J. of Ramanujan Society of Math. and Math. Sc. **ISSN : 2319-1023**
Vol.3, No.2 (2014), pp. 17-22

A note on some summations due to Ramanujan, their generalization and some allied series

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Abstract: In this short note, we aim to discuss some summations due to Ramanujan, their generalizations and some allied series.
Keywords and Phrases: generalized hypergeometric series, Gauss summation theorem, Karlsson-Minton summation formula.

1. Introduction

We start with the following summations due to Ramanujan [6]

$$1+\frac{1}{5}\left(\frac{1}{2}\right)^2+\frac{1}{9}\left(\frac{1\cdot 3}{2\cdot 4}\right)^2+\cdots=\frac{\pi^2}{4\Gamma^2(\frac{3}{4})} \tag{1.1}$$

and

$$1+\frac{1}{5^2}\left(\frac{1}{2}\right)+\frac{1}{9^2}\left(\frac{1\cdot 3}{2\cdot 4}\right)+\cdots=\frac{\pi^{5/2}}{8\sqrt{2}\Gamma^2(\frac{3}{4})}. \tag{1.2}$$

As pointed out by Berndt [1] the above summations can be obtained quite simply by putting (i) $a=b=\frac{1}{2}, c=\frac{1}{4}$ and (ii) $a=\frac{1}{2}, b=c=\frac{1}{4}$ in Dixon's summation theorem [8, p.52] for the ${}_3F_2$ series, viz.

$${}_3F_2\left[\begin{array}{l} a,b,c;1 \\ 1+a-b,1+a-c \end{array}\right]=\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}$$

valid provided $Re(\frac{1}{2}a-b-c)>-1$.

A similar series evaluation

$$1+\frac{1}{5}\left(\frac{1}{2}\right)+\frac{1}{9}\left(\frac{1.3}{2.4}\right)+...=\frac{\pi^{3/2}}{2\sqrt{2}\Gamma^2(\frac{3}{4})} \tag{1.3}$$

was also obtained by Ramanujan [7] using an integral representation. However, a more direct approach makes use of the fact that this series can be expressed as ${}_2F_1(\frac{1}{2}, \frac{1}{4}; \frac{5}{4}; 1)$ combined with the well-known Gauss summation theorem [8,p. 28]

$$ {}_2F_1\left[\begin{matrix} a, b; 1 \\ c \end{matrix}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{1.4} $$

valid provided $\mathrm{Re}\,(c-a-b) > 0$, by setting $a = \frac{1}{2}$, $b = \frac{1}{4}$ and $c = \frac{5}{4}$.

Next, let us consider the series

$$ S = \frac{1}{b} + \frac{1}{2}\frac{1}{b+\mu} + \left(\frac{1.3}{2.4}\right)\frac{1}{b+2\mu} + ..., \tag{1.5} $$

where $\mu > 0$ (This can be extended to complex values of μ provided $|\arg\mu| < \pi$). Then S can be written as

$$ S = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!}\frac{1}{b+n\mu} = \frac{1}{b}\sum_{n=0}^{\infty}\frac{(\frac{1}{2})_n}{n!}\frac{(b/\mu)_n}{(b/\mu+1)_n} $$

$$ = \frac{1}{b}\,{}_2F_1\left(\frac{1}{2}, \frac{b}{\mu}; \frac{b}{\mu}+1; 1\right) $$

which can again be evaluated with the help of the Gauss summation theorem to yield the sum

$$ S = \frac{\pi^{1/2}\Gamma(b/\mu)}{\mu\Gamma(b/\mu+\frac{1}{2})} \tag{1.6} $$

The case $\mu = 2$ was considered by Ramanujan [7] using an integral representation. Clearly, the series (1.5) reduces to (1.3) by taking $\mu = 4$, $b = 1$. Thus the series (1.5) may be regarded as a generalization of (1.3).

2. Generalizations and other allied series

In this section we shall mention some generalizations of Ramanujan's summations and also consider some allied series. For this, we apply the following results [2,3]

$$ {}_3F_2\left[\begin{matrix} a, b, c; 1 \\ b+m, c+1 \end{matrix}\right] $$

$$ = \frac{c\Gamma(1-a)(b)_m}{(b-c)_m}\left\{\frac{\Gamma(c)}{\Gamma(1+c-a)} - \frac{\Gamma(b)}{\Gamma(1+b-a)}\sum_{k=0}^{m-1}\frac{(1-a)_k(b-c)_k}{(1+b-a)_k k!}\right\} \tag{2.1} $$

for positive integer m, and the generalized Karlsson-Minton summation formula for positive integers (m_r) [4,5]

$$ {}_{r+2}F_{r+1}\left[\begin{array}{l} a, b, (f_r+m_r); 1 \\ c, (f_r) \end{array}\right] = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \sum_{k=0}^{m} \frac{(-)^k (a)_k (b)_k C_k(r)}{(1+a+b-c)_k} \qquad (2.2) $$

provided $\mathrm{Re}\,(c-a-b) > m$, where $m = m_1 + \cdots + m_r$, (f_r) denotes the parameter sequence $(f_1, \ldots, f_r)$ and the coefficients $C_k(r)$ are given by

$$ C_k(r) = \frac{(-1)^k}{k!}\ {}_{r+1}F_r\left[\begin{array}{l} -k, (f_r+m_r); 1 \\ (f_r) \end{array}\right]. $$

