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GENERALIZED *H*- RESOLVENT EQUATION WITH $H - \phi - \eta$ ACCRETIVE OPERATOR

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Abstract: In this paper, we consider extended variational-like inclusion problem (for short EVLIP) which contains many known variational inclusions existing in literature. In connection with EVLIP we consider a generalized resolvent equation problem with H-ϕ-η-accretive operator called generalized H-resolvent equation problem (for short H-REP). To compute the approximate solution of H-REP, we introduce an algorithm. Convergence of sequences procreated by algorithm are also studied.

Keywords and Phrases: Extended Variational-Like Inclusion, Resolvent Operator, Generalized H-Resolvent Equation, Algorithm, Convergence.

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1. Introduction, Notations, Definitions and Known Results

In past years, generalized forms of variational inequalities, variational inclusions and variational-like inclusions, have been expansively studied and extended in different directions to study the practical problems arising in optimization, economics, finance, applied science etc. See, for example [1, 3-6, 10, 12-19, 24] and references therein. As we all know that, develop an adept iterative algorithm for approximation solution of variational inclusions is most interesting aspect of variational inclusion theory. It is well known that projection method and Wiener-Hopf equation can not be improved to solve nonlinear variational inequalities and variational inclusions. Then resolvent operator technique is strategic and useful for

approximation solvability of variational inclusions. Lot of studies and research has been done on several techniques for computing the solution of the variational inclusion and variational-like inclusion in the setting of different spaces, see [1, 3-8, 11-19, 21, 22, 24] and references therein.
Fang and Huang [16] have extended the concept of resolvent operators to the new H-accretive operators, which was associated with the m-accretive operators. Ahamd and Ansari [5] have considered generalized variational inclusions (GVI) and they also considered generalized resolvent equation with H-accretive operator called H-resolvent equation (H-RE) and suggested the algorithm for unique solution of GVI and H-RE and studied the convergence of iterative sequences generated by the proposed algorithm.
In this paper, we consider extended variational-like inclusion problem (for short EVLIP) which contains many known variational inclusions existing in literature. In connection with EVLIP we consider a generalized resolvent equation problem with H-ϕ-η-accretive operator called generalized H-resolvent equation problem (for short H-REP). To compute the approximate solution of H-REP, we introduce an algorithm. Convergence of sequences procreated by algorithm are also studied.

Now, we present some basic notations, definitions and known results of functional analysis relevant to our paper. Throughout the paper unless otherwise specified, we assume that X is a real Banach space endowed with a norm $\|.\|$ and topological dual X^*. d is a metric induced by the norm $\|.\|$, $CB(X)$ is the family of all non-empty closed and bounded subsets of X, 2^X is family of all non-empty subsets of X, and $\mathcal{D}(.,.)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(E,F) = \max\{\sup_{x\in E} d(x,F), \sup_{y\in F} d(E,y)\},$$

where $d(x,F) = \inf_{y\in F} d(x,y)$ and $d(E,y) = \inf_{x\in E} d(x,y)$.

Definition 1.1. [9] *Let X be a real Banach space then generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by*

$$J_q(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, $J_q(x) = \|x\|^{q-1}J_2(x)$ for $x \neq 0$ and J_q is a single-valued if X is strictly convex. If X is real Hilbert space, J_2 becomes the identity mapping on X.

Definition 1.2. [2] *A Banach space X is said to be uniformly smooth if for any given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{\|x+y\| + \|x-y\|}{2} - 1 \leq \epsilon\|y\|$$

holds.
The function

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t\}.$$

is called the modulus of smoothness of the space X.

Remark 1.3. *The space X is uniformly convex if and only if $\rho_X(\epsilon) > 0$ for all $\epsilon > 0$, and it is called uniformly smooth if and only if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$.*

Definition 1.4. [2] *The space X is called q-uniformly smooth, if there exist a constant $C > 0$ such that*

$$\rho_X(t) \leq Ct^q, q > 1.$$

Note that J_q is single valued if X is uniformly smooth. The following inequality in q-uniformly smooth Banach spaces has been proved by Xu [25].

