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# HOPF BIFURCATION ANALYSIS IN THREE SPECIES ECOLOGICAL MODEL

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**Abstract:** In this work we proposed a three species ecological model with a prey, predator and competitor. Distributed type of delay is incorporated in the interaction of prey and competitor species is taken for investigation. The system dynamics is studied at its interior equilibrium point with exponential type of delay kernel. The effect of Time delay on the dynamical behavior of the system is studied using Numerical simulation. It is observed that Hopf bifurcation exist for the system for different kernel strengths.

**Keywords and Phrases:** Co-existing state, local stability, global stability, Time delay, hopf bifurcation.
**2010 Mathematics Subject Classification:** 54A05.

## 1. Introduction

Mathematical modeling in Ecology gains importance in recent decades. The stability analysis of ecosystems is quite intersecting and complex in nature. Differential equations are widely used in the stability analysis. Braun [8] and Simon's [9] explain the applications of differential equations in this area. Lokta [1] and Volterra [2] studied the different models in population ecology. Kapur [3, 4] discussed the models in biology, medicine, epidemiology, ecology etc. May [5], Freedman [6], Paul colinvaux [7] contributed a lot to this field. Time delays are natural in ecological phenomenon. The stability analysis of time delay models are widely explained by Cushing, J. M [10], Sreehari Rao [11], Gopalaswamy. K [12]. Time delay in interactions in three species models with a prey, predator and competitor models are discussed by paparao [13 - 15]. In spite of that we proposed three species ecological model with distributed type delay model to investigate instability tendencies using different delay kernel strengths. The model is described by system of integro-differential equations and system dynamics is studied at co-existing state. Numerical simulation is carried out in support of stability analysis using MAT LAB simulation.

## 2. Mathematical Model

A three species ecological model with a prey ($N_1$) predator ($N_2$) and competitor ($N_3$) are considered for investigation. Here $N_2$ is praying on $N_1$. Apart from this all prey, predator species are competing with third species (competitor $N_3$). Death rates are also considered for three species. A time delay is introduced in the interaction of prey and competitor. Keeping the above aspects in view, the model is characterized by the following system of integro- differential equations.

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 - \alpha_{13} N_1 \int_{-\infty}^{t} k_1(t-u) N_3(u)\, du - d_1 N_1,$$
$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 - \alpha_{22} N_2 N_1 - \alpha_{23} N_2 N_3 - d_2 N_2,$$
$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 - \alpha_{31} N_3 \int_{-\infty}^{t} k_2(t-u) N_1(u)\, du - \alpha_{32} N_2 N_3 - d_3 N_3, \tag{2.1}$$

where the parameters in the above model is described as follows
$N_i$ : Population strengths of three species, $a_i$: Growth rates of three species, $\alpha_{ij}$ interspecies competition rate $\alpha_{ij}, (i \neq j)$ Intra species competition rate: di: Death rates of three species, kernel weights $k_1(t-u)$ and $k_2(t-u)$. Assume that all parameters are positive and Put $t-u=z$, we get the following system of equations

$$\frac{dN_1}{dt} = a_1 N_1 - \alpha_{11} N_1^2 - \alpha_{12} N_1 N_2 - \alpha_{13} N_1 \int_{0}^{\infty} k_1(z) N_3(t-z) dz - d_1 N_1,$$
$$\frac{dN_2}{dt} = a_2 N_2 - \alpha_{22} N_2^2 - \alpha_{21} N_2 N_1 - \alpha_{23} N_2 N_3 - d_2 N_2,$$
$$\frac{dN_3}{dt} = a_3 N_3 - \alpha_{33} N_3^2 - \alpha_{31} N_3 \int_{0}^{\infty} k_2(z) N_1(t-z) dz - \alpha_{32} N_2 N_1 - d_3 N_3 \tag{2.2}$$

Choose the kernels $k_1$ and $k_2$ such that

$$\int_0^{\infty} k_1(z)dz = 1, \int_0^{\infty} k_2(z)dz = 1, \int_0^{\infty} zk_1(z)dz < \infty, \int_0^{\infty} k_1(z)dz << \infty, \quad (2.3)$$

