South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 2 (2020), pp. 13-22

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

ESSENTIAL ASCENT AND ESSENTIAL DESCENT OF LINEAR OPERATORS AND COMPOSITION OPERATORS

Harish Chandra and Pradeep Kumar

Department of Mathematics,
DST-CIMS, Banaras Hindu University,
Varanasi - 221005, Uttar Pradesh, INDIA

E-mail : harish_pathak2004@yahoomail.com, pradeep28_bhu@yahoo.co.in

(**Received:** *Nov. 28, 2019* **Accepted:** *May. 18, 2020* **Published:** *Aug. 30, 2020)*

Abstract: In this paper we prove certain results relating to essential ascent and essential descent of linear operator on an arbitrary vector space. Further, we give a complete characterization of essential ascent and essential descent of composition operators on l^p spaces.

Keywords and Phrases: Essential Ascent, Essential Descent, Composition Operator.

2010 Mathematics Subject Classification: 47B33.

1. Introduction

Let X denote an arbitrary vector space and T be a linear operator on X. Let $D(T)$, $N(T)$ and $R(T)$ denote domain, kernel and range of T respectively. Let $\mathbb{N}$ denote the set of natural numbers.

Definition 1.1. *If there is some integer $n \geq 0$ such that $N(T^n) = N(T^{n+1})$, the smallest such integer is called the ascent of T and is denoted by a(T). If no such integer exists then $a(T) = \infty$; see [13].*

Definition 2.2. *If there is some integer $n \geq 0$ such that $R(T^{n+1}) = R(T^n)$, the smallest such integer is called the descent of T and is denoted by d(T). If no such integer exists then $d(T) = \infty$; see [13].*

Note that if X is a finite dimensional, then a(T) and d(T) are both finite. We state the following well-known result about ascent and descent of linear operators.

Theorem 1.1. *If $D(T) = X$, $a(T)$ and $d(T)$ are both finite, then $a(T) = d(T)$; see [13].*

2. Essential Ascent and Essential Descent of Linear Operators

Definition 2.1. *If there is some integer $n \geq 0$ such that dim $(N(T^{n+1})/N(T^n))$ is finite, the smallest such integer is called the essential ascent of T and is denoted by $a_e(T)$. If no such integer exists then $a_e(T) = \infty$; see [10].*

Definition 2.2. *If there is some integer $n \geq 0$ such that dim $(R(T^n)/R(T^{n+1}))$ is finite, the smallest such integer is called the essential descent of T and is denoted by $d_e(T)$. If no such integer exists then $d_e(T) = \infty$; see [10].*

In [4] Grabiner and Zemanek have proved that if essential ascent and essential descent of a bounded linear operator on Banach space X are both finite, then they must be equal. In this section we give an elementary proof of the same result for a linear operator on an arbitrary vector space X.

Theorem 2.1. *If $a_e(T)$is finite and $d_e(T) = 0$, then $a_e(T) = 0$.*

Proof. Suppose that $a_e(T) \neq 0$, i.e. dim $(N(T)/N(T^0)) = \infty$. This implies that dim $N(T) = \infty$. Hence there exist a sequence of linearly independent vectors $\{y_i\}_{i=1}^{\infty}$ in $N(T)$. Also it is given that $d_e(T) = 0$, i.e. dim $(X/R(T))$ is finite $= n$(say). It follows that $\{y_i + R(T)\}_{i=1}^{n+1}$ are linearly dependent vectors in $X/R(T)$. Hence there exist scalars $\{\alpha_i\}_{i=1}^{n+1}$ not all of them zero such that

$$\sum_{i=1}^{n+1} \alpha_i(y_i + R(T)) = 0 + R(T) = R(T)$$

Then

$$\sum_{i=1}^{n+1} \alpha_i y_i + R(T) = R(T)$$

It follows that

$$\sum_{i=1}^{n+1} \alpha_i y_i \in R(T)$$

Let

$$z_1 = \sum_{i=1}^{n+1} \alpha_i y_i$$

Clearly $z_1 \neq 0$ and $z_1 \in R(T)$. Similarly since $\{y_i + R(T)\}_{i=n+2}^{2n+2}$ are linearly dependent vectors in $X/R(T)$. Then there exist scalars $\{\alpha_i\}_{i=n+2}^{2n+2}$ not all of them