Lemma 1.5. [25] *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 1.6. [3] *Let $A, B : X \to X$ and $\eta, H : X \times X \to X$ be the single valued mappings.*

(i) A is said to be η-accretive, if $\langle Ax - Ay, J_q(\eta(x,y))\rangle \geq 0, \forall x, y \in X$;

(ii) A is said to be strictly η-accretive, if A is η-accretive and equality holds if and only if $x = y$;

(iii) $H(A,.)$ is said to be α-strongly η-accretive with respect to A, if there exist a constant $\alpha > 0$ such that $\langle H(Ax,u) - H(Ay,u), J_q(\eta(x,y))\rangle \geq \alpha\|x - y\|^q, \forall x, y, u \in X$;

(iv) $H(.,B)$ is said to be β-relaxed η-accretive with respect to B, if there exist a constant $\beta > 0$ such that $\langle H(u,Bx) - H(u,By), J_q(\eta(x,y))\rangle \geq (-\beta)\|x - y\|^q, \forall x, y, u \in X$;

(v) $H(.,.)$ is said to r_1-Lipschitz continuous with respect to A, if there exist a constant $r_1 > 0$ such that $\|H(Ax,u) - H(Ay,u)\| \leq r_1\|x - y\|, \forall x, y, u \in X$. In a similar way, we can define the Lipschitz continuity of the mapping $H(.,.)$ with respect to B.

(vi) *η is said to be τ-Lipschitz continuous, if there exist a constant $\tau > 0$ such that $\|\eta(x,y)\| \leq \tau\|x-y\|, \forall x,y \in X$.*

Definition 1.7. [3] *Let $N, P : X \times X \times X \to X$ and $\eta : X \times X \to X$ be the single valued mappings. Let $M : X \times X \to 2^X$ be multi-valued mapping.*

(i) *M is said to be η-accretive, if $\langle u - v, J_q(\eta(x,y))\rangle \geq 0, \forall x,y \in X, u \in M(x,z), v \in M(y,z)$, for each fixed $z \in X$;*

(ii) *M is said to be strictly η-accretive, if M is η-accretive and equality holds if and only if $x = y$;*

(iii) *N is said to be t-relaxed η-accretive in the first argument, if there exist a constant $t > 0$ such that $\langle N(x,u,v) - N(y,u,v), J_q(\eta(x,y))\rangle \geq -t\|x - y\|^q, \forall x,y,u,v \in X$;*

(iv) *N is said to be ξ-Lipschitz continuous in the first argument, if there exists a constant $\xi > 0$ such that $\|N(x,u,v) - N(y,u,v)\| \leq \xi\|x-y\|, \forall x,y,u,v \in X$. Similarly, we can define the lipschitz continuity of N in the second and third argument.*

(v) *P is said to be ζ-Lipschitz continuous in the first argument, if there exists a constant $\zeta > 0$ such that $\|P(x,u,v) - P(y,u,v)\| \leq \zeta\|x-y\|, \forall x,y,u,v \in X$. Similarly, we can define the lipschitz continuity of P in the second and third argument.*

Definition 1.8. [16] *The operator $H : X \to X$ is said to be*

(i) *accretive if $\langle H(x) - H(y), J_q(x-y)\rangle \geq 0, \forall x,y \in X$*

(ii) *strongly accretive if there exists a constant $r > 0$ such that $\langle H(x) - H(y), J_q(x-y)\rangle \geq r\|x-y\|^q, \forall x,y \in X$*

Definition 1.9. [3] *Let $\phi, A, B : X \to X$ and $H, \eta : X \times X \to X$ be the single-valued mappings. Let $M : X \times X \to 2^X$ be a multi-valued mapping. M is said to be $H(.,.) - \phi - \eta$-accretive operator with respect to mappings A and B, if for each fixed $z \in X$, $\phi \circ M(.,z)$ is η-accretive in the first argument and $(H(A,B) + \phi \circ M(.,z))(X) = X$.*

Theorem 1.10. [3] *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B, $\alpha > \beta$. Let M be an $H(.,.) - \phi - \eta$-accretive operator with respect to mappings A and B. Then the operator $(H(A,B) + \phi \circ$*

$M(.,z))^{-1}$ is single-valued for each fixed $z \in X$.