## 3. Equilibrium States

The co-existing state is obtained by solving system of equations (2.1) is given by E: Co-existing state:

$$\bar{N_1} = \frac{(a_1-d_1)(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})-\alpha_{12}(((a_2-d_2)\alpha_{33}-(a_3-d_3)\alpha_{23})+\alpha_{13}((a_2-d_2)\alpha_{32}-(a_3-d_3)\alpha_{22})}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23})-\alpha_{13}(\alpha_{21}\alpha_{32}+\alpha_{31}\alpha_{22})}$$

$$\bar{N_2} = \frac{((a_2-d_2)(\alpha_{11}\alpha_{33}+(a_1-d_1)\alpha_{21}\alpha_{33}+(a_1-d_1)\alpha_{31}\alpha_{23})-((a_3-d_3)\alpha_{11}\alpha_{23}+(a_3-d_3)\alpha_{21}\alpha_{13}+(a_2-d_2)\alpha_{13}\alpha_{31}))}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23})-\alpha_{13}(\alpha_{21}\alpha_{32}+\alpha_{31}\alpha_{22})}$$

$$\bar{N_3} = \frac{((a_3-d_3)\alpha_{11}\alpha_{22}+(a_3-d_3)\alpha_{21}\alpha_{12}+(a_2-d_2)\alpha_{12}\alpha_{31})-((a_2-d_2)\alpha_{11}\alpha_{32}+(a_1-d_1)\alpha_{21}\alpha_{32}+(a_1-d_1)\alpha_{31}\alpha_{22})}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}(\alpha_{21}\alpha_{33}+\alpha_{31}\alpha_{23})-\alpha_{1}3(\alpha_{21}\alpha_{32}+\alpha_{31}\alpha_{22})}$$

(3.1)

This equilibrium state exist only when, $\bar{N_1} > 0, \ \bar{N_2} > 0, \bar{N_3} > 0$.

## 4. Stability of the Equilibrium Point

**Theorem 4.1.** *The co-existing state* $(\bar{N_1}, \ \bar{N_2}, \bar{N_3})$ *is locally asymptotically stable if* $\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}k_1(\lambda) \ k_2(\lambda) > 0$.

**Proof.** Let the variational matrix is given by

$$J = \begin{bmatrix} -\alpha_{11}\bar{N_1} & \alpha_{12}\bar{N_1} & -\alpha_{13}\bar{N_1} \ k_1(\lambda) \\ \alpha_{21}\bar{N_2} & -\alpha_{22}\bar{N_2} & -\alpha_{23}\bar{N_2} \\ -\alpha_{31}\bar{N_3} \ k_2(\lambda) & -\alpha_{32}\bar{N_3} & -\alpha_{33}\bar{N_3} \end{bmatrix} \quad (4.1)$$

with The characteristic equation

$$\lambda^3 + b_1 \ \lambda^2 \ + b_2 \ \lambda + b_3 = 0 \quad (4.2)$$

where $b_1 = (\alpha_{11}\bar{N_1} + \alpha_{22}\bar{N_2} + \alpha_{33}\bar{N_3})$

$$b_2 = (\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})\bar{N_1}\bar{N_2} + (\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}k_1(\lambda)k_2(\lambda))\bar{N_1}\bar{N_3} + (\alpha_{22}\alpha_{33} - \alpha_{23}\alpha_{32})\bar{N_2}\bar{N_3}$$

$$b_3 = \bar{N_1}\bar{N_2}\bar{N_3}(\alpha_{11}\alpha_{22}\alpha_{33} + \alpha_{12}\alpha_{21}\alpha_{33} - \alpha_{13}\alpha_{22}\alpha_{31}k_1(\lambda)k_2(\lambda) - \alpha_{11}\alpha_{23}\alpha_{32} + \alpha_{12}\alpha_{23}\alpha_{31}k_2(\lambda) + \alpha_{13}\alpha_{21}\alpha_{32}k_1(\lambda)) \quad (4.3)$$

By Routh-Hurwitz criteria, the system is stable if $b_1 > 0, \ (b_1 \ b_2 - b_3)$ and $b_3 \ (b_1 \ b_2 - b_3) > 0$ $b_1 = (\alpha_{11} \ \bar{N_1} + \alpha_{22} \ \bar{N_2} + \alpha_{33} \ \bar{N_3}) > 0$
By algebraic calculations
$(b_1 \ b_2 - b_3) = (\alpha_{11}^2 \ \alpha_{22} + \alpha_{11}\alpha_{12}\alpha_{21})\bar{N_1}^2\bar{N_2} + (\alpha_{11}^2 \ \alpha_{33} - \alpha_{11}\alpha_{13} \ \alpha_{31}k_1(\lambda)k_2(\lambda))\bar{N_1}^2\bar{N_3} +$

