South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 2 (2020), pp. 41-56

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ON EXISTENCE OF $\psi-$HILFER HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

Shabna M. S. and Ranjini M. C.

Department of Mathematics,
MES Mampad College, Malappuram, Kerala - 676542 INDIA

E-mail : shabnasaidh@gmail.com, ranjiniprasad@gmail.com

(**Received:** *Dec. 04, 2019* **Accepted:** *May. 11, 2020* **Published:** *Aug. 30, 2020)*

Abstract: In this paper we derive existence results for the solutions to the first order hybrid fractional differential equations with perturbation of first kind and second kind involving $\psi-$Hilfer fractional derivative using different fixed point theorems. Finally the result is illustrated with an example.

Keywords and Phrases: Fractional differential equation, Hybrid, Fixed point theorem, $\psi-$Hilfer fractional derivative.

2010 Mathematics Subject Classification: 26A33, 34A38, 47H10, 34A12, 34K37.

1. Introduction

For the last few decades, many researchers attracted towards the study of fractional calculus motivated by it's wide application both in pure and applied mathematics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. For studying about the dynamical systems described by non linear differential and integral equations, the perturbation techniques are very useful. The perturbed differential equations are categorized into various types. Quadratic perturbations of nonlinear fractional differential equations, which is an important type of these perturbations (Hybrid Differential Equations) have achieved a great deal of interest and attention of several researchers. Dhange and Lakshmikantham [14, 15] and Dhange and Jadhav [16] discussed the existence and uniqueness theorems of the solution to the ordinary first order hybrid differential equations with perturbation of first and second kind respectively. Much work has done in this theory and we refer the readers to the articles [17, 18, 19, 20, 21, 22].

Fractional Calculus is a rich field and we can find several definitions for fractional integrals and fractional derivatives. One who interested in the study of fractional calculus may confuse to select operators. One way to overcome this problem is to consider more general definitions. Recently Sousa and Oliveira [23] proposed a new general fractional derivative, which is named as $\psi-$Hilfer fractional derivative. They derived around 22 types of fractional derivatives and integrals from $\psi-$Hilfer operator. Many works has done on the fractional equations involving $\psi-$Hilfer fractional operator [23, 24, 25, 26, 27, 28].

In this paper we discuss the existences of hybrid fractional differential equations of first and second type involving $\psi-$Hilfer fractional derivative, which are given by

$$\begin{cases} {}^H\mathbb{D}_{0+}^{\alpha,\beta;\psi} \frac{x(t)}{f(t,x(t))} = g(t,x(t)), \quad a.e \quad t \in I = [0,T] \\ I_{0+}^{1-\gamma,\psi} \frac{x(0)}{f(0,x(0))} = x_0 \end{cases} \tag{1}$$

and

$$\begin{cases} {}^H\mathbb{D}_{t_0+}^{\alpha,\beta;\psi}[x(t) - f(t,x(t))] & = g(t,x(t)),\ a.e.\ t \in J = [t_0, t_0 + a] \\ I_{t_0}^{1-\gamma;\psi}[x(t_0) - f(t_0, x(t_0))] & = \sigma \in \mathbb{R} \end{cases} \tag{2}$$

where ${}^H\mathbb{D}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative with $0 < \alpha < 1, 0 \leq \beta \leq 1$, $\alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1$ and $f \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R}|\{0\})$ and $g \in \mathcal{C}_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$. $J = [t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some t_0 and $a \in \mathbb{R}$. $I = [t_0, t_0 + a]$,with $t_0 = 0$ and $a = T$.

This paper is organized as follows: Some basic definitions and lemmas are introduced in section 2. It also includes some results required to prove our main result. In section 3 we give the existence result for $\psi-$Hilfer Fractional Hybrid differential equation of first type based on Dhange fixed point theorem. In section 4 we give the existence result for $\psi-$Hilfer Fractional Hybrid differential equation of second type based on fixed point theorem is given. We finished the section with an example and plotted graphs for different functions.