zero such that

$$\sum_{i=n+2}^{2n+2} \alpha_i(y_i + R(T)) = 0 + R(T) = R(T)$$

Put

$$z_2 = \sum_{i=n+2}^{2n+2} \alpha_i y_i$$

Clearly $z_2 \neq 0$ and $z_2 \in R(T)$. Continuing we get a sequence of linearly independent vectors $\{z_m\}_{m=1}^{\infty}$ such that $z_m \in R(T)$, since $\{y_i\}_{i=1}^{\infty}$ are linearly independent vectors in $N(T)$. Hence there exist a sequence $\{x_m\}_{m=1}^{\infty}$ of linearly independent vectors in X such that $Tx_m = z_m \neq 0$ but $T^2x_m = 0$ for all $m \geq 1$. Then $\{x_m\}_{m=1}^{\infty} \in N(T^2)$ but $\{x_m\}_{m=1}^{\infty} \notin N(T)$. Therefore $\{x_m + N(T)\}_{m=1}^{\infty}$ are linearly independent vectors in $N(T^2)/N(T)$. Then dim $(N(T^2)/N(T)) = \infty$. By using above argument recursively, we get dim $(N(T^{k+1})/N(T^k)) = \infty$ for each $k \geq 0$. Hence $a_e(T) = \infty$. This is a contradiction. Therefore $a_e(T) = 0$

Theorem 2.2. *If $D(T) = X$, $a_e(T)$ and $d_e(T)$ are both finite, then $a_e(T) = d_e(T)$.*
Proof. Let $p = d_e(T)$; i.e.dim $(R(T^p)/R(T^{p+1}))$ is finite. Let $T_1(= T\big|_{R(T)})$ be the mapping from $R(T^p)$ to $R(T^{p+1})$. Then $T_1x = Tx$ for each $x \in R(T^p)$. Since $N(T_1^{n+1}) - N(T_1^n) \subseteq N(T^{n+1}) - N(T^n)$, $a_e(T_1) \leq a_e(T)$. Hence $a_e(T_1)$ is finite. Also dim $(R(T_1^0)/R(T_1))$ = dim $(R(T^p)/R(T^{p+1})) < \infty$. Therefore $d_e(T_1) = 0$, by Theorem (2.1) we get $a_e(T_1) = 0$. Therefore dim $N(T_1)$ is finite. Now we claim that dim $(N(T^{p+1})/N(T^p)) \leq$ dim $N(T_1) < \infty$. Let $\{x_i + N(T^p)\}_{i=1}^n$ are linearly independent vectors in $N(T^{p+1})/N(T^p)$. Clearly $\{x_i\}_{i=1}^n \in N(T^{p+1}) - N(T^p)$. Then $T^{p+1}(x_i) = 0$ but $T^p x_i \neq 0$ for each i, $1 \leq i \leq n$. Let $y_i = T^p x_i \neq 0$ for each i, $1 \leq i \leq n$. Then $\{y_i\}_{i=1}^n \in N(T_1)$. We show that $\{y_i\}_{i=1}^n$ are linearly independent vectors in $N(T_1)$. Suppose $\sum_{i=1}^n \alpha_i y_i = 0$. This implies that $T^p(\sum_{i=1}^n \alpha_i x_i) = 0$, since $y_i = T^p x_i$ for each i, $1 \leq i \leq n$. Hence $\sum_{i=1}^n \alpha_i x_i \in N(T^p)$. Therefore $\sum_{i=1}^n \alpha_i(x_i + N(T^p)) = N(T^p)$ Now, since $\{x_i + N(T^p)\}_{i=1}^n$ are linearly independent vectors in $N(T^{p+1})/N(T^p)$, it follows that $\alpha_i = 0$ for each i, $1 \leq i \leq n$. Hence $\{y_i\}_{i=1}^n$ are linearly independent vectors in $N(T_1)$. Thus dim $(N(T^{p+1})/N(T^p)) \leq$ dim $N(T_1) < \infty$. Hence

$$a_e(T) \leq p = d_e(T) \tag{1}$$

We have only to proof $a_e(T) \geq d_e(T)$. The case when $d_e(T) = 0$ is proved in Theorem (2.1). Therefore we assume that $p = d_e(T) \geq 1$. This implies that

dim $(R(T^{p-1})/R(T^p)) = \infty$. Let $\{y_i + R(T^p)\}_{i=1}^{\infty}$ are linearly independent vectors in $R(T^{p-1})/R(T^p)$. Then $\{y_i\}_{i=1}^{\infty}$ are linearly independent vectors in $R(T^{p-1})$. Clearly $y_i \in R(T^{p-1}) - R(T^p)$ for each $i \geq 1$. Then $Ty_i \in R(T^p)$ for each $i \geq 1$. Let $I = \{i \in \mathbb{N} : Ty_i \in R(T^{p+1})\}$.