$(\alpha_{22}^2\alpha_{33}+\alpha_{22}\alpha_{23}\alpha_{32})\bar{N_2}^2\bar{N_3}+(\alpha_{22}^2\alpha_{11}+\alpha_{22}\alpha_{12}\alpha_{21})\bar{N_2}^2\bar{N_1}+(\alpha_{11}\alpha_{33}^2-\alpha_{33}\alpha_{13}\alpha_{31}$ $k_1(\lambda)k_2(\lambda))\bar{N_3}^2\bar{N_1}+(\alpha_{22}\alpha_{33}^2+\alpha_{33}\alpha_{23}\alpha_{32})\bar{N_2}\bar{N_3}^2+\bar{N_1}\bar{N_2}\bar{N_3}(2\alpha_{11}\alpha_{22}\alpha_{33}+\alpha_{12}\alpha_{23}\alpha_{31}$ $k_2(\lambda)+\alpha_{13}\alpha_{21}\alpha_{32}k_1(\lambda)\ (b_1\ b_2-b_3)>0$ if $\alpha_{11}\ \alpha_{33}-\alpha_{13}\alpha_{31}\ k_1(\lambda)k_2(\lambda)>0$ (4.4)

Also $b_3(b_1\ b_2-b_3)>0$ if $\alpha_{11}\ \alpha_{33}-\alpha_{13}\alpha_{31}\ k_1(\lambda)k_2(\lambda)>0$

Hence the co-existing state $(\bar{N_1},\ \bar{N_2},\bar{N_3})$ is locally asymptotically stable if $\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31}k_1(\lambda)k_2(\lambda)>0$. Let us define the kernel as follows $k_1(z)=k_2(z)=a\ e^{-az}$ for $a>0$, then the Laplace transform of $k_1(z)$ and $k_2(z)$ are defined as $k_1(\lambda)=k_2(\lambda)=\int_0^\infty e^{-\lambda t}\ a\ e^{-at}dt=\frac{a}{a+\lambda}$.

Then the system is locally asymptotically stable if $\alpha_{11}\ \alpha_{33}>\alpha_{13}\ \alpha_{31}\left[\frac{a}{a+\lambda}\right]^2$.

## 5. Global Stability

**Theorem 5.1.** *The co-existing state* $(\bar{N_1},\ \bar{N_2},\bar{N_3})$ *is locally asymptotically stable.*

**Proof.** Let the Lyapunov function be

$$V(N_1,N_2,N_3)=\sum_{i=1}^{3}N_i-\bar{N_i}-\bar{N_i}\ log(\frac{N_i}{\bar{N_i}})+\frac{1}{2}\ \alpha_{13}\int_0^\infty k_1(z)\int_{t-z}^{t}[N_3-\bar{N_3}]^2du\ dz$$
$$+\frac{1}{2}\ \alpha_{31}\ \int_0^\infty\ k_2(z)\ \int_{t-z}^{t}[N_1-\bar{N_1}]^2du\ dz \quad (5.1)$$

The time derivative of 'V' along the solutions of equations (2.1) is

$$V'(t)=\sum_{i=1}^{3}\frac{N_i-\bar{N_i}}{N_i}N_i'+\frac{1}{2}\ \alpha_{13}\ \int_0^\infty\ k_1(z)[N_3(t-z)-\bar{N_3}]^2dz+\frac{1}{2}\ \alpha_{31}\ \int_0^\infty\ k_2(z)[N_1-\bar{N_1}]^2dz-\frac{1}{2}\ \alpha_{31}\ \int_0^\infty\ k_2(z)[N_1(t-z)-\bar{N_1}]^2dz \quad (5.2)$$