In the particular case $r = 1$, Vandermonde's summation theorem can be used to show that

$$ C_k(1) = \binom{m}{k} \frac{1}{(f)_k}. $$

Let us consider the extension of (1.5) in the form

$$ S = \sum_{n=0}^{\infty} \frac{(\frac{1}{2})_n}{n!} \frac{1}{(1+n/b)(1+n/c)} = {}_3F_2\left[\begin{array}{l} \frac{1}{2}, b, c; 1 \\ b+1, c+1 \end{array}\right]. \qquad (2.3) $$

This can be evaluated by means of (2.1) with $m = 1$ to give

$$ S = \begin{cases} \frac{\pi^{1/2} bc}{b-c} \left\{ \frac{\Gamma(c)}{\Gamma(c+\frac{1}{2})} - \frac{\Gamma(b)}{\Gamma(b+\frac{1}{2})} \right\} & (b \neq c) \\ \\ \frac{\pi^{1/2} b^2 \Gamma(b)}{\Gamma(b+\frac{1}{2})} \left\{ \psi(b+\frac{1}{2}) - \psi(b) \right\} & (b = c) \end{cases} $$

The special case $b = c$ is obtained by a limiting process with ψ denoting the logarithmic derivative of the gamma function. If we let $b = c = \frac{1}{4}$ in (2.3), we immediately obtain Ramanujan's summation (1.2), upon noting that $\psi(\frac{3}{4}) - \psi(\frac{1}{4}) = \pi$.

Further, consider the series

$$ S_p = \sum_{n=0}^{\infty} \left(\frac{(\frac{1}{2})_n}{n!} \right)^2 \frac{1}{(n+1)\cdots(n+p)}, \qquad (2.4) $$

where integer $p \geq 1$. This corresponds to

$$ S_p = \frac{1}{p!}\ {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; p+1; 1 \right) $$

which upon use of Gauss' summation theorem (1.4) reduces to

$$S_p = \frac{\Gamma(p)}{\Gamma^2(p+\frac{1}{2})}. \tag{2.5}$$

Thus, from (2.4) and (2.5) it follows that

$$S_1 = \frac{4}{\pi}, \qquad S_2 = \frac{16}{9\pi}, \qquad S_3 = \frac{128}{225\pi}$$

and so on.

If we let $a = b = \frac{1}{2}$ and $c = p+1$ for positive integer p in (2.2), then we have when $r = 1$ (with $m = m_1$)

$${}_3F_2\left[\begin{array}{l} \frac{1}{2}, \frac{1}{2}, f+m; 1 \\ p+1, f \end{array}\right] = \frac{p!\Gamma(p)}{\Gamma^2(p+\frac{1}{2})}\sum_{k=0}^{m}(-)^k \left(\begin{array}{c} m \\ k \end{array}\right) \frac{((\frac{1}{2})_k)^2}{(f)_k(1-p)_k} \quad (p > m).$$

When m=1, we therefore find

$$\sum_{n=0}^{\infty}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2 \frac{n+f}{(n+1)\cdots(n+p)} = \frac{\Gamma(p)}{\Gamma^2(p+\frac{1}{2})}\left(f + \frac{1}{4(p-1)}\right) \tag{2.6}$$

for p=2,3,..., and when m=2

$$\sum_{n=0}^{\infty}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2 \frac{(n+f)(n+f+1)}{(n+1)\cdots(n+p)}$$

$$= \frac{\Gamma(p)}{\Gamma^2(p+\frac{1}{2})}\left(f(f+1) + \frac{f+1}{2(p-1)} + \frac{9}{16(p-1)(p-2)}\right) \tag{2.7}$$

for p=3,4,....

When r=2 and $m_1 = m_2 = 1$ (so that m=2), we obtain from (2.2)

$${}_4F_3\left[\begin{array}{l} \frac{1}{2}, \frac{1}{2}, f_1+1, f_2+1; 1 \\ p+1, f_1, f_2 \end{array}\right] = \frac{p!}{f_1 f_2}\sum_{n=0}^{\infty}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2 \frac{(n+f_1)(n+f_2)}{(n+1)\cdots(n+p)}$$

$$= \frac{p!\Gamma(p)}{\Gamma^2(p+\frac{1}{2})}\left\{1 + \frac{C_1(2)}{4(p-1)} + \frac{9C_2(2)}{16(p-1)(p-2)}\right\},$$

where $C_1(2) = (1+f_1+f_2)/(f_1 f_2)$ and $C_2(2) = 1/(f_1 f_2)$. Hence

$$\sum_{n=0}^{\infty}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2 \frac{(n+f_1)(n+f_2)}{(n+1)\cdots(n+p)}$$

$$= \frac{\Gamma(p)}{\Gamma^2(p+\frac{1}{2})}\left(f_1 f_2 + \frac{f_1+f_2+1}{4(p-1)} + \frac{9}{16(p-1)(p-2)}\right) \tag{2.8}$$

for p=3,4,....

We remark that series such as (2.6) can also be obtained by a 'telescoping' process applied to the series S_p in (2.4). For example, it easy to see that

$$\sum_{n=0}^{\infty}\left(\frac{(\frac{1}{2})_n}{n!}\right)^2 \frac{(n+f)}{(n+1)\cdots(n+p)} = S_{p-1} + (f-p)S_p.$$

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