2. Preliminaries

Let $[a, b]$, $(0 < a < b < \infty)$ be finite interval on the half axis $\mathbb{R}^+$ and let $C[a, b]$ be the space of continuous function f on $[a, b]$ with the norm. Define,

$$||f||_{C[a,b]} = \max_{x \in [a,b]} |f(t)|. \tag{3}$$

The weighted space $C_{1-\gamma;\psi}[a,b]$ of continuous f on $(a,b]$ is defined by

$$C_{1-\gamma;\psi}[a,b]=\{f:(a,b]\to\mathbb{R};(\psi(t)-\psi(a))^{1-\gamma}f(t)\in C[a,b]\}, \tag{4}$$

$0\le\gamma<1$ with the norm

$$||f||_{C_{1-\gamma;\psi}}[a,b]=||(\psi(t)-\psi(a))^{1-\gamma}f(t)||_{C[a,b]}=\max_{x\in[a,b]}|(\psi(t)-\psi(a))^{1-\gamma}f(t)|. \tag{5}$$

The weighted space $C^{n}_{\gamma;\psi}[a,b]$ of continuous f on (a,b] is defined by

$$C^{n}_{\gamma;\psi}[a,b]=\{f:(a,b]\to\mathbb{R};f(t)\in C^{n-1}[a,b];f^{(n)}(t)\in C_{\gamma,\psi}[a,b]\}, \tag{6}$$

$0<\gamma<1$ with the norm

$$\|f\|_{C^{n}_{\gamma;\psi}[a,b]}=\sum_{k=0}^{n-1}\|f^{(k)}\|_{C[a,b]}+\|f^{(n)}\|_{C_{\gamma;\psi}[a,b]}. \tag{7}$$

For the weighed function $\mathcal{C}_{1-\gamma;\psi}[a,b]$,

(i) The map $t\to g(t,x)$ is measurable for each $x\in\mathbb{R}$;

(ii) The map $x\to g(t,x)$ is continuous for each $t\in[a,b]$;

(iii) For each $g\in\mathcal{C}_{1-\gamma;\psi}[a,b]$, $g(t,x(t))$ is $\psi-$integrable.

Definition 1. [1] *Let (a,b), $(-\infty\le a<b\le\infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha>0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on (a,b). The left-sided fractional integral of a function f with respect to a function ψ on [a,b] is defined by:*

$$I^{\alpha;\psi}_{a+}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}f(s)ds. \tag{8}$$

The right-sided fractional integral is defined in an analogous form.

Definition 2. [23] *Let $n-1<\alpha<n$ with $n\in\mathbb{N}$, let $I=[a,b]$ be an interval such that $-\infty\le a<b\le\infty$ and let $f,\psi\in C^{n}[a,b]$ be two functions such that ψ is increasing and $\psi'(t)\neq0$, for all $t\in I$. The left-sided $\psi-$Hilfer fractional derivative ${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}(\cdot)$ of a function f of order α and type $0\le\beta\le1$, is defined by*

$${}^{H}\mathbb{D}^{\alpha,\beta;\psi}_{a+}f(t)=I^{\beta(n-\alpha);\psi}_{a+}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^{n}I^{(1-\beta)(n-\alpha);\psi}_{a+}f(t). \tag{9}$$

The right-sided $\psi-$Hilfer fractional derivative is defined in an analogous form.

Lemma 1. [23] *If $f \in C^n[a,b], 0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then*

$$I_{a+}^{\alpha;\psi}\ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\phi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a). \tag{10}$$

Lemma 2. [23] *If $f \in C^1[a,b], 0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then*

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi}\ f(t) = f(t). \tag{11}$$

Lemma 3. [6] *Let $\alpha > 0$ and $\delta > 0$. If $f(t) = (\psi(t)-\psi(a))^{\delta-1}$, then*

$$I_{a+}^{\alpha;\psi} f(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(t)-\psi(a))^{\alpha+\delta-1}. \tag{12}$$

Lemma 4. *Let $\psi \in C^1([a,b],\mathbb{R})$ be a function such that ψ is increasing and $\psi'(t) \neq 0, \forall t \in [a,b]$. If $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then $\psi-$Riemann-Liouville fractional integral operator $I_{a+}^{\alpha;\psi}(\cdot)$ is bounded from $C_{1-\gamma;\psi}[a,b]$ to $C_{1-\gamma;\psi}[a,b]$.*

$$\|I_{a+}^{\alpha;\psi} h\|_{C_{1-\gamma;\psi}[a,b]} \leq M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(t)-\psi(a))^{\alpha},$$

where M is the bound of a bounded function $(\psi(\cdot)-\psi(a))^{1-\gamma} h(\cdot)$.