From the equation (2.1) we have

$$V'(t)=[N_1-\bar{N_1}](a_1-\alpha_{11}N_1-\alpha_{12}N_2-\alpha_{13}\ \int_0^\infty\ k_1(z)N_3(t-z)dz-d_1)+[N_2-\bar{N_2}](a_2-\alpha_{22}N_2+\alpha_{21}N_1-\alpha_{23}N_3-d_2)+[N_3-\bar{N_3}](a_3-\alpha_{32}N_3--\alpha_{31}\ \int_0^\infty\ k_2(z)N_1(t-z)dz-\alpha_{32}N_2-d_3)+\frac{1}{2}\alpha_{31}[N_1-\bar{N_1}]^2+\frac{1}{2}\alpha_{31}[N_3-\bar{N_3}]^2-\frac{1}{2}\alpha_{13}\ \int_0^\infty\ k_1(z)[N_3(t-z)-\bar{N_3}]^2dz-\frac{1}{2}\alpha_{31}\ \int_0^\infty\ k_2(z)[N_1(t-z)-\bar{N_1}]^2dz$$

By proper choice of $a_1,\ a_2,\ a_3$ and using the inequality $ab\le\frac{a^2+b^2}{2}$ $\int_0^\infty\ k_1(z)[N_3(t-z)-\bar{N_3}]^2dz\le\int_0^\infty\ k_1(z)dz=1$ and $\int_0^\infty\ k_2(z)[N_1(t-z)-\bar{N_1}]^2dz\le\int_0^\infty\ k_2(z)dz=1$

$$=-\alpha_{11}(N_1-\bar{N_1})^2-\alpha_{22}(N_2-\bar{N_2})^2-\alpha_{33}(N_3-\bar{N_3})^2-\frac{(\alpha_{32}+\alpha_{23})}{2}\left[(N_2-\bar{N_2})^2+(N_3-\bar{N_3})^2\right]+\frac{1}{2}\alpha_{31}[N_1-\bar{N_1}]^2+\frac{1}{2}\alpha_{31}[N_3-\bar{N_3}]^2+\frac{(\alpha_{21}-\alpha_{12})}{2}\left[(N_2-\bar{N_2})^2+(N_1-\bar{N_1})^2\right]-\frac{1}{2}(\alpha_{31}+\alpha_{13})$$
$$\le-\left\|\left(\alpha_{11}+\frac{1}{2}\alpha_{31}+\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{12}\right)\right\|(N_1-\bar{N_1}]^2)\|-\left\|\left(\alpha_{22}-\frac{1}{2}\alpha_{12}+\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{32}-\frac{1}{2}\alpha_{23}\right)\right\|(N_2-\bar{N_2}]^2)\|-\left\|\left(\alpha_{33}+\frac{1}{2}\alpha_{13}-\frac{1}{2}\alpha_{32}-\frac{1}{2}\alpha_{23}\right)\right\|(N_3-\bar{N_3})^2)\|-\frac{1}{2}\|(\alpha_{31}+\alpha_{13})\|\ V^1(t)\le-\mu\ \sum_{i=1}^{3}\left[N_i-\bar{N_i}^2\right]<0$$

where, $\mu=min\left(\alpha_{11}+\alpha_{22}+\alpha_{33}+\frac{1}{2}\alpha_{13}+\frac{1}{2}\alpha_{31}+\frac{1}{2}\alpha_{21}-\frac{1}{2}\alpha_{12}-\frac{1}{2}\left(\alpha_{31}+\alpha_{13}\right)\right)$

Therefore the system is globally stable at co-existing state $E_1\left(\bar{N_1},\bar{N_2},\bar{N_3}\right)$

## 6. Numerical Example

Graphs Description: Fig A: Time series plot Fig B: Phase portrait

**Example 6.1:** Let $a_1 = 1.5, a_2 = 2.65, a_3 = 3.45, \alpha_{11} = 0.1, \alpha_{12} = 0.5, \alpha_{13} = 0.01$, $\alpha_{21} = 0.5$, $\alpha_{22} = 0.2$, $\alpha_{23} = 0.4$, $\alpha_{31} = 0.2$, $\alpha_{32} = 0.2$, $\alpha_{33} = 0.2$, $N_1 = 10, N_2 = 15$, $N_3 = 15$, $d_1 = 0.05$, $d_2 = 0.05$ and $d_3 = 0.05$. The solution curves for system (2.2) are shown below for the above parameters The systems of equations (2.2) are simulated using MATLAB using ode45 for the delay kernels $k_1(z) = k_2(z) = a\ e^{-az}$ for $a > 0$ with the parameters shown in Example 1 with different kernel values are plotted below.