Definition 1.11 [3] *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B, $\alpha > \beta$. Let M be an $H(.,.)-\phi-\eta$-accretive operator with respect to mappings A and B. Then for each fixed $z \in X$, the resolvent operator $R_{M(.,z)}^{H(.,.)-\phi-\eta} : X \to X$ is defined by*

$$R_{M(.,z)}^{H(.,.)-\phi-\eta}(u) = (H(A,B) + \phi \circ M(.,z))^{-1}(u), \forall u \in X.$$

Theorem 1.12. [3] *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B, $\alpha > \beta$ and η is τ-Lipschitz continuous. Let $M : X \times X \to 2^X$ is a $H(.,.)-\phi-\eta$-accretive operator with respect to mappings A and B. Then the resolvent operator $R_{M(.,z)}^{H(.,.)-\phi-\eta} : X \to X$ is $\frac{\tau^{q-1}}{\alpha-\beta}$-Lipschitz continuous i.e.,*

$$\|R_{M(.,z)}^{H(.,.)-\phi-\eta}(u) - R_{M(.,z)}^{H(.,.)-\phi-\eta}(v)\| \leq \frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|, \forall u,v \in X \text{ and each fixed } z \in X.$$

2. Extended Variational-Like Inclusion Problem

Let $G, J, K, L, R, S, T : X \to CB(X)$ be multi-valued mappings. $A, B, \phi : X \to X$, $H, \eta : X \times X \to X$ and $N, P : X \times X \times X \to X$ be single valued mappings. Suppose $M : X \times X \to 2^X$ be a multi-valued mapping such that M is $H(.,.)-\phi-\eta-$ accretive operator.

We consider the following problem of finding $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ and

$$0 \in N(u,v,w) - P(j,k,l) + M(x,z) \tag{2.1}$$

Problem (2.1) is called extended variational-like inclusion problem.

Below are some special cases of our problem:

(i) If $P \equiv R \equiv 0$ and $N(.,.,.) = N(.,.)$ then our problem reduces to the problem considered by Ahmad et al. [3].

(ii) If $P \equiv R \equiv 0$ and $N(.,.,.) = N(.,.)$, X is real Hilbert space and $M(.,z)$ is maximal monotone operator then problem similar to (3.1) was introduced and studied by Huang et al. [20].

(iii) If $P \equiv T \equiv R \equiv G \equiv 0$, S is single-valued and identity mapping and $N(.,.,.) = N(.), M(.,.) = M(.)$ then our problem reduces to the problem considered by Bi et al. [11], that is find $u \in X$ such that $0 \in N(u) + M(u)$.

It is easy to see that (2.1) includes many more known variational inclusions considered and studied in the literature.

3. Generalized H-Resolvent Equation Problem

In this section, we propose the *generalized H-resolvent equation problem* for the case when $H(.,.) = H(.)$ along with some suitable assumption. we consider the following *generalized H-resolvent equation problem* to find $s, x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ such that

$$N(u, v, w) - P(j, k, l) + \phi^{-1} J^{H-\phi-\eta}_{M(.,z)}(s) = 0 \tag{3.1}$$

where $J^{H-\phi-\eta}_{M(.,z)} = I - H(R^{H-\phi-\eta}_{M(.,z)}), I$ is the identity operator, $R^{H-\phi-\eta}_{M(.,z)}$ is the *H-resolvent operator*. The equation (3.1) is called *generalized H-resolvent equation.*

Lemma 3.1. *Let X be a q-uniformly smooth Banach space. $G, J, K, L, R, S, T : X \to CB(X)$ be multi-valued mappings, $H : X \to X$ be single valued mapping and $\phi : X \to X$ be a mapping satisfying $\phi(x + y) = \phi(x) + \phi(y)$ and $ker(\phi) = 0$, where $ker(\phi) = \{x \in X : \phi(x) = 0\}$. Let $\eta : X \times X \to X$ be single valued mappings and $N, P : X \times X \times X \to X$ be also single valued mappings. Let $M : X \times X \to 2^X$ be a multi-valued mapping such that M is $H - \phi - \eta -$ accretive operator. Then (x, u, v, w, z, j, k, l) where $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$, is a solution of problem (3.1) if and only if (x, u, v, w, z, j, k, l) satisfies*

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)]$$

Proof. Let (x, u, v, w, z, j, k, l) where $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ satisfies the above equation, i.e.,

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)]$$

Using the definition of resolvent operator, we have

$$\begin{aligned}
x &= (H(.) + \phi \circ M(., z))^{-1}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)] \\
&\Leftrightarrow H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l) \in H(x) + \phi \circ M(x, z) \\
&\Leftrightarrow 0 \in \phi \circ N(u, v, w) - \phi \circ P(j, k, l) + \phi \circ M(x, z) \\
&\Leftrightarrow 0 \in \phi(N(u, v, w) - P(j, k, l)) + M(x, z)) \\
&\Leftrightarrow \phi^{-1}(0) \in N(u, v, w) - P(j, k, l) + M(x, z) \\
&\Leftrightarrow 0 \in N(u, v, w) - P(j, k, l) + M(x, z).
\end{aligned}$$

This completes the proof.
Now we present an equivalence between (2.1) and (3.1).