Theorem 1. [15, 16] *Let S be a non empty, closed, convex and bounded subset of the Banach algebra X, and let $A : X \to X$ and $B : X \to X$ be two operators such that,*

(a) *A is Lipschitzian with a Lipschitz constant α;*

(b) *B is completely continuous;*

(c) *$x = Ax\,Bx \implies x \in S$ for all $y \in S$;*

(d) *$M\zeta(r) < r$, where $M = \|B(S)\| = sup\|B(x)\| : x \in S$;*

then the operator equation $Ax\,Bx = x$ has a solution in S.

Theorem 2. [15, 16] *Let S be a closed convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that,*

(a) *A is a nonlinear contraction;*

(b) B is continuous and compact;

(c) $x = Ax + Bx$ for all $y \in S \implies x \in S$;

Then the operator equation $Ax + By = x$ has a solution in S.

3. $\psi-$Hilfer Fractional Hybrid Differential Equation of the First Type

We take $X = C_{1-\gamma;\psi}([0,T]), T > 0$ through out this section.
we have the following lemma.

Lemma 5. *Any function satisfies IVP (1) will also satisfy the integral equation*

$$\begin{aligned}
x(t) & \\
&= f(t,x(t))\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}\left[\frac{x(0)}{f(0,x(0))}\right] \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds, \\
&= f(t,x(t))\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds\right], \qquad (13)
\end{aligned}$$

$$t \in [0,T].$$

In addition if the function $x \to \frac{x}{f(0,x)}$ is injective, and $I_{0+}^{\alpha;\psi}g(t,x(t))$ is an absolutely continuous function, then the converse is true.

Proof. From lemma (1), the proof is clear [23, 24].

Theorem 3. *Assume the following.*

(H_1) *The function $x \to \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$, for all $t \in I$.*

(H_2) *There exists a constant $L_f > 0$ such that $|f(t,x) - f(t,y)| \leq L_f|x-y|$, for all $t \in I$ and $x, y \in \mathbb{R}$.*

(H_3) *$K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| < 1$, then the $\psi-$Hilfer Hybrid Fractional differential equation has a solution defined on I, where K is the bound of a bounded function $(\psi(\cdot)-\psi(a))^{1-\gamma}g(\cdot)$.*

Then IVP (1) has a mild solution on I.

Proof. We define a subset S of X by $S = \{x \in X : \|x\| \leq N\}$,
where,

$$N = \frac{F_0\left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)x_0 + \|h\|\frac{1}{\Gamma(\alpha)}\left(\frac{(\psi(T)-\psi(0))^{\alpha}}{\alpha}\right)\right]}{1 - L_f\left(\frac{\psi(T-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right) + \|h\|\frac{1}{\Gamma(\alpha)}\left(\frac{\psi(T)-\psi(0))^{\alpha}}{\alpha}\right)}. \qquad (14)$$

and $F_0 = \sup_{t\in I}\|f(t,0)\|_{C_{1-\gamma;\psi}[0,T]}$.

It is clear that S satisfies the hypotheses of Theorem(1).

Also $IVP(1)$ is equivalent to the $\psi-$Hilfer Hybrid volterra Integral equation:

$$x(t) = f(t,x(t))\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds\right], \tag{15}$$

$$t \in [0,T].$$

Define two operators $A : X \to X$ and $B : S \to X$ by:

$$Ax(t) = f(t,x(t)),\ t \in I. \tag{16}$$

$$Bx(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds. \tag{17}$$

Then equation (15) is transformed into the operator equation as

$$x(t) = Ax(t)Bx(t),\ t \in I. \tag{18}$$

We shall show that the operators A,B satisfy all the conditions of Theorem(1).