Case(a): Suppose I is infinite. If $Ty_i \in R(T^{p+1})$ for infinitely many $i \geq 1$. Then $Ty_i = T^{p+1}w_i$ for some $w_i \in X$ and infinitely many $i \geq 1$. It follows that $T(T^{p-1}x_i) = T^{p+1}w_i$ for infinitely many $i \geq 1$, since $y_i \in R(T^{p-1})$ for infinitely many $i \geq 1$. Then $T^p(x_i - Tw_i) = 0$,for infinitely many $i \geq 1$. It follows that $(x_i - Tw_i) \in N(T^p)$ for infinitely many $i \geq 1$. But $T^{p-1}(x_i - Tw_i) = T^{p-1}x_i - T^pw_i = y_i - T^pw_i \neq 0$ for infinitely many $i \geq 1$, since $y_i \in R(T^{p-1}) - R(T^p)$ and $T^pw_i \in R(T^p)$ for infinitely many $i \geq 1$. Then $(x_i - Tw_i) \notin N(T^{p-1})$ for infinitely many $i \geq 1$. We claim that $\{(x_i - Tw_i) + N(T^{p-1}) : i \in I\}$ are linearly independent vectors in $N(T^p)/N(T^{p-1})$. It is sufficient if we prove that every finite subset $\{(x_i - Tw_i) + N(T^{p-1}) : i \in I\}$ are linearly independent vectors in $N(T^p)/N(T^{p-1})$. Suppose $\sum_{i=1}^{k} \alpha_i(x_i - Tw_i) + N(T^{p-1}) = 0 + N(T^{p-1}) = N(T^{p-1})$. Then $\sum_{i=1}^{k} \alpha_i(x_i - Tw_i) \in N(T^{p-1})$. It follows that $T^{p-1}\left\{\sum_{i=1}^{k} \alpha_i(x_i - Tw_i)\right\} = 0$. Then $\sum_{i=1}^{k} \alpha_i(y_i - T^pw_i) = 0$. Therefore $\sum_{i=1}^{k} \alpha_i y_i = \sum_{i=1}^{k} \alpha_i(T^pw_i)$. Since $\{y_i\}_{i=1}^{\infty}$ are linearly independent vectors in $R(T^{p-1}) - R(T^p)$ and $T^pw_i \in R(T^p)$ for infinitely many $i \leq 1$. Hence $\sum_{i=1}^{k} \alpha_i y_i = 0$. It follows that $\alpha_i = 0$ for each $1 \leq i \leq k$. Therefore $\{(x_i - Tw_i) + N(T^{p-1}) : i \in I\}$ are linearly independent vectors in $N(T^p)/N(T^{p-1})$. Since I is infinite set. Thus dim $(N(T^p)/N(T^{p-1})) = \infty$. Hence $a_e(T) \geq p = d_e(T)$.

Case(b) : suppose that I is finite. Leaving the finite members of the of the Sequence $\{y_i\}$ for $i \in I$, the sequence $\{y_i\}_{i \notin I}$ satisfies that $y_i \notin R(T^{p-1}) - R(T^p)$ and $Ty_i \notin R(T^{p+1})$ for each $i \notin I$. So without loss of generality, we may assume that $I = \phi$. Let dim $(R(T^{p-1})/R(T^p))$ is finite $= m$(say). It follows that $\{Ty_i + R(T^p)\}_{i=1}^{m+1}$ are linearly dependent vectors in $R(T^{p-1})/R(T^p)$. Then there exist scalars $\{\alpha_i\}_{i=1}^{m+1}$ not all of them zero such that