Proposition 3.2. [5] *The (2.1) has a solution (x, u, v, w, z, j, k, l) with $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ if and only if (3.1) has a solution $(s, x, u, v, w, z, j, k, l)$ with $s, x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$, where*

$$x = R^{H-\phi-\eta}_{M(.,z)}(s) \tag{3.2}$$

and

$$s = H(x) - \phi \circ (N(u, v, w) - P(j, k, l)) \tag{3.3}$$

Proof. Let (x, u, v, w, z, j, k, l) be the solution of (2.1) then by lemma 3.1 it is a solution of following equation

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ (N(u, v, w) - P(j, k, l))] \tag{3.4}$$

Let $s = H(x) - \phi \circ (N(u, v, w) - P(j, k, l))$, then from (3.4), we have

$$x = R^{H-\phi-\eta}_{M(.,z)}(s).$$

By using the fact that $J^{H-\phi-\eta}_{M(.,z)} = I - H(R^{H-\phi-\eta}_{M(.,z)})$, we obtain

$$\begin{aligned}
s &= H(R^{H-\phi-\eta}_{M(.,z)}(s)) - \phi \circ (N(u, v, w) - P(j, k, l)) \\
\Leftrightarrow s - H(R^{H-\phi-\eta}_{M(.,z)}(s)) &= -\phi \circ (N(u, v, w) - P(j, k, l)) \\
\Leftrightarrow [I - H(R^{H-\phi-\eta}_{M(.,z)})](s) &= -\phi \circ (N(u, v, w) - P(j, k, l)) \\
\Leftrightarrow J^{H-\phi-\eta}_{M(.,z)}(s) &= -\phi \circ (N(u, v, w) - P(j, k, l))
\end{aligned}$$

Hence $N(u, v, w) - P(j, k, l) + \phi^{-1} J^{H-\phi-\eta}_{M(.,z)}(s) = 0$
Based on proposition 3.2, we suggest the following iterative method to compute the approximate solution of (3.1).

Algorithm 3.3. *For any given $s_0, x_0 \in X$, $u_0 \in S(x_0)$, $v_0 \in T(x_0)$, $w_0 \in R(x_0)$, $z_0 \in G(x_0)$, $j_0 \in J(x_0)$, $k_0 \in K(x_0)$, and $l_0 \in L(x_0)$, compute the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$, $\{j_n\}$, $\{k_n\}$ and $\{l_n\}$ by the following iterative schemes*

$$x_{n+1} = R^{H-\phi-\eta}_{M(.,z)}(s_{n+1}) \tag{3.5}$$

$$u_n \in S(x_n), \|u_n - u_{n+1}\| \leq \mathcal{D}(S(x_n), S(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.6}$$

$$v_n \in T(x_n), \|v_n - v_{n+1}\| \leq \mathcal{D}(T(x_n), T(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.7}$$

$$w_n \in R(x_n), \|w_n - w_{n+1}\| \leq \mathcal{D}(R(x_n), R(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.8}$$

$$z_n \in G(x_n), \|z_n - z_{n+1}\| \leq \mathcal{D}(G(x_n), G(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.9}$$

$$j_n \in J(x_n), \|j_n - j_{n+1}\| \leq \mathcal{D}(J(x_n), J(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.10}$$

$$k_n \in K(x_n), \|k_n - k_{n+1}\| \leq \mathcal{D}(K(x_n), K(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.11}$$

$$l_n \in L(x_n), \|l_n - l_{n+1}\| \leq \mathcal{D}(L(x_n), L(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.12}$$

$$s_{n+1} = H(x_n) - \phi \circ (N(u_n, v_n, w_n) - P(j_n, k_n, l_n)) \tag{3.13}$$

$n = 0, 1, 2, 3, \ldots$
Now we study the existence of the solution of (3.1) and the convergence of iterative sequences generated by the above algorithm to the exact solution of (3.1).