Claim I:

Let $x, y \in X$, then by Hypothesis (H_2)

$$|Ax(t) - Ay(t)| = |f(t,x(t)) - f(t,y(t))| \le L_f|x(t)-y(t)| \le L_f\|x-y\|,\ \forall t \in I.$$

Taking supremum over t, we obtain

$$\|Ax - Ay\|_{C_{1-\gamma;\psi}[I\times\mathbb{R},\mathbb{R}|\{0\}]} \le L_f\|x-y\|,$$

$$\text{where } x, y \in X.$$

Claim II:

We show that B is continuous in S. Let x_n be a sequence in S converging to a point $x \in S$, Then by Lebesgue Dominated Convergence Theorem:

$$\begin{aligned}\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty}\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x_n(s))ds\right],\\ &= \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\lim_{n\to\infty} g(s,x_n(s))ds.\end{aligned}$$

Hence

$$\lim_{n\to\infty} Bx_n(t) = Bx(t),\ \forall t \in I.$$

Claim III:

B is a Compact Operator on S.

First, we show that $B(S)$ is a uniformly bounded set in X. Let $x \in S$, then by hypothesis (H_3), $\forall t \in I$.

$$\begin{aligned}
&|Bx(t)| \\
&\quad \le \left|\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds, \\
&\quad \le \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(t)-\psi(0))^{\alpha}\Gamma(1-\alpha)\frac{(\psi(t)-\psi(0))^{\alpha}}{\alpha\Gamma(\alpha)}, \\
&\quad \le \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

Thus

$$\|Bx\| \le \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)},\ \forall x \in X.$$

This shows that B is uniformly bounded on S.

Next we show that $B(S)$ is an equicontinuous set on X.

Let $t_1, t_2 \in I$, then for any $x \in S$,

$$\begin{aligned}
&|Bx(t_1) - Bx(t_2)| = \frac{1}{\Gamma(\alpha)}\times \\
&\left|\int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds - \int_0^{t_2} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds\right|, \\
&\qquad\qquad \le \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds, \\
&\qquad\qquad \le K\frac{1}{\Gamma(\alpha)}\frac{(\psi(t_1)-\psi(t_2))^{\alpha}}{\alpha}.
\end{aligned}$$

Hence for $\epsilon > 0$, there exist a $\delta > 0$ such that, whenever $|t_1 - t_2| < \delta$, then $|B(x(t_1)) - B(x(t_2))| < \epsilon$, $\forall\ t_1, t_2 \in I$ and $\forall\ x \in X$. This shows that $B(S)$ is an equicontinuous set in X. Then by the Arzelá-Ascoli theorem, B is a continuous and compact operator on S.

Claim IV:
The hypothesis (c) of theorem (1) is satisfied.

Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$, then:

$$
\begin{aligned}
|x(t)| &= |Ax(t)||By(t)|,\\
&\le |f(t,x(t))| \left| \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right|,\\
&\le [f(t,x(t)) - f(t,0)] +\\
&\qquad |f(t,0)| \left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right],\\
&\le (L_f|x(t)| + F_0) \left[\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right].
\end{aligned}
$$

Thus

$$
|x(t)| \le \frac{F_0 \left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \right) x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right]}{1 - L_f \left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 \right) + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}}. \tag{19}
$$

Taking supremum over $t \in I$,

$$
||x|| \le \frac{F_0 \left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} \right) x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right]}{1 - L_f \left(\frac{\psi(T-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 \right) + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}} = N. \tag{20}
$$

Thus $x \in S$ and hypothesis (c) of Theorem(1) is satisfied. Finally we have,

$$
\begin{aligned}
M = ||B(S)|| = \sup ||Bx|| : x \in S\\
\le K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} |x_0|.
\end{aligned}
$$

and so

$$
\alpha M \le K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} |x_0| < 1.
$$

Thus all conditions of Theorem(1) are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result equation(1) has a solution defined on I. This completes the proof.