$$\sum_{i=1}^{m+1} \alpha_i(Ty_i + R(T^p)) = 0 + R(T^p) = R(T^p)$$

It follows that

$$\sum_{i=1}^{m+1} \alpha_i(Ty_i) + R(T^p) = R(T^p)$$

Hence

$\sum_{i=1}^{m+1} \alpha_i(Ty_i) \in R(T^p)$
Let
$z_1 = \sum_{i=1}^{m+1} \alpha_i y_i$
Clearly $z_1 \neq 0$ and $Tz_1 \in R(T^p)$. Similarly since $\{Ty_i + R(T^p)\}_{i=m+2}^{2m+2}$ are linearly dependent vectors in $R(T^{p-1})/R(T^p)$. Then there exist scalars $\{\alpha_i\}_{i=m+2}^{2m+2}$ not all of them zero such that

$$\sum_{i=m+2}^{2m+2} \alpha_i(Ty_i + R(T^p)) = 0 + R(T^p) = R(T^p)$$

Put

$$z_2 = \sum_{i=m+2}^{2m+2} \alpha_i y_i$$

Clearly $z_2 \neq 0$ and $Tz_2 \in R(T^p)$. Continuing we get a sequence of linearly independent vectors $\{z_i\}_{i=1}^{\infty}$ in $R(T^{p-1})$, such that $\{Tz_i\}_{i=1}^{\infty} \in R(T^p)$. Now we apply case (a)and,it follows that dim $(N(T^{p-1})/N(T^p)) = \infty$. Hence

$$a_e(T) \geq p = d_e(T) \tag{2}$$

Combining equation (1) and (2), we get $a_e(T) = d_e(T)$.

3. Essential Ascent and Essential Descent of Composition Operators

The composition operator C_ϕ on l^p where $1 \leq p < \infty$ induced by a function ϕ on $\mathbb{N}$ into itself, is defined by $C_\phi(f) = f \circ \phi$ for all $f \in l^p$. It is well known that a necessary and sufficient condition that a function ϕ on $\mathbb{N}$ into itself induces a composition operator on l^p is that $\left\{\overline{\overline{\phi^{-1}(n)}} : n \in \mathbb{N}\right\}$ is bounded. Here $\overline{\overline{\phi^{-1}(n)}}$ denotes the number of elements in $\phi^{-1}(n)$; see [12]. We denote by χ_n, the characteristic function of $\{n\}$. Note that a linear operator T belongs to exactly one of the following classes:
1. $a_e(T) = d_e(T) = finite.$
2. $a_e(T) = \infty$ and $d_e(T) = finite.$
3. $a_e(T) = finite$ and $d_e(T) = \infty.$
4. $a_e(T) = d_e(T) = \infty.$
We give examples of composition operators, exactly one for each of the above type, as follows:

Example 3.1. Let ϕ be a self-map on $\mathbb{N}$ defined as:

$$\phi(n) = \begin{cases} \text{n}, & \text{if } n \text{ is odd} \\ \text{n-1}, & \text{if } n \text{ is even} \end{cases}.$$

Then $a_e(C_\phi) = 1$ and $d_e(C_\phi) = 1$.

Example 3.2. Let ϕ be the self-map on $\mathbb{N}$ defined as:

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$

Where $\{p_k : k \in \mathbb{N}\}$ denote the enumeration of primes.

$$\text{and } \phi(n) = n \text{ when } n \in \left(\mathbb{N} - \bigcup_{k\in\mathbb{N}} E_k\right)$$
$$\text{where } E_k = \{p_k^n : n \geq 1\} \text{ for each } k \in \mathbb{N}.$$

Then $a_e(C_\phi) = \infty$ and $d_e(C_\phi) = 0$.

Example 3.3. Let ϕ be the self-map on $\mathbb{N}$ defined as:

$$\phi(3n) = 2n \ ;$$

and

$$\phi(3n-2) = \phi(3n-1) = 2n-1.$$

Then $a_e(C_\phi) = 0$ and $d_e(C_\phi) = \infty$.

Example 3.4. Let $P = \bigcup_{k\in\mathbb{N}} \{p_k^n : n \in \mathbb{N}\}$ where p_k denote the k-th prime and $\mathbb{N} - P = \{q_k : k \geq 1\} = \{1, 6, 10, 12, \ldots\ldots\}$. Clearly $\mathbb{N} - P$ is an infinite subset of $\mathbb{N}$ and ϕ be the self-map on $\mathbb{N}$ defined as :

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$

$$\phi(q_1) = \phi(q_2) = q_1$$

and

$$\phi(q_{2k-1}) = \phi(q_{2k}) = q_{2k-2} \text{ for each } k \geq 2.$$

Then it is easy to show that $a_e(C_\phi) = \infty$ and $d_e(C_\phi) = \infty$.