Case (1): for $\lambda = 0.005, a = 1$

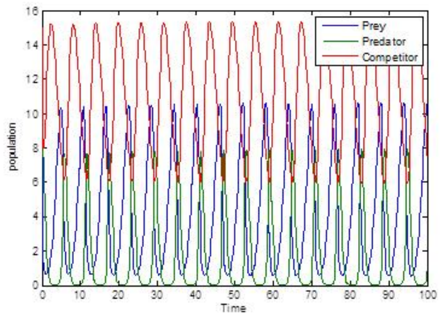


Figure 1: Figure 6.1.1(A)

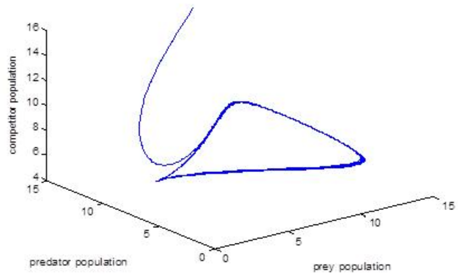


Figure 2: Figure 6.1.1(B)

The system is unstable due to unbounded oscillations in three species population for $\lambda = 0.005, a = 1$. For fixed value of $\lambda = 0.005$ and the value of a is increased from 1 to 100, the system still exhibits unstable nature. For fixed $\lambda = 0.005$ and $a = 0.01$ the system is stable and converge to an equilibrium point E(5, 2, 12). The plots are shown below

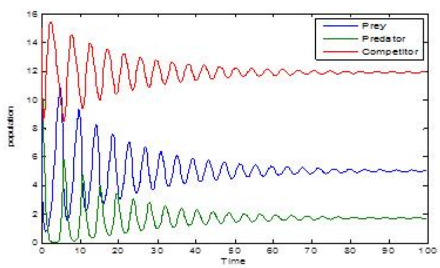


Figure 3: Figure 6.1.2(A)

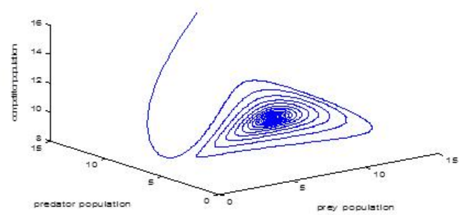


Figure 4: Figure 6.1.2(B)

The system is stable for $\lambda = 0.005$ and the range of afrom $[0.01, 0.023]$. So the delay argument for fixed $\lambda = 0.005$ and afrom $[0.01, 0.023]$, the system dynamics becomes stable and for a greater than 0.023 for this fixed $\lambda = 0.005$, the system becomes unstable. Hence the system exhibit hopf bifurcation for $\lambda = 0.005$ and $a > 0.023$.

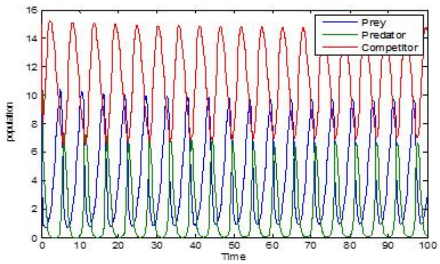


Figure 5: Figure 6.2.1(A)

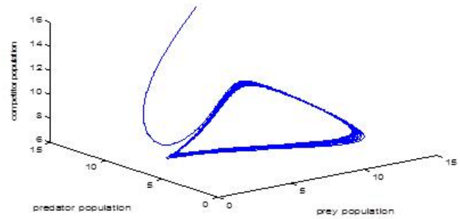


Figure 6: Figure 6.2.1(B)

The system is unstable due to unbounded oscillations in three species population for $\lambda = 0.05, a = 1$. For fixed value of $\lambda = 0.05$ and the value of a is increased from 1 to 100, the system still exhibits unstable nature. For$\lambda = 0.05$ and a $= 0.05$ the system is asymptotically stable to the fixed equilibrium point E(6, 2, 13) and the time series plot and phase plane is given below.