4. $\psi-$Hilfer Fractional Hybrid Differential Equation of second type

Consider the $\psi-$Hilfer Fractional Hybrid Differential Equation of the form:

$$\begin{cases} {}^{H}\mathbb{D}_{t_0+}^{\alpha,\beta;\psi}[x(t)-f(t,x(t))] & = g(t,x(t)),\ a.e.\ t\in J=[t_0,t_0+a], \\ I_{t_0}^{1-\gamma;\psi}[x(t_0)-f(t_0,x(t_0))] & =\sigma\in\mathbb{R}. \end{cases} \tag{21}$$

Lemma 6. *Any function satisfies IVP(2) will also satisfy the integral equation*

$$x(t)= f(t,x(t))+\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}I_{t_0+}^{1-\gamma}(x(t_0)-f(t_0,x(t_0)))+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds, \tag{22}$$

$$= f(t,x(t))+\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds, \tag{23}$$

$$t\in[t_0,t_0+a].$$

In addition if the function $x\to x-f(0,x)$ is injective, and $I_{0+}^{\alpha;\psi}g(t,x(t))$ is an absolutely continuous function, then the converse is true.

Proof. From lemma (1), the proof is clear [23, 24].

Theorem 4. *Assume the following*

($\mathcal{A}1$) *There exists constants $M_f\geq L_f>0$ such that* $|f(t,x)-f(t,y)|\leq\frac{L_f|x-y|}{(M_f+|x-y|)}$ *for all $t\in J$ and $x,y\in\mathbb{R}$;*

($\mathcal{A}2$) *There exists $r>0$ such that* $r>L_f+F_0+\left|\frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|+\frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\psi(t_0+a)-\psi(t_0))^{2\alpha}$, *where $F_0=\sup_{t\in J}|f(t,0)|$;*

($\mathcal{A}3$) *K is the bound of a bounded function $(\psi(\cdot)-\psi(a))^{1-\gamma}g(\cdot)$.*

Then equation (21) has a mild solution on J.

Proof. Let $X=C_{1-\gamma;\psi}([t_0,t_0+a]),T>0$ and define the set $S\subset X$ by $S=\{x\in X:||x||\leq r\}$.

We prove the existence of a mild solution to problem (21) by discussing the solution to the integral equation (23) which is equivalent to the operator equation,

$$Ax(t) + Bx(t) = x(t), \ t \in J. \tag{24}$$

where,

$$\begin{aligned} Ax(t) &= f(t, x(t)). \\ Bx(t) &= \frac{(\psi(t) - \psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s, x(s))ds. \end{aligned}$$

Now we prove our Theorem by proving that the conditions of Theorem(2) are satisfied.

Step I:

Using the hypothesis ($\mathcal{A}1$) we get:

$$\begin{aligned} |Ax(t) - Ay(t)| =& |f(t, x(t)) - f(t, y(t))|, \\ \le& \frac{L_f |x(t) - y(t)|}{M_f + |x(t) - y(t)|}, \\ \le& \frac{L_f ||x - y||}{M_f + ||x - y||}. \end{aligned}$$

Thus the operator A is a nonlinear contraction with the function ϕ defined by $\phi(r) = \frac{L_f r}{M_f + r}$.

Step II:

Similarly by Theorem(3), we can prove that B is continuous and compact.

Step III:

Let $x \in X$ be fixed and $y \in S$ be arbitrary such that $x = Ax + By$, then we get:

$$
\begin{aligned}
&|x(t)|\\
&\quad\leq |Ax(t)| + |By(t)|,\\
&\quad\leq |f(t,x(t))| + \left|\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|\\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds,\\
&\quad\leq |f(t,x(t))-f(t,0)| + |f(t,0)| + \left|\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|\\
&\qquad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds,\\
&\quad\leq L_f + F_0 + \left|\frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right| + \frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\psi(t_0+a)-\psi(t_0))^{2\alpha},\\
&\quad\leq r.
\end{aligned}
$$

which proves that $||x|| \leq r$. Thus $x \in S$.