In this section we prove results about essential ascent and essential descent of composition operators on l^p spaces where $1 \leq p < \infty$.

Theorem 3.1. *$a_e(C_\phi) = \infty$ if and only if there exists a sequence $\{E_k\}_{k=1}^\infty$ of subsets of $\mathbb{N}$ such that each E_k is infinite, $E_k \subseteq R(\phi^{k-1})$ and $R(\phi^k) \bigcap E_k = \phi$ for*

each $k \in \mathbb{N}$.

Proof. Suppose that $a_e(C_\phi) = \infty$. Let $E_k = \{m : m \in R(\phi^{k-1}) - R(\phi^k)\}$. By construction of E_k, it is clear that $E_k \subseteq R(\phi^{k-1})$ and $R(\phi^k) \bigcap E_k = \phi$ for each $k \in \mathbb{N}$. We claim that E_k is infinite set. Suppose on the contrary, there exist a natural number N for which E_N is finite set. Clearly $\{\chi_n + N(C_\phi^{N-1}) : n \in E_N\}$ forms a basis for $N(C_\phi^N)/N(C_\phi^{N-1})$. Let $f = g + N(C_\phi^{N-1})$, where $g = \sum \alpha_n \chi_n$. Then
$N(C_\phi^N)/N(C_\phi^{N-1}) = span\{\chi_n + N(C_\phi^{N-1}) : n \in E_N\}$.
Therefore dim $(N(C_\phi^N)/N(C_\phi^{N-1})) \leq \overline{\overline{E_N}} < \infty$. Thus $a_e(C_\phi) \leq (N-1)$. This is a contradiction. Hence E_k is infinite set.
Conversely, assume that there exists a sequence $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ such that each E_k is infinite, $E_k \subseteq R(\phi^{k-1})$ and $R(\phi^k) \cap E_k = \phi$ for each $k \in \mathbb{N}$. Hence $E_k \subseteq \{m : m \in R(\phi^{k-1}) - R(\phi^k)\}$. Now we claim that $\{\chi_n + N(C_\phi^{k-1}) : n \in E_k\}$ is linearly independent sequence of $(N(C_\phi^k)/N(C_\phi^{k-1}))$. Claim is achieved if we prove that every finite subset $\{\chi_n + N(C_\phi^{k-1}) : n \in E_k\}$ is linearly independent in $N(C_\phi^k)/N(C_\phi^{k-1})$. Let $\beta_1(\chi_{n_1} + N(C_\phi^{k-1}) + \ldots\cdots + \beta_l(\chi_{n_l} + N(C_\phi^{k-1}) = N(C_\phi^{k-1})$. This implies that $\beta_1\chi_{n_1} + \ldots\cdots + \beta_l\chi_{n_l} \in N(C_\phi^{k-1})$. Therefore $C_\phi^{k-1}(\beta_1\chi_{n_1} + \ldots\cdots + \beta_l\chi_{n_l}) = 0$. $E_k \subseteq R(\phi^{k-1}) - R(\phi^k)\}$ provides that $\{(\phi^{k-1})^{-1}(n_j)\}_{1\leq j\leq l}$ is a collection of non-empty pairwise disjoint sets. Thus $\beta_j\chi_{(\phi^{k-1})^{-1}(n_j)} = 0$ for $1 \leq j \leq l$. Thus $\{\chi_n + N(C_\phi^{k-1}) : n \in E_k\}$ is linearly independent sequence of $N(C_\phi^k)/N(C_\phi^{k-1})$. Since each E_k is infinite set. Therefore dim $(N(C_\phi^k)/N(C_\phi^{k-1})) = \infty$ for each $k \geq 1$. Hence $a_e(C_\phi) = \infty$.

Remark 3.1. *The following example shows that for each* $n \in \mathbb{N}$ *there exist a composition operator* C_ϕ *on* l^p *such that* $a_e(C_\phi) = n - 1$.

Example 3.5. Let n be any fixed natural number and ϕ be a self-map on $\mathbb{N}$ defined as:

$$\phi(m) = \begin{cases} \text{m}, & \text{if } n/(m-1) \\ \text{m-1}, & \text{otherwise.} \end{cases}$$

Then $a_e(C_\phi) = n - 1$ and $d_e(C_\phi) = n - 1$.