Theorem 3.4. *Let X be a real q-uniformly smooth Banach space and $H : X \to X$ be a strongly accretive and Lipschitz continuous operator with constant r and γ, respectively. Let $\phi \circ N$ and $\phi \circ P$ be both Lipschitz continuous in all three arguments with constants ξ_1, ξ_2, ξ_3 and $\zeta_1, \zeta_2, \zeta_3$ respectively, also let G, J, K, L, R, S, T be $\mathcal{D}$-Lipschitz continuous with constants $\lambda_G, \lambda_J, \lambda_K, \lambda_L, \lambda_R, \lambda_S$ and λ_T, respectively. Suppose that $M : X \times X \to 2^X$ is $H - \phi - \eta$-accretive multivalued map such that*

$$0 < \frac{1}{r}[\gamma^q - (q - c_q)\{\xi_1(\lambda_S + \varepsilon^n) + \xi_2(\lambda_T + \varepsilon^n) + \xi_3(\lambda_R + \varepsilon^n)\}^q - (q - c_q)\{\zeta_1(\lambda_J + \varepsilon^n) \\ + \zeta_2(\lambda_K + \varepsilon^n) + \zeta_3(\lambda_L + \varepsilon^n)\}^q]^{\frac{1}{q}} < 1 \tag{3.14}$$

holds. Then there exists a unique solution $(s, x, u, v, w, z, j, k, l)$ with $s, x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$, and the iterative sequences $\{s_n\}, \{x_n\}, \{u_n\}, \{v_n\}, \{w_n\}, \{z_n\}, \{j_n\}, \{k_n\}$ and $\{l_n\}$ generated by Algorithm 3.3 converge to s,x,u,v,w,z,j,k,l strongly in X, respectively.
Proof.

$$\|s_{n+1} - s_n\| = \|H(x_n) - H(x_{n-1}) - \phi \circ [(N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})) \\ - (P(j_n, k_n, l_n) - P(j_{n-1}, k_{n-1}, l_{n-1}))]\| \tag{3.15}$$

By Lemma 1.5, we have

$$\begin{aligned}
&\|H(x_n)-H(x_{n-1})-\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le \|H(x_n)-H(x_{n-1})\|^q-q\langle\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1})),J_q(H(x_n)-H(x_{n-1}))\rangle+c_q\|\phi\circ(N(u_n,v_n,w_n)\\
&\qquad -N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q
\end{aligned}\tag{3.16}$$

Again by Lemma 1.5,

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le \|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|^q-(q-c_q)\|\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1}))\|^q
\end{aligned}\tag{3.17}$$

Now,

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|\\
&\le \|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_n,w_n))+\phi\circ(N(u_{n-1},v_n,w_n)-N(u_{n-1},v_{n-1},w_n))\\
&\qquad +\phi\circ(N(u_{n-1},v_{n-1},w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|\\
&\le \xi_1\|u_n-u_{n-1}\|+\xi_2\|v_n-v_{n-1}\|+\xi_3\|w_n-w_{n-1}\|\le\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)\\
&\qquad +\xi_3(\lambda_R+\varepsilon^n)\}\|x_n-x_{n-1}\|
\end{aligned}$$

So,

$$\begin{aligned}
\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|^q&\le\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)\\
&\qquad +\xi_3(\lambda_R+\varepsilon^n)\}^q\|x_n-x_{n-1}\|^q
\end{aligned}\tag{3.18}$$

Similarly,

$$\begin{aligned}
\|\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q&\le\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)\\
&\qquad +\zeta_3(\lambda_L+\varepsilon^n)\}^q\|x_n-x_{n-1}\|^q
\end{aligned}\tag{3.19}$$

Using (3.18) and (3.19), (3.17) becomes

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q-(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)\\
&\qquad +\zeta_3(\lambda_L+\varepsilon^n)\}^q]\|x_n-x_{n-1}\|^q
\end{aligned}$$

Therefore (3.16) becomes

$$\begin{aligned}
&\|H(x_n)-H(x_{n-1})-\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)\\
&\qquad\qquad -P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le \gamma^q\|x_n-x_{n-1}\|^q-(q-c_q)[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q\\
&\qquad -(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)+\zeta_3(\lambda_L+\varepsilon^n)\}^q]^q\|x_n-x_{n-1}\|^q
\end{aligned}$$

So from (3.15), we have

$$\begin{aligned}
\|s_{n+1}-s_n\| &\le [\gamma^q-(q-c_q)[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q\\
&-(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)+\zeta_3(\lambda_L+\varepsilon^n)\}^q]^q]^{\frac{1}{q}}\|x_n-x_{n-1}\| \quad (3.20)
\end{aligned}$$