Thus the conditions of Theorem (2) are satisfied; then the operator equation $Ax(t) + Bx(t) = x(t)$ has a solution in S which proves the existence of a mild solution to problem (21) in J.

We finish the section with the following example.

Example 1. Consider the $\psi-$Hilfer fractional Hybrid differential equation

$$
\begin{aligned}
&{}^{H}\mathbb{D}_{0+}^{0.5,0;x}\left(x(t) - \frac{sin(t)|x(t)|}{2+|x(t)|}\right) = \frac{tx(t)}{1+|x(t)|},\\
&I_{0+}^{0.5;x}(x(0) - f(0,x(0))) = 1,\ t \in [0,\pi].
\end{aligned}
$$

We get that,

$$
\begin{aligned}
|f(t,x(t)) - f(t,y(t))| &\leq \frac{|x(t)-y(t)|}{2+|x(t)-y(t)|+|y(t)|},\\
&\leq \frac{|x(t)-y(t)|}{2+|x(t)-y(t)|}.
\end{aligned}
$$

and where $|g(t,x(t))| \leq t$, we get that all hypotheses of theorem (4) are satisfied with

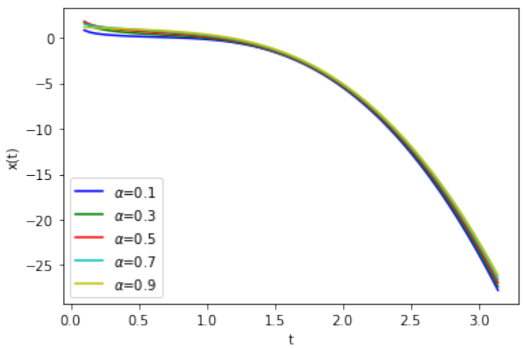
$$L_f = 1, M_f = 2, T = \pi, F_0 = 0.$$

We conclude that

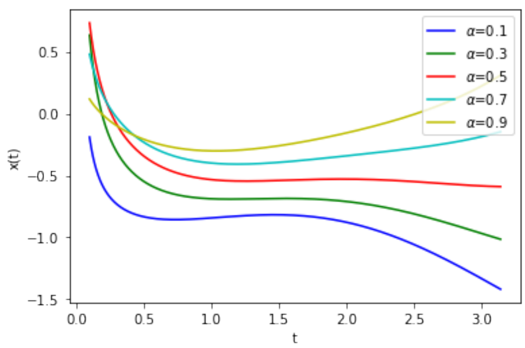
$$L_f + F_0 + \left| \frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)} \sigma \right| + \frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} (\psi(t_0+a)-\psi(t_0))^{2\alpha} =$$

$$1 + \frac{1}{\pi} + \pi^2\sqrt{\pi} < 19.$$

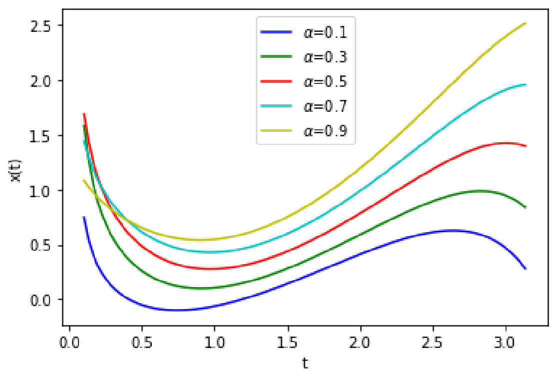
For different $x(t)$ and α the solutions corresponding to the problem are plotted below. In each case red colour indicated the solution of the above problem.