Theorem 3.2. $d_e(C_\phi) = \infty$ *if and only if for each* $k \geq 0$; $\overline{\overline{\phi^{-1}(n)}} > 1$ *for infinitely many* $n \in R(\phi^k)$.

Proof. If possible, suppose $A = \left\{n \in R(\phi^N) : \overline{\overline{\phi^{-1}(n)}} > 1)\right\}$ is finite for some natural number N. We claim that dim $(R(C_\phi^N)/R(C_\phi^{N+1})) < \infty$. Let $f \in R(C_\phi^N)$.

Then $f = C_\phi^N(g)$ for some $g \in l^p$. Let $g = \sum \alpha_n \chi_n$. Then

$$\begin{aligned} C_\phi^N(g) &= \sum_{n \in\ R(\phi^N)} \alpha_n \chi_{(\phi^N)^{-1}(n)} \\ &= \sum_{\substack{n' \in\ R(\phi^N) \\ and\ \overline{\overline{\phi^{-1}(n')}} > 1}} \alpha_{n'} \chi_{(\phi^N)^{-1}(n')} + \sum_{\substack{n'' \in\ R(\phi^N) \\ and\ \overline{\overline{\phi^{-1}(n'')}} = 1}} \alpha_{n''} \chi_{(\phi^N)^{-1}(n'')} \end{aligned}$$

i.e.

$$C_\phi^N(g) = h_1 + h_2(say) \tag{3}$$

We claim that $h_2 \in R(C_\phi^{N+1})$. Let $g' = \sum \beta_n \chi_n$, where

$$\beta_n = \begin{cases} \alpha_{\phi^{-1}(n)}, & \text{when } n \in R(\phi^{N+1}), \\ \overline{\overline{\phi^{-1}(n)}} = 1 \text{ and } \overline{\overline{\phi^{-1}(n'')}} = 1\ whenever n'' \in \phi^{-1}(n) \\ 0, \quad \text{Otherwise} \end{cases}$$

Then, clearly $g' \in l^p$. Now

$$\begin{aligned} C_\phi^{N+1}(g') &= \sum_{\substack{n \in\ R(\phi^{N+1}) \\ and\ \overline{\overline{\phi^{-1}(n)}} = 1}} \beta_n \chi_{(\phi^{N+1})^{-1}(n)} + \sum_{\substack{n \notin\ R(\phi^{N+1}) \\ or\ \overline{\overline{\phi^{-1}(n)}} > 1}} \beta_n \chi_{(\phi^{N+1})^{-1}(n)} \\ &= \sum_{\substack{n \in\ R(\phi^{N+1}) \\ and\ \overline{\overline{\phi^{-1}(n)}} = 1}} \alpha_{\phi^{-1}(n)} \chi_{(\phi^{N+1})^{-1}(n)} \end{aligned}$$

Now put $n'' = \phi^{-1}(n)$, then $n'' \in R(\phi^N)$ and by our assumption we get $\overline{\overline{\phi^{-1}(n)}} = 1 \Leftrightarrow \overline{\overline{\phi^{-1}(n'')}} = 1$(fallows from the definition of β_n). Therefore

$$C_\phi^{N+1}(g') = \sum_{\substack{n'' \in\ R(\phi^N) \\ and\ \overline{\overline{\phi^{-1}(n'')}} = 1}} \alpha_{n''} \chi_{(\phi^N)^{-1}(n'')} = h_2 \tag{4}$$

Thus $h_2 \in R(C_\phi^{N+1})$.

Combining equation (3) and (4) we get dim $(R(C_\phi^N)/R(C_\phi^{N+1}))$is finite. Thus $d_e(C_\phi) \leq N$.