By (3.5), we obtain

$$\begin{aligned}
\|x_n-x_{n-1}\| &= \|x_n-x_{n-1}+x_n-x_{n-1}-R^{H-\phi-\eta}_{M(.,z)}(s_n)+R^{H-\phi-\eta}_{M(.,z)}(s_{n-1})\|\\
&\le 2\|x_n-x_{n-1}\|-\|R^{H-\phi-\eta}_{M(.,z)}(s_n)-R^{H-\phi-\eta}_{M(.,z)}(s_{n-1})\|\\
&\le 2\|x_n-x_{n-1}\|-\frac{1}{r}\|s_n-s_{n-1}\|
\end{aligned}$$

Where $R^{H-\phi-\eta}_{M(.,z)}$ is $\frac{1}{r}$-Lipschitz continuous.
Therefore,

$$\|x_n-x_{n-1}\|\le\frac{1}{r}\|s_n-s_{n-1}\| \quad (3.21)$$

By combining (3.20) and (3.21), we get

$$\|s_{n+1}-s_n\|\le b\|s_n-s_{n-1}\| \quad (3.22)$$

where

$$\begin{aligned}
b=\frac{1}{r}[\gamma^q-(q-c_q)[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q-(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)\\
+\zeta_2(\lambda_K+\varepsilon^n)+\zeta_3(\lambda_L+\varepsilon^n)\}^q]^q]^{\frac{1}{q}} \quad (3.23)
\end{aligned}$$

From (3.14), it follows that $0\le b<1$. Consequently, from (3.22), we see that the sequence $\{s_n\}$ is cauchy sequence in a Banach space X. So there exist $s\in X$ such that $\{s_n\}\to s$ as $n\to\infty$. From (3.21), we know that the sequence $\{x_n\}$ is a cauchy sequence in X, so there exist $x \in X$ such that $\{x_n\} \to x$. Also from Algorithm 3.3, we have

$$\begin{aligned}
\|u_n - u_{n+1}\| &\leq (\lambda_S + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|v_n - v_{n+1}\| &\leq (\lambda_T + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|w_n - w_{n+1}\| &\leq (\lambda_R + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|z_n - z_{n+1}\| &\leq (\lambda_G + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|j_n - j_{n+1}\| &\leq (\lambda_J + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|k_n - k_{n+1}\| &\leq (\lambda_K + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|l_n - l_{n+1}\| &\leq (\lambda_L + \varepsilon^n)\|x_n - x_{n+1}\|.
\end{aligned}$$

and hence $\{u_n\}, \{v_n\}, \{w_n\}, \{z_n\}, \{j_n\}, \{k_n\}$ and $\{l_n\}$ are also cauchy sequences in X, so that there exist u, v, w, z, j, k, l in X such that $\{u_n\} \to u, \{v_n\} \to v, \{w_n\} \to w, \{z_n\} \to z, \{j_n\} \to j, \{k_n\} \to k$ and $\{l_n\} \to l$ as $n \to \infty$. Now using the continuity of operators $R, S, T, G, J, K, L, H, \phi\circ N, \phi\circ P, \eta$ and M and by Algorithm 3.3, we have

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)].$$

Now, we shall show that $u \in S(x)$
$d(u, S(x)) \leq \|u - u_n\| + d(u, S(x)) \leq \|u - u_n\| + \mathcal{D}(S(x_n), S(x))$
$\leq \|u - u_n\| + \lambda_S\|x_n - x\| \to 0$ as $n \to \infty$.
$\Rightarrow d(u, S(x)) = 0$, since $S(x) \in CB(X)$ [23], it follows that $u \in S(x)$.
Similarly we can prove that $v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$. Let $(s^*, x^*, u^*, v^*, w^*, z^*, j^*, k^*, l^*)$ be another solution of (H-REP). Then by Lemma 3.1, we have

$$x^* = R^{H-\phi-\eta}_{M(.,z)}[H(x^*) - \phi \circ N(u^*, v^*, w^*) + \phi \circ P(j^*, k^*, l^*)]$$

From above two equations and by using the same argument given above we get

$$\|x - x^*\| \leq b\|x - x^*\|,$$

where b is defined by (3.23). Since $0 \leq b < 1$, we get $x = x^*$, then by algorithm 3.3 $(s^*, x^*, u^*, v^*, w^*, z^*, j^*, k^*, l^*)$ is unique solution of (H-REP). This completes the proof.

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