(a) $x(t) = t^3$

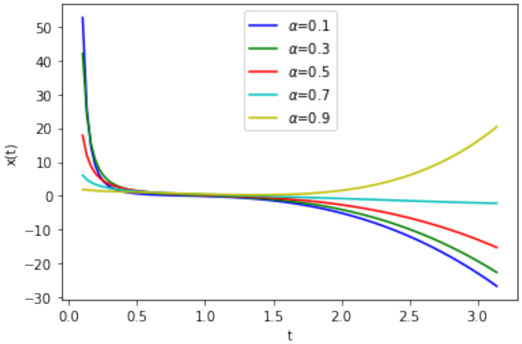


(b) $x(t) = 1 + t$

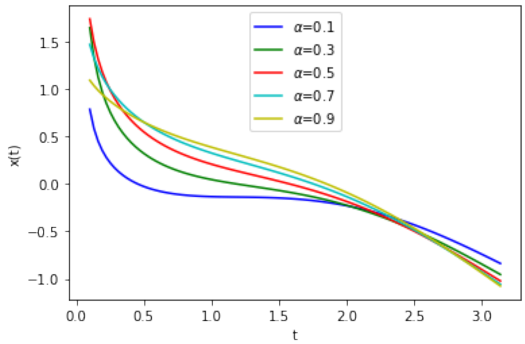


(c) $x(t) = sint$

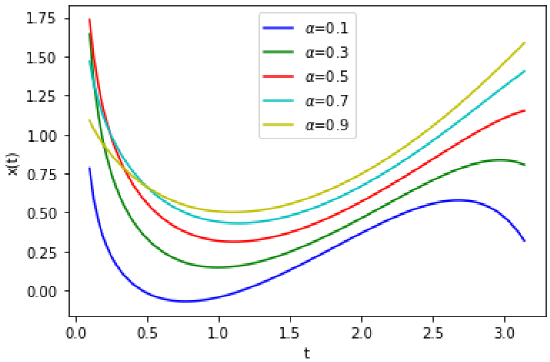
Corresponding to the above example with $f(t) = \frac{sin|x(t)|}{2+|x(t)|}$ and $g(t) = \frac{tx(t)}{1+|x(t)|}$ we plotted some graphs with different $x(t)$ and $\psi(t)$.



(a) $x(t) = t^3, \psi(t) = t^3$



(b) $x(t) = t, \psi(t) = ln(t+1)$



(c) $x(t) = sint, \psi(t) = ln(t+1)$

5. Conclusions

In this paper we proved the existences of two types of Hybrid fractional differential equation involving $\psi-$Hilfer fractional Derivative, which is a generalised fractional derivative using different fixed point theorems and concluded the paper with an example. we plotted some graphs for different values of $\psi(t)$ and $x(t)$ corresponding to the example.

6. Acknowledgements

We are thankful to the editor and reviewers for their useful corrections and suggestions, which improved the quality of the paper.

References

[1] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo., Theory and applications of Fractional Differential equations, North-Holland Mathematics Studies, 204, Elsevier Science B. V., Amsterdam, 2006.

[2] Yong Zhou, Jinrong Wang and Lu ZhangG., Basic Theory of Fractional Differential Equations, Second Edition, WSPC World Scientific Co. Pte, Ltd (2017).

[3] R. P. Agarwal, Yong Zhou and Yunyun He., Existence of fractional neutral functional differential equations, Computers and Mathematics with Applications, Vol. 59(2010) No. 3, pp 1095-1100.

[4] W. R. Melvin., A class of Neutral Functional Differential Equations, Journal of Differential Equations, Vol. 12(1972), No. 3, pp 524-534.

[5] Runping Ye and Guowei Zhang., Neutral Functional Differential Equations of Second order with infinite Delays, Electronic Journal of Differential Equations, Vol. 2010(2010), No.36, pp 1-12.

[6] Ricardo Almeida., A Caputo fractional derivative of a function with respect to another function, Communications in Nonlinear Science and Numerical Simulation, Vol. 44(2017), pp 460-481.

[7] Ricardo Almeida, Agnieszka B. Malinowska and M. Teresa T. Monteiro., Fractional differential equations with a Caputo derivative with respect to a Kernel function and their applications, Mathematical Methods in the Applied Sciences, WILEY, Vol. 41(2018), No. 1, pp. 336-352.

[8] Ricardo Almeida., Fractional differential equations with mixed boundary conditions, The Bulletin of the Malaysian Mathematical Society, Series 2(2018).

