad . \tag{3.13}$$

Again differentiate the equation (3.13) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0 \ . \tag{3.14}$$

Put $x = \sqrt{t}$, then use values of equations (1.4) and (1.5) in above differential equation (3.14), after simplification we obtain

$$t(1-t)\frac{d^2y}{dt^2} + \left\{\frac{1}{2} - t\right\}\frac{dy}{dt} = 0 \ . \tag{3.15}$$

Now substitute $y = z(\sqrt{t})$ and put the values of equations (1.7) and (1.8) in above differential equation (3.15), after simplification we have

$$t(1-t)\frac{d^2z}{dt^2} + \left\{\frac{3}{2} - 2t\right\}\frac{dz}{dt} - \frac{1}{4}z = 0 \ . \tag{3.16}$$

Now compare the coefficients of above differential equation (3.16) with Gauss' standard differential equation (1.13), we get

$$c = \frac{3}{2},\ a + b + 1 = 2 \ and \ ab = \frac{1}{4} \quad .$$

Now solve the above algebraic equations simultaneously, we get

$$a = \frac{1}{2},\ b = \frac{1}{2} \quad .$$

Therefore the solution of above differential equation (3.16) is given by

$$z = {}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ \frac{3}{2};\end{array} t\right],$$

$$y = \sqrt{t}\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ \frac{3}{2};\end{array} t\right],$$

$$\sin^{-1}(x) = x\ {}_2F_1\left[\begin{array}{c}\frac{1}{2},\ \frac{1}{2};\\ \\ \frac{3}{2};\end{array} x^2\right].$$

This completes the proof of hypergeometric form (2.3).

4. Conclusion

In our present investigation, we derived the hypergeometric forms of some functions involving arcsine function by using differential equation approach. Moreover, the results derived in this paper are expected to have useful applications in wide range of problems of Mathematics, Statistics and Physical sciences. Similarly, we can derive the hypergeometric forms of other functions in an analogous manner.

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