Conversely, assume that $\overline{\overline{\phi^{-1}(n)}} > 1$ for infinitely many $n \in\ R(\phi^k)$. Let $\{n_m\}_{m=1}^\infty \in R(\phi^k)$ such that $\overline{\overline{\phi^{-1}(n_m)}} > 1$ for each $m \geq 1$. Let $\{\alpha_{n_m}, \beta_{n_m}\} \subseteq \phi^{-1}(n_m)$ Define a sequence $\{f_m\}_{m=1}^\infty$ as follows :

$$f_m(n) = \begin{cases} 1, & \text{if } \phi^{k-1}(n) = \alpha_{n_m} \\ -1, & \text{if } \phi^{k-1}(n) = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{f_m\}_{m=1}^{\infty} \in l^p$. And also we define a sequence $\{h_m\}_{m=1}^{\infty}$ as follows:

$$h_m(n) = \begin{cases} 1, & \text{if } n = \alpha_{n_m} \\ -1, & \text{if } n = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{h_m\}_{m=1}^{\infty} \in l^p$ and $f_m = C_\phi^{k-1}(h_m) \in R(C_\phi^{k-1})$. We claim that $\{f_m\}_{m=1}^{\infty} \notin R(C_\phi^k)$. If possible, assume that $\{f_{m_0}\}_{m_0=1}^{\infty} \in R(C_\phi^k)$ for some $m_0 \geq 1$. This implies that $f_{m_0} = C_\phi^k(h_0)$ for some $h_0 \in l^p$. Then $f_{m_0} = h_0 \circ \phi^k$. Let $n_m^{(1)}$ and $n_m^{(2)}$ be such that $\phi^{k-1}(n_m^{(1)}) = \alpha_{n_m}$ and $\phi^{k-1}(n_m^{(2)}) = \beta_{n_m}$, where $\phi(\alpha_{n_m}) = \phi(\beta_{n_m}) = n_m$. Then $1 = f_{m_0}(n_m^{(1)}) = h_0(\phi^k(n_m^{(1)})) = h_0(\phi(\alpha_{n_m})) = h_0(\phi(\beta_{n_m})) = h_0(\phi^k(n_m^{(2)})) = f_{m_0}(n_m^{(2)}) = -1$. This is a contradiction. Hence $\{f_m\}_{m=1}^{\infty} \notin R(C_\phi^k)$. Thus sequence $\left\{f_m + R(C_\phi^k)\right\}_{m=1}^{\infty}$ are linearly independent in $R(C_\phi^{k-1})/R(C_\phi^k)$. Thus dim $(R(C_\phi^{k-1})/R(C_\phi^k))$ is not finite. Since $k \geq 1$ is arbitrary it follows that $d_e(C_\phi) = \infty$

Remark 3.2. *From example (3.5) it follows that for each $n \in \mathbb{N}$ there exist a composition operator C_ϕ on l^p such that $d_e(C_\phi) = n - 1$.*

References

[1] Aupetit B., Primer on Spectral Theory, Springer-Verlag, New-York, 1991.

[2] Burgos M., Kaidi A., Mbekhta M., Oudghiri M., The Descent Spectrum and Perturbations, J. Operator Theory, 56, 2(2006), 259-271.

[3] Grabiner S., Uniform Ascent and Descent of Bounded Operators, J. Math. Soc. Japan, 34(1982), 317-337.

[4] Grabiner S. Zemanek J., Ascent, Descent, and Ergodic Properties of Linear Operators, J. Operator Theory, 48(2002), 69-81.

[5] Kaashoek M. A., Ascent, Descent, Nullity and Defect: A Note On a Paper by A. E. Taylor Math. Ann, 172(1967), 105-115.

[6] Kaashoek M. A., Lay D. C., Ascent, Descent, and Commuting Perturbations, Trans. Amer. Math. Soc., 169(1972), 35-47.

[7] Lay D. C., Spectral Analysis using Ascent, Descent, Nulity and Defect, Math. Ann., 184(1970), 197-214.

[8] Lal N., Tripathi G. P., Composition Operators on l^2 of the form Normal Plus Compact, J. Indian. Math. Soc., 72(2005), 221-226.

[9] Mbekhta M., Ascent,Descent et Spectre Essential Quasi-Fredholm, Rend. Circ. Math. Palermo, (1997), 175-196.

[10] Mbekhta M., Muller V., On the Axiomatic Theory of Spectrum II, Studia Math., (1996), 129-147.

[11] Nordgren E. A., Composition Operators on Hilbert Spaces, J. Math. Soc. Japan, 34(1982), 317-337.

[12] Singh L., A Study of Composition Operators on l^2, Thesis, Banaras Hindu University, 1987.

[13] Taylor A. E., Lay D. C., Introduction to Functional Analysis, John- Wiley, New York-Chichester-Brisbane, 1980.