[9] Krishnan Balachandran, Juan J. Trujillo., The nonlocal Cauchy problem for nonlinear fractional integro-differential equations in Banach Spaces, Nonlinear Analysis, 72(2010), pp. 4587-4593.

[10] William R. Melvin., Some extensions of Krasnoselskii Fixed point theorem, Journal of Differential Equations, 11(1972), pp. 335-348.

[11] J. A Tenreiro Machado, Manuel F. Silva, Ramiros S. Barbosa, Isabel S. Jesus, Cecilia M Reis, Maria G. Marcos and Alexandra F. Galhano., Some applications of Fractional calculus in Engineering, Mathematical Problems in Engineering, Vol. 2010, Article ID 639801.

[12] Mehdi Dalir and Majid Bashour., Applications of Fractional Calculus, Applied Mathematical Sciences, Vol. 4(2010), No. 21, pp. 1021-1032.

[13] Yong Zhang and Samantha E. Hansen., A review of applications of fractional calculus in Earth system dynamics, Chaos, Solitons and Fractals, Vol. 102(2017), pp. 29-46.

[14] Dhage. B. C., On a fixed point theorem in Banach algebras with applications, Appl. Math. Lett., 18 (2005) pp. 273-280.

[15] B. C. Dhange and V. Lakshmikantham., Basic results on hybrid differential equations, Nonlinear Analysis: Hybrid Systems, vol. 4, no. 3(2010), pp. 414-424.

[16] B. Dhange and N. Jadhav., Basic results in the theory of hybrid differential equations with linear perturbations of second type, Tamkang Journal of Mathematics, vol. 44, no. 2(2013), pp. 171-186.

[17] Mohamed A. E Herzallah and Dumitru Baleanu., On Fractional Order Hybrid Differential Equations.

[18] Khalid Hilal and Ahmed Kajouni, Boundary value problems for hybrid differential equations with fractional order, Advances in differential equations, (2015) 2015:183.

[19] Tahereh Bashiri, Seiyed Mansour Vaezpour and Choonkil Park, Existence results for fractional hybrid differential systems in Banach algebras, Advances in Differential Equations, (2016) 2016:57.

[20] Azmat Ullah Khan Niazi, Jiang Wei, Mujeeb Ur Rehman and Du Jun, Existence results for hybrid fractional neutral differential equations, Advances in Differential Equations, (2017) 2017:353.

[21] Mohammad Esmael Samei, Vahid Hedayati and Shahram Rezapour., Existence results for a fraction hybrid differential inclusion with Caputo -Hadamard type fractional derivative, Advances in Differential Equations, (2019) 2019:63.

[22] Choukri Derbazi, Hadda Hammouche, Mouffak Benchohra and Yong Zhou, Fractional hybrid differential equations with three-point boundary hybrid conditions, Advances in Differential Equations, (201) 201:125.

[23] J. Vanterler da C. Sousa and E. Capelas de Oliveira., On the $\psi-$Hilfer fractional derivative, Commun. Nonlinear Sci. Numer. Simulat, 60(2018), 72-91.

[24] J. Vanterler da C. Sousa and E. Capelas de Oliveira., On a new operator in fractional calculus and applications, Computers and Mathematics with Application, 10 (2017).

[25] J. Vanterler da C. Sousa and E. Capelas de Oliveira, Ulam-Hyers stability of a nonlinear fractional Volterra integro-differential equation, Appl. Math. Lett., 81 (2018), 50-56.

[26] J. Vanterler da C. Sousa and E. Capelas de Oliveira., On the Ulam-Hyers-Rassias stability for nonlinear fractional differential equations using the $\psi-$Hilfer operator, Journal of Fixed Point Theory and Applications, 20 no. 96 (2018).

[27] J. Vanterler da C. Sousa and E. Capelas de Oliveira., Leibniz type rule: $\psi-$Hilfer fractional operator, Commun. Nonlinear Sci. Numer. Simulat., 77(2019), 305-311.

[28] E. Capelas de Oliveira and J. Vanterler da C. Sousa., Ulam-Hyers-Rassias Stability for a Class of Fractional Integro-Differential Equations, Results Math., (2018), 73: 111.