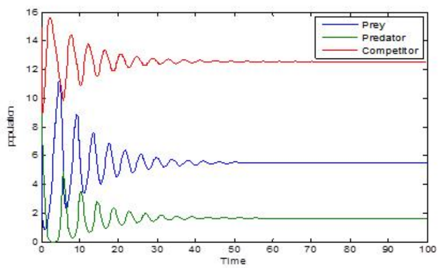


Figure 7: Figure 6.2.2(A)

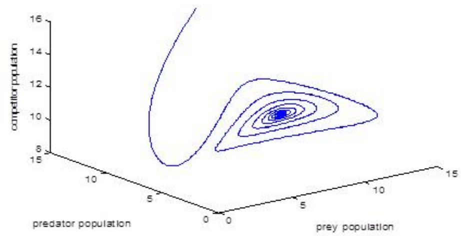


Figure 8: Figure 6.2.2(B)

The system is stable for $\lambda = 0.05$ and the range of afrom [0.01, 0.23]. So the delay argument for fixed $\lambda = 0.05$ and afrom [0.01, 0.23], the system dynamics becomes stable and for a greater than 0.23 for this fixed $\lambda = 0.05$, the system becomes unstable. Hence the system exhibit hopf bifurcation for $\lambda = 0.05$ and $a > 0.23$.

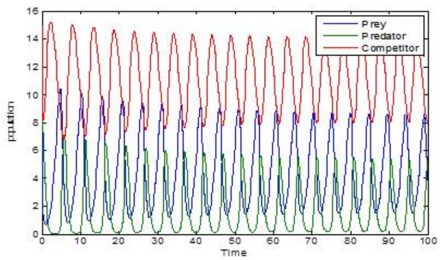


Figure 9: Figure 6.3.1(A)

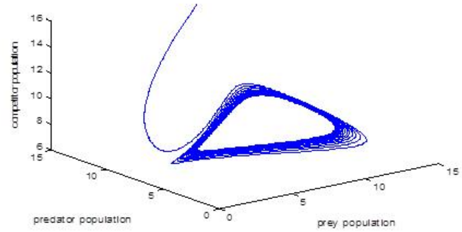


Figure 10: Figure 6.3.1(B)

The system is unstable due to unbounded oscillations in three species population for fixed $\lambda = 0.5, a = 5$. For $\lambda = 0.5, a = 2$ the system is stable and quenching to the equilibrium point E(6, 1, 11). The plots are shown below

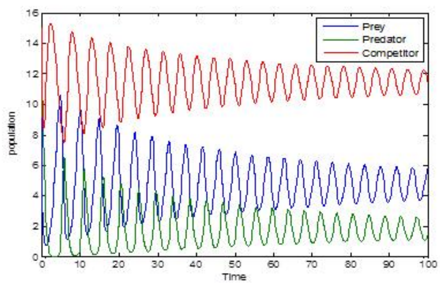


Figure 11: Figure 6.3.2(A)

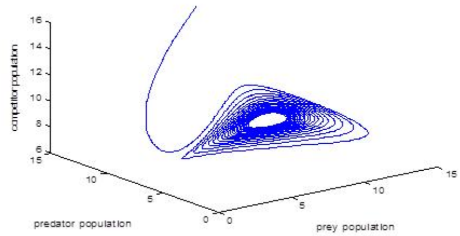


Figure 12: Figure 6.3.2(B)

The system is stable for $\lambda = 0.5$ and the range of afrom [0.1, 2.3]. So the delay argument for fixed $\lambda = 0.5$ and afrom[0.1, 2.3], the system dynamics becomes stable and for a greater than 2.3 for this fixed $\lambda = 0.5$, the system becomes unstable. Hence the system exhibit hopf bifurcation for $\lambda = 0.5$ and $a > 2.3$.

## 7. Conclusion

An Ecological model is proposed with a prey, predator and a competitor species with death rates. A time delay (distributed type) was introduced in the interaction of prey and competitor species. Exponential type delay kernel is chosen for investigation. The system is locally stable if at co-existing state. A suitable Lyapunov's function is identified in pursuit of global stability. Numerical simulation is carried out by choosing suitable parameters with exponential delay kernel. For different values of kernel strengths ($\lambda$ and a) for the same parameters taken in example 6.1. The dynamics of the system is change from stable to unstable vice-versa. So the delay arguments play a key role in switch over stability analysis of the system which leads to hopf bifurcation. The Hopf bifurcation parameters is identified for this model for (i)$\lambda = 0.005$ and $a >= 0.02$(ii)$\lambda = 0.05$ and $a > 0.23$ (iii) $\lambda = 0.5$ and $a > 2.3$. Hence the delay arguments can change the stable equilibrium to unstable or vice-